CONVEX SETS AND KÄHLER MANIFOLDS

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0.1. *Brunn-Minkowski inequality.* Recall that the *Minkowski sum* X+Y of subsets X and Y in the Euclidean space $\mathbb{R}^n$ is the set of the sums $x+y \in \mathbb{R}^n$ for all $x \in X$ and $y \in Y$. An equivalent definition is

$$X+Y = \bigcup_{y \in Y} X+y$$

where X+y denotes the y-translate of X which is the same thing as the sum of X with the one-point set {y}. Note that
X+Y = Y+X as x+y = y+x in $\mathbb{R}^n$.

0.1 A. *Example.* Let X_ε be the ε-ball in $\mathbb{R}^n$ around the origin. Then, by the second definition, $X_\varepsilon+y$ equals the union of the ε-balls in $\mathbb{R}^n$ with centers in Y which is customary called the *ε-neighborhood* of Y.

0.1.B. **Brunn-Minkowski theorem**. The n-dimensional volume (i.e. Lebegue's measure) *of* X+Y *is bounded from below by*

$$[\mathrm{Vol}\,(X+Y)]^{1/n} \geq (\mathrm{Vol}\,X)^{1/n} + (\mathrm{Vol}\,Y)1/n\,. \qquad (*)$$

*Remarks and corollaries.*0.1 B_1. We are most interested here in the classical case of (*) where X and Y are *bounded convex* subsets in $\mathbb{R}^n$. Yet, (*) remains valid for arbitrary (measurable) subsets X and Y in $\mathbb{R}^n$ (see 3.1.).

0.1.B_2. Let X and Y be rectangular solids with mutually parallel edges of lengths $a_1,\ldots,$ a_n and $b_1,\ldots, b_n$. Say,

$$X = [o,a_1] \times \ldots \times [o,a_n],$$
$$Y = [o,b_1] \times \ldots \times [o,b_n].$$

Then

$$X+Y = [o,a_1+b_1] \times \ldots \times [o,a_n+b_n]$$

and (*) reduces to the following well-known algebraic inequality,

$$\left(\prod_{i=1}^{n} (a_i+b_i)\right)^{1/n} \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n} + \left(\prod_{i=1}^{n} b_i\right)^{1/n} \qquad (*)$$

0. 1 B_3. Let Y have *smooth* boundary ∂Y and take the ε-ball B_ε in r^n for X. Then, one easily sees (compare 0.1.A) that the (n-1)-dimensional volume of ∂Y satisfies

$$\mathrm{Vol}\,\partial Y = \lim_{\varepsilon \to 0} \varepsilon^{-(n-1)} (\mathrm{Vol}\,(Y+B_\varepsilon) - \mathrm{Vol}\,Y).$$

Thus, (*) yields the *Euclidean isoperimetric inequality,*

$$\mathrm{Vol}\,\partial Y \geq C_n\,(\mathrm{Vol}\,Y)^{\frac{n}{n-1}}$$

where C_n denotes the (n-1)-dimensional volume of the boundary sphere of the ball $B \subset \mathbb{R}^n$ normalized by the condition Vol B = 1.

0.2. *Hodge-Teissier-Hovanski inequality.* Consider the Cartesian product of two complex projective spaces $P_1 \times P_2$ with the standard metric and let V be a complex algebraic subvariety in $P_1 \times P_2$ of complex dimension n. (The reader unfamiliar with this terminology is addressed to section 3.3.). Denote by $V_1 \subset P_1$ and $V_2 \subset P_2$ the projections of V to P_1 and to P_2.

0.2.A. **Alqebraic Brunn-Minkowski.** *If* V *is irreducible* (see 3.3.), *then the* 2n-*dimensional volumes of* V, V_1 and V_2 *satisfy*

$$(\mathrm{Vol}\,V)^{1/n} \geq (\mathrm{Vol}\,V_1)^{1/n} + (\mathrm{Vol}\,V_2)^{1/n}. \qquad (+)$$

Remark and Corollaries. 0.2.A_1. If n=1, then (+) is trivial. In fact one has equality in this case.

0.2.A_2. If n=2 then (+) is equivalent to the *Hodge index theorem* (see 3.3.). Note that (+) may easily fail if V is reducible. For example, take

$$V = (V_1 \times v_2) \cup (v_2 \times V_2)$$

for $V_i \subset P_i$ and $V_i \subset P_i$ for i = 1,2. Then

$$(\text{Vol } V)^{1/n} = (\text{Vol } V_1 + \text{Vol } V_2)^{1/n} < (\text{Vol } V_1)^{1/n} + (\text{Vol } V_2)^{1/n}.$$

0.2.A_3. The inequality (+) for $n \geq 3$ was discovered by Hovanski and Teissier. Their proof (see 3.3.) goes by induction on n = dim V which starts with n = 2, where the inductive step for $n \geq 3$ is realized by intersecting V with an appropriate hypersurface H in $P_1 \times P_2$, and where the irreducibility of the intersection $V \cap H$ (having the dimension by one less than V) is achieved with the *Bertini* theorem (see 3.3.). In fact, Teissier and Hovanski proved a refinement of (+) which is parallel to the *Alexandrov-Fenchel inequality* for convex sets (see 1 .6.). Alexandrov gave two proofs of his inequality. The first proof (see $[Al]_1$) is combinatorial and resembles the algebra-geometric argument by Hovanski and Teissier (instead of Hodge index theorem for n = 2 Alexandrov uses a corresponding geometric inequality of Minkowski). The second proof by Alexandrov (see $[Al]_2$) appeals to the elementary theory of second order elliptic operators. We shall see in §2 that a modern reedition of Alexandrov's proof (exterior products of differential forms instead of mixed discriminants of quadratic forms) yields the Hodge-Teisdsier-Hovanski inequality as readily (even faster) as it yields the Alexandrov-Fenchel inequality (for n = 2 Alexandrov's argument is essentially equivalent to Hodge's proof of his index theorem).

0.2.B. *Moment map, Legendre transform and the implication* (+) $\Rightarrow$ (*). A variety V is called *toral* if it admits an isometric (for the metric induced from $P_1 \times P_2 \supset V$) action of the torus T^n. Such an action induces what is called *the moment map* $M : V \Rightarrow \mathbb{R}^n$ which is defined with the induced *symplectic* (Kähler) *form* on V (see 3.2.). Similar (moment) maps, also denoted M, are defined for V_1 and V_2. One shows (see 3.2.) that M preserves volumes (up to a normalizing constant) and that the image M(V) is the Minkowski sum of the moment-images of V_1 and V_2,

$$M(V) = M(V_1) + M(V_2).$$

Thus (*) $\Rightarrow$ (+) for toral varieties V. On the other hand one knows (see 2.4. and 3.3) that for any pair of *convex polyhedra* X_1 and X_2 in $\mathbb{R}^n$ with vertices in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, there exists a toral variety V such that $M(V_i) = X_i$ for i = 1,2. Using this along with an approximation of convex sets by polyhedra with rational vertices one derives (*) from (+) for all *convex* subsets in $\mathbb{R}^n$.

0.2.B_1. *REMARK*. The correspondence between toral varieties and convex polyhedra goes back to Newton and Minding (see the discussion by Hovanski in chapter 4 of [Bu-Za]). The relation between (+) and (*) was discovered by Teissier and Hovanski (see [T] and [B-z]). The approach using the moment map is due to Arnold and Atiyah (see $[At]_2$).

0.2.B_2. The action of T^n on V can be complexified to an action of $(\mathbb{C}^\times)^n = T^n \times (\mathbb{R}^\times_+)^n$ on V (see 3.2.). Then the restriction of the moment map to the $(\mathbb{R}^\times_+)^n$-orbits can be identified with the *Legendre transform* for the *Kähler potential* on V (see 3.2. and $[At]_2$). Note that this kind of Legendre transform is built in into Alexandrov's argument as it applies to *supporting functions* of the convex sets in question (see $[Al]_2$).

§1. *Legendre transform, mixed volumes and Kähler forms.* Consider a C^1-function f on a linear space L and let us interprete the differential of f as a map of L into the dual space L', say $D_f : L \to L'$ (If L is a Hilbert space one can use instead the *gradient* map $L \to L$ for $x \to \mathrm{grad}_x f$ that some people find more geometric).

Recall that a map $\varphi : L \to L'$ is called *monotone increasing* if

$$< \varphi(x_1) - \varphi(x_2), x_1 - x_2 > \geq 0$$

for all x_1 and x_2 in L. One calls φ strictly increasing if the above inequality is strict for all x_1 and $x_2 \neq x_1$. It is obviou that every strictly increasing map is one-to-one. In particular, if such a φ is continuous and L is finite dimensional, then φ is a homeomorphism. Also observe that the map $\varphi = D_f$ is (strictly) increasing if and only if f is (strictly) convex. Thus we obtain the

1.1. *Homeomorphism property. If is a strictly convex C^1-function on $\mathbb{R}^n$ then the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (onto an open subset in $\mathbb{R}^n$).*

The following property is somewhat more exciting.

1.2. *Convexity theorem. If* f *is convex then the closure of the image* $D_f(\mathbb{R}^n) \subset \mathbb{R}^n$ *is convex.*

Proof. For an arbitrary function f on L denote by $L'_f \subset L'$ the set of those linear functions y on L for which

$$f - y \geq \text{const} > -\infty . \qquad (1)$$

This means that the graph $\Gamma_f \subset L \times \mathbb{R}$ lies above that of the function y-const, and so L'_f contains the image $D_f(L) \subset L'$ for convex functions f. In fact if $y = d_{x_0} f$ then the graph $H_y \subset L \times \mathbb{R}$ of the function $y(x) + f(x_0) - y(x_0)$ is tangent to the graph $\Gamma_f \subset L \times \mathbb{R}$ at the point $(x_0, f(x_0))$. Hence Γ_f lies above the hyperplane H_y for all $y \in D_f(L)$ in the case where f is convex.

Next we observe the subset $L'_f \subset L$ is convex, as the inequalities

$$f - y_1 > -\infty \text{ and } f - y_2 > -\infty$$

obviously imply the same inequality for convex combinations of y_1 and y_2,

$$f - (ty_1 + (1-t)y_2) > -\infty .$$

To conclude the proof we must show that L'_f is *contained* in the closure of $D_f(\mathbb{R}^n)$. This is equivalent to

$$\inf_{x \in L} \|d_x \; g\| = 0$$

for the functions $g = f-y$ satisfying the above (boundness away from $-\infty$) condition (1). In fact, if (2) is violated and
$\|d_x \; g\| > \varepsilon > 0$ for all $x \in L$, then $\inf g = -\infty$ as the following trivial lemma shows.

1 .2.A. *Let* X *be a complete metric space and* $g : X \to \mathbb{R}$ *a continuous function, such that for every* $x \in X$ *there exists* $x' \in x$ *different from* x, such that

$$g(x) - g(x') \geq \varepsilon \, \mathrm{dist}(x, x'),$$

where ε is a fixed positive number. Then

$$\inf_{X} g(x) = -\infty .$$

1.2.B. *Remark.* The above discussion presents a tiny piece of the convex duality theory going back to Legendre whose name is attached to the transform of f from L to $D_f(L) \subset L'$ by the map D_f,

$$f'(y) = f(D_f^{-1}).$$

The Legendre transform f' of f is correctly defined for strictly convex functions f as D_f is one-to-one. In this case f' also is strictly convex and satisfies *Legendre duality relation* $D_{f'} = D_f^{-1}$ under an appropriate (reflexivity) condition on L (which is obviously satisfied for $L = \mathbb{R}^n$).

1.3. *Minkowski additivity* of L'_f and $D_f(L)$. If $y_1 \in L_{f_1}$ and $y_2 \in L_{f_2}$ (see (1) above), then, obviously) $\inf_L(f_1+f_2-y_1-y_2) > -\infty$, that is y_1+y_2 is contained in the Minkowski sum of L_{f_1} and L_{f_2}.

In other words

$$L'_{f_1} + L'_{f_2} \subset L'_{f_1+f_2} . \qquad (2)$$

It is equally obvious that

$$D_{f_1}(L) + D_{f_2}(L) \supset D_{f_1+f_2}(L), \qquad (3)$$

as $d(f_1 + f_2) = df_1 + df_2$.

Thus we obtain the following

1.3.A. **Additivity.** *If* f_1 *and* f_2 *are strictly convex functions on* $\mathbb{R}^n$, *then*

$$D_{f_1+f_2}(\mathbb{R}^n) = D_{f_1}(\mathbb{R}^n) + D_{f_2}(\mathbb{R}^n). \qquad (4)$$

1.4. ***Brunn-Minkowski theorem for convex functions*** f. Let $[D^2f]^n$ denote the determinant of the Hessian D^2f of f ,

$$[D^2f]^n = \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

and note that $[D^2f]^n$ equals the Jacobian of the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$.
Therefore

$$\mathrm{Vol}\, D_f(\mathbb{R}^n) = \int_{\mathbb{R}^n} [D^2 f]^n \qquad (5)$$

for all strictly convex C^2 -functions f on $\mathbb{R}^n$.

1.4.A. *Remark.* For an arbitrary (non-smooth) ccnvex function f one can define $[D^2f]^n$ as a measure on $\mathbb{R}^n$ and show that

$$\mathrm{Vol}\, L' = \int_{R^n} [D^2 f]^n .$$

Now we apply (Brunn-Minkowski) inequality (*) in 0.1.B. to $D_{f_1}(\mathbb{R}^n)$ i = 1,2 and obtain the following

1 .4.B. **Theorem.** *Every two strictly convex* C^2*-functions* f_1 *and* f2 on $\mathbb{R}^n$ *satisfy*

$$\left(\int_{R^n}\left[D^2(f_1+f_2)\right]^n\right)^{1/n} \geq \left(\int_{R^n}\left[D^2 f_1\right]^n\right)^{1/n} + \left(\int_{R^n}\left[D^2 f_2\right]^n\right)^{1/n} \cdot \quad (**)$$

1.4.B$_1$. *Remark.* This inequality remains valid for *all* convex functions on $\mathbb{R}^n$. This can be derived from (**) by a simple approximation argument or proved more directly using (*) and 1.4.A.

1.4.C. *Implication* (**) $\Rightarrow$ (*) *for convex sets in* $\mathbb{R}^n$. Let Y be a convex bounded open subset in $L' = \mathbb{R}^n$ and define f (x) on $L = \mathbb{R}^n$ by

$$f(x) = \log \int_R \exp\langle x,y\rangle dy .$$

One checks by a straightforward computation that f is real analytic and strictly convex, and that the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$ sends each x to the center of gravity of the measure $\exp\langle x,y\rangle dy$ on Y. It follows that

$$D_f(\mathbb{R}^n) \subset Y.$$

To show that $D_f(\mathbb{R}^n) \supset Y$ take a point x_0 such that the linear function $\langle x_0,y\rangle$ on $L' \supset Y$ has *only one* maximum point, say y_0, in the closure $ClY \subset L'$ of Y. Note that these points y_0 are exactly the extremal points of ClY. Now we look at the measures $\exp(\lambda x_0,y)\, dy$ on Y and see that these concentrate at y_0 as $\lambda \to \infty$. Hence, the closure of the image of D_f contains *all* extremal points y_0 of Cl Y. Since the image $D_f(\mathbb{R}^n) \subset Y$ is convex, it necessarily equals Y.

1.4.C_1. **Conclusion.** *Every convex bounded open subset* Y *in* $\mathbb{R}^n$ *admits a surjective diffeomorphism* $D_f : \mathbb{R}^n \to Y$ *for some strictly convex* C^2*-function* f *on* $\mathbb{R}^n$. *Thus* (**) $\Rightarrow$ (*) *for convex bounded open subsets. This trivially implies that* (**) $\Rightarrow$ (*) *for all convex subsets in* $\mathbb{R}^n$.

1.4.C_2. *Remark.* There are many convex functions f with $D_f(\mathbb{R}^n) = Y$. For example, instead of the Lebegue's measure dy on Y one can take an arbitrary measure $d\mu$ on Y, such that the convex hull of the support of μ equals the closure of Y. Then one sees as earlier that $D_f(\mathbb{R}^n) = Y$ for

$$f(x) = \log \int_{\mathbb{R}^n} \exp\langle x,y\rangle d\mu .$$

However, for every compact convex subset Y, there is a distinguished convex function f_0 (which is *non-smooth* and not *strictly* convex), called the support function of Y, such that $L'_{f_0} = Y$. This function is characterized by the homogeneity, $f_0(\alpha x) = \alpha f_0(x)$ for all $\alpha \geq 0$ (as well as by convexity and the relation $L'_{f_0} = Y$). It is easy to see that f_0 equals the infimum of the convex functions f, such that $L'_f \supset Y$). Usually, one defines f_0 as the infimum of the *linear* functions y (x) on L over all $y \in L' \setminus Y$.

Our choice of $f = \log \int \exp$ is motivated by the Kähler geometry in $\mathbb{C}P^n$ (see 2.4.).

1.5. *Kähler formulation of* (**). Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way,

$$\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n = \mathbb{C}^n,$$

and let us denote by $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the operator corresponding to the multiplication by $\sqrt{-1}$ in $\mathbb{C}^n$. We also denote by J the induced operator on vector fields and differential forms on $\mathbb{R}^{2n}$. We call an exterior 2-form ω on $(\mathbb{R}^{2n}, J)$ *positive* if $\omega(\partial, J\partial) \geq 0$ for all vector fields ∂ on $\mathbb{R}^{2n}$ and call ω *strictly positive* is if $\omega(\partial, J\partial) > 0$ for all non-vanishing fields ∂. Say that ω is a *(1,1)-form* if $J\omega = \omega$, that is $\omega(J\partial_1, J\partial_2) = \omega(\partial_1, \partial_2)$. Since $J^2 = -\mathrm{Id}$ and ω is antisymmetric, this is equivalent to the symmetry of the form h defined by $h(\partial_1, \partial_2) = \omega(\partial_1, J\partial_2)$. Note that such a ω is positive if and only if the (quadratic) form h is positive semidefinite.

Recall that the *second differential* $H = D^2f$ of a function f is the quadratic form defined by the formula

$$H(\partial_1,\partial_1) = \partial_1(\partial_2 f)$$

for all *translation invariant* (parallel) vector fields ∂_1 and ∂_2, where ∂f stands for the (Lie) derivative of f along ∂.

Another useful second order differential operator, now from functions to exterior 2-forms, is

$$f \to \omega = dJdf,$$

where d is the exterior differential first applied to f and then to the 1-form Jdf.

A straightforward computation expresses dJd in terms of D^2 by

$$h = H + JH \qquad (6)$$

where $H = D^2f$ and h is defined along with $\omega = dJdf$ by $h(\partial_1,\partial_2) = \omega(\partial_1, J\partial_2)$. Note that the definition of h on the left hand side of identity (6) uses only J while the definition of H via D^2 needs the *affine structure* of $\mathbb{R}^{2n}$.

Since the form H is symmetric, the form h also is symmetric as well as J-invariant. Hence, the *form* $\omega = dJdf$ *is* (1,1) *for all functions* f *on* $\mathbb{R}^{2n}$.

Let us divide $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n$ by the n-dimensional lattice $\sqrt{-1}\ \mathbb{Z}^n$ and denote by $V = \mathbb{R}^n \times T^n$ the resulting manifold for the torus

$$T^n = \sqrt{-1}\, \mathbb{R}^n / \sqrt{-1}\, \mathbb{Z}^n.$$

The notions of a (1,1)-form and of positivity descend from $\mathbb{R}^{2n}$ to V along with the operator J which acts on the tangent bundle of V. Now we can formulate the following

1.5.A. **Brunn-Minkowski inequality for T^n-invariant forms on** V.

Let ω_1 *and* ω_2 *be exact positive* (1,1)*-forms on* V *which are invariant under the natural* T^n*-action on* V. *Then the top-dimensional exterior power* $(\omega_1+\omega_2)^n$ *satisfies,*

$$\left(\int_V (\omega_1 + \omega_2)^n\right)^{1/n} \geq \left(\int_V \omega_1^n\right)^{1/n} + \left(\int_V \omega_2^n\right)^{1/n}.$$

Let us show that (***) is equivalent to (**) in 1.4.B. First we prove the implication (***) $\rightarrow$ (**) by relating to each function f on $\mathbb{R}^n$ the form ω on V which is the djd $\tilde{f}$ of the pull-back $\tilde{f}$ of f to V for the projection $V \rightarrow \mathbb{R}^n = V/T^n$. It is clear that

$$\int_V \omega^n = \int_{R^n} [D^2 f]^n$$

for all f on $\mathbb{R}^n$ and that ω is positive if and only if f is convex. Since $\omega = dJd\widehat{f}$ it is an *exact* (1,1)*-form* and so (***) $\rightarrow$ (**).

To prove that (**) $\rightarrow$ (***) we start with an exact (1,1)-form ω on V. The exactness obviously implies that ω vanishes on every T^n-orbit in V. It follows that the associated quadratic form h admits a unique decomposition

$$h = \tilde{H} + J\tilde{H}$$

where $\tilde{H}$ is induced from a quadratic form H on $\mathbb{R}^n$. The equality $d\omega = 0$ implies (by a straightforward computation) that

$$\partial_1 H(\partial_2, \partial_3) = \partial_2 H(\partial_1, \partial_3)$$

for all parallel fields ∂_1, ∂_2 and ∂_3 on $\mathbb{R}^n$. Hence $H = D^2 f$ for some function on $\mathbb{R}^n$ which is convex as ω is positive.

Hence, (***) does follow from (**).

1.6. *Mixed volumes.* Let us use notation y^I for the monomial $y_1^{i_1} \dots y_k^{i_k}$, where $y = (y_1, \dots, y_k)$ is a string of variables and $I = (i_1, \dots, i_k) \in Z_+^k$ denotes the multiindex with non-negative integer entries. We denote by $|I| = i_1 + \dots + i_k$ the *degree* of y^I and observe that the monomials $\{y^I\}_{|I|=n}$ constitute a basis in the space of polynomials in $y_1, \dots y_k$ of degree n. Next, we write $Iy = i_1 y_1 + \dots + i_k y_k$ and observe that (by a trivial argument) the polynomials $\{(Iy)^n\}_{|I|=n}$ also constitute a basis in the space of polynomials. In other words every monomial is a linear combination of some Iy with universal coefficients. For example,

$$y_1 y_2 = 1/2((y_1+y_2)^2 - y_1^2 - y_2^2).$$

Similarly, for strings of differential 2-forms, $\Omega = (\omega_1, \dots, \omega_k)$ we write

$$\Omega^I = \omega_1^{i_1} \wedge \dots \wedge \omega^{i_k}$$

and we are interested in the integrals $\int_V \Omega^I$, where dim $V = 2n = 2|I|$.

We note that every such integral is a linear combination of the integrals

$$\int (I\Omega)^n \text{ for } I\Omega = i_1\omega_1 + \dots + i_k\omega_k,$$

with the above universal coefficients.

Now, let $V = \mathbb{R}^n \times T^n$ and $\omega_1, \dots, \omega_k$ correspond to convex bodies $Y_1, \dots, Y_k$. Namely $\omega_j = dJd\tilde{f}_j$ for $j = 1, \dots, k$ where $\tilde{f}$ is a smooth T^k -invariant function on V, such that the corresponding function on $\mathbb{R}^n$ is convex and $D_{f_j}(\mathbb{R}^n) = Y_j$ for $j = 1, \dots, k$.

1.6.A. **Proposition-Definition.** *The integral* $\int_V \Omega^I$, *where* $|I|=n$ *only depends on* $Y = (Y_1, \dots, Y_k)$ *but not on a choice of the functions* f_j. *This integral is called the* I^{th} *mixed volume of* $Y_1, \dots, Y_k$, *and denoted* $[y^I] = \left[Y_1^{i_1} \dots Y_k^{i_k}\right]$.

Proof. By the previous discussion each mixed volume is a universal linear combination of the volumes of the Minkowski sums $IY = i_1Y_1 + \dots + i_kY_k$ where iX denotes $\underbrace{X+X+\dots+X}_{i}$.

1.6.A_1. *Remark.* As it is clear from this definition, the volume Vol(IY) expands in the usual way into the sum of mixed volumes. For example,

$$\mathrm{Vol}(Y_1+Y_2) = [(Y_1+Y_2)^n] = \sum_{i=0}^{n} b_i \left[Y_1^i Y_2^{n-i}\right],$$

where $b_i = \frac{n!}{i!(n-i)!}$.

In order to state the Alexandrov Fenchel inequality concerning mixed volumes we need the following notion of convexity for real functions on the *discrete simplex*

$$\Delta_n^{k-1} = \{I \in Z_+^k \mid |I| = n\} \subset \mathbb{R}^k.$$

In other words, Δ_n^{k-1} is the set of multiindices $(i_1,...,i_k)$ with $i_1+...+i_k = n$. We say that a function l (I) is *1-concave* on Δ_n^{k-1} if the restriction of l to every line parallel to one of the edges of Δ_n^{k-1} is concave. For example, if k-1 = 1, then this is the usual concavity,

$$l\left(\sum_{\nu} a_\nu I_\nu\right) \geq \sum_{\nu} a_\nu l(I_\nu)$$

for $I_\nu = (\nu, n-\nu) \in \Delta_n^1$ and all those convex combinations, where $\sum_\nu a_\nu I_\nu$ lies in Δ_n^1 (i.e. is integral).

In general, Δ_n^{k-1} has $\frac{k(k-1)}{2}$ edges. A line parallel to an edge is given by fixing k-2 (out of k) coordinates $(i_1,...,i_k)$. For example, a line parallel to the "first" edge is given by fixing the last (k-2) coordinates $i_3, i_4, ... i_k$. If $i_3+i_4+...+i_k = m \leq n$, then this line is $\{\nu, n-m-\nu, i_3,...,i_k\}$ for $\nu = 0,1,...,n-m$, and the 1-concavity condition on this line amounts to the above (7).

1.6.B. *Alexandrov-Fenchel theorem.* Let $Y = (Y_1,...Y_k)$ *be a sequence of convex bounded open subsets in* $\mathbb{R}^n$. *Then the mixed volumes* $[Y^I]$ *for* $|I|=n$ *are positive and the function*

$$l_Y(I) = \log\,[Y^I]$$

is 1-concave.

Remark and corollaries. 1 .6. B_1. *The mixed volume* $[Y^I]$ *for* $I = (i_1,\dots,i_k)$ *is bounded from below by the following weighted product of the volumes of* $Y_1,\dots,Y_k$,

$$[Y^I] \geq (\text{Vol } Y_1)^{\frac{i_1}{n}},\dots,(\text{Vol } Y_k)^{\frac{i_k}{n}}. \qquad (8)$$

In fact, every 1-concave function $l(I)$ (obviously) satisfies $l(I) \geq \frac{i_1}{n} l(n, 0,\dots,0) + \frac{i_2}{n} l(0, n, 0,\dots,0) + \frac{i_k}{n} l(0,\dots,0,n)$.

1.6.B_2. Inequality (8) for k=2 reads

$$\left[Y_1^i, Y_2^{n-i}\right] \geq (\text{Vol}Y_1)^{\frac{i}{n}} (\text{Vol}Y_2)^{\frac{n-i}{n}}$$

which implies the Brunn-Minkowski inequality as

$$\text{Vol}\,(Y_1 + Y_2) = \sum_i b_i \left[Y_1^i Y_2^{n-i}\right].$$

1.6.B_3. I do not know if $lY(I)$ is a concave function in I.

1.6.B_4. We shall prove the 1-concavity of $\log[K^I]$ along with the following

1.6.C. *Alexandrov-Fenchel inequality on compact manifolds.* Recall that the mixed volume $[Y^I]$ is defined as the integral $\int_V \Omega^I$, where the string of 2-forms, $\Omega = (\omega_1,\dots,\omega_k)$ on $V = \mathbb{R}^n \times T^n$, corresponds to convex sets $Y_1,\dots,Y_k$ in $\mathbb{R}^n$.
Thus the Alexandrov-Fenchel theorem amounts to 1-concavity of $\log \int\Omega^I$ for exact positive (1,1)-forms $\omega_1,\dots,\omega_k$ on V. This is proven in §2 where we start with the following result concerning *compact* manifolds V.

1.6.C_1. **Theorem.** *Let* V *be a compact complex manifold. Then for every sequence of strictly positive closed* (1,1)*-forms* $\Omega = (\omega_1,\dots,\omega_k)$ *on* V *the function*

$$\log \int_V \Omega^I, \quad \text{for } |I| = \dim_{\mathbb{C}} V$$

is 1-concave.

Remarks 1.6.C_2. This theorem does not *directly* imply Alexandrov-Fenchel inequality since the manifold $\mathbb{R}^n \times T^n$ is non-compact. Yet this manifold can be approximated in a certain way by compact manifolds (this idea is due to Teissier and Hovanski) and then Alexandrov-Fenchel inequality reduces to the compact case.

1.6.C_3. If V is an *algebraic* manifold then 1.6.C_1 is equivalent to the following property of the index of the intersection of divisors on V, denoted $D_1^{i_1} \cap ... \cap D_k^{i_k}$ for $i_1+...+i_k = \dim_{\mathbb{C}} V$, where D^i stands for $\underbrace{D \cap ... \cap D}_{i}$.

Teissier-Hovanski theorem. *If the divisors* $D_1,...,D_k$ *are ample then the function*

$$\log\left(D_1^{i_1} \cap ... \cap D_k^{i_k}\right)$$

is 1-concave in $I = (i_1,...,i_k)$.

Note that this result was proven by Teissier and Hovanski over an *arbitrary* ground field.

§2. *The proof of Alexandrov-Fenchel inequality.*

2.1. *Algebraic inequality of Alexandrov.* Every 2n-linear antisymmetric form Λ on $\mathbb{C}^n$ is proportional to the standard oriented volume form Λ_0 on $\mathbb{C}^n$, say $\Lambda = \lambda\Lambda_0$ and we set $[\Lambda] \underset{\text{def}}{=} \lambda$.

2.1.A. *Alexandrov's Lemma. Let* ω_0 *be a positive* (1,1)*-form, let* Ω *be the exterior product of* (n-2) *positive* (1,1)*-forms and let a* (1,1)*-form* ω *satisfy*

$$\Omega \wedge \omega_0 \wedge \omega = 0. \qquad (\perp)$$

Then

$$[\Omega \wedge \omega \wedge \omega] \leq 0.$$

Proof. First, let n=2 and use the fact that ω_0 and ω can be simultaneously diagonalized (as the corresponding quadratic forms $\omega_0(z,Jz)$ and $\omega(z,Jz)$ can be diagonalized) by a complex linear transformation of $\mathbb{C}^n$. Such a transformation makes

$$\omega_0 = a_0\, dx_1 \wedge dy_1 + b_0\, dx_2 \wedge dy_2$$

and

$$\omega = a dx_1 \wedge dy_1 + b\, dx_2 \wedge dy_2\,.$$

Then the relation

$$\omega_0 \wedge \omega = a_0\, b + a\, b_0 = 0$$

implies that

$$[\omega_0 \wedge \omega_0] = ab \leq 0$$

as $a_0 \geq 0$ and $b_0 \geq 0$ (because of the positivity of ω_0).

Next, assume the lemma is true for some $n \geq 2$ and pass to n+1 in two steps.

Step 1. Let $\Omega = \omega' \wedge \Omega'$, where ω' is a *monomial* (1,1)-form, that is ω' induced from a (1,1)-form on $\mathbb{C}^1$ by a $\mathbb{C}$-linear map $l : \mathbb{C}^{n+1} \to \mathbb{C}^1$. Denote by $L \subset \mathbb{C}^{n+1}$ the kernel of l and assume without loss of generality than $\dim_{\mathbb{C}} L = n$ (i.e. $\omega' \neq 0$). Then for every J-invariant (2n-2)-form, say, Ω', the exterior product $\omega' \wedge \Omega''$ has the same sign as the restriction of Ω'' to L. (This becomes more obvious if ω' is transformed by a $\mathbb{C}$-linear automorphism of $\mathbb{C}^{n+1}$ to the form $dx_{n+1} \wedge dy_{n+1}$ where $L = \mathbb{C}^n \subset \mathbb{C}^{n+1}$). Hence,

$$\Omega \wedge \omega_0 \wedge \omega = 0 \quad \Leftrightarrow \quad \Omega' \wedge \omega_0 \wedge \omega \mid L = 0$$

and

$$[\Omega \wedge \omega \wedge \omega] \leq 0 \quad \Leftrightarrow \quad [\Omega' \wedge \omega \wedge \omega \mid L] \leq 0.$$

Thus the lemma for $\Omega = \Omega' \wedge \omega'$ follows from the inductive assumption applied to the forms Ω', ω, and ω_0 restricted to $L = \mathbb{C}^n$.

Step 2. We need the following

2.1.A1. **Trivial Lemma.** *Let* A *be a linear function on the space of antisymmetric two forms on* $\mathbb{C}^{n+1}$. *Then every positive* (1,1)*-form* ω' *can be decomposed into a sum of positive monomial forms,*

$$\omega' = \sum_{i=1}^{k} \omega'_i$$

such that

$$A(\omega'_i) = A(\omega') \text{ for } i = 1,\dots,k.$$

Proof. Since ω' is J-invariant, the kernel K of ω' is a $\mathbb{C}$-linear subspace in $\mathbb{C}^{n+1}$ of dimension $\dim_{\mathbb{R}} K = 2(n+1)$-rank ω'. Then for every $\mathbb{C}$-hyperplane $L \supset K$ there exists a unique positive monomial form ω'_L such that Ker $\omega'_L = L$ and $\omega'' = \omega' - \omega'_L$ is positive of rank by one less than that of ω'. (The reader who feels more comfortable with quadratic forms will see this by looking at $\omega'(z,Jz)$ which is a positive semidefinite quadratic form). Then by induction on $k = 1/2$ rank ω', one sees that $\omega' = \sum_{i=1}^{k} \omega'_{L_i}$ for some hyperplanes in $L_i \supset K$. Since

$$A\omega' = \sum_{i=1}^{k} A(\omega'_{L_i})$$

the function $A(\omega') - A(\omega'_L)$ necessarily changes sign as L runs over all hyperplanes containing K and so that function vanishes for some L. This gives

$$\omega' = \omega'' + \omega'_L$$

where $A(\omega'') = A(\omega'_L) = A(\omega')$ and the proof is concluded by induction on $k = 1/2$ rank ω'. (One can also derive the lemma from Kakutani's theorem which says that every continuous function on S^n is constant on some orthonormal n-frame).

Now we can complete the inductive step (n) $\rightarrow$ (n+1) by writing $\Omega = \omega' \wedge \Omega'$ and by observing that

$$\omega' \rightarrow [\Omega \wedge \omega_o \wedge \omega]$$

is a linear function on the space of forms. Then the trivial lemma allows a monomial decomposition $\omega' = \sum_{i=1}^{k} \omega'_i$ such that

$$\omega'_i \wedge \Omega' \wedge \omega_o \wedge \omega = 0$$

for all i = 1,..., k and by applying Step 1 to all ω'_i we obtain

$$[\Omega \wedge \omega \wedge \omega] = \sum_{i=1}^{k} [\omega'_i \wedge \Omega' \wedge \omega \wedge \omega] \leq 0.$$

Q.E.D.

2.1.A_2. *Remark.* Observe that

$$\Omega(\omega, \omega) \underset{\text{def}}{=} [\Omega \wedge \omega \wedge \omega]$$

is a *quadratic form* on the space of (1,1)-forms ω. Then the condition ($\perp$) of the lemma claims Ω-orthogonality between ω_o and ω. Note also that Ω is (non-strictly) positive on positive forms as is seen by taking monomial decomposition of all forms in question. In particular,

$$\Omega \wedge \omega_o \wedge \omega_o \geq 0$$

and the Lemma states that Ω is negative on the orthogonal complement of ω_o. This implies by elementary theory of quadratic form, that

$$[\Omega \wedge \omega_o \wedge \omega_1]^2 \geq [\Omega \wedge \omega_o \wedge \omega_o]\,[\Omega \wedge \omega_1 \wedge \omega_1] \qquad (**)$$

for all (1, 1)-forms ω_1.

2.1.B. **Corollary**. *Let* $\omega_1,...,\omega_k$ *be positive* (1,1)*-forms on* $\mathbb{C}^n$ *and* $I = (i_1,..., i_k)$ *with* $i_1+... +i_k = n$. *Then the function*

$$l(I) = \log\left[\omega_1^{i_1} \wedge ... \wedge \omega_k^{i_k}\right]$$

is 1*-concave*.

Proof. Let

$$\Omega = \omega_1^{i_1-1} \wedge \omega_2^{i_2-1} \wedge \omega_3^{i_3} \wedge ... \wedge \omega_k^{i_k},$$

write

$$\omega_1^{i_1} \wedge ... \wedge \omega_k^{i_k} = \Omega \wedge \omega_1 \wedge \omega_2$$

apply (**) and take logarithms. Then one sees the 1-concavity along the "first" edge in Δ_k^{k-1} (compare 1.6.A_1) and the rest of edges are taken care of in the same way.Q.E.D.

Remarks 2.1.B_1. It is probably not hard to figure out whether the above $l(I)$ is concave.

2.1.B_2. One can generalize the proof of 2.1.A. Using more general *Hodge bilinear relations* as the base of induction. This gives, for example, the following result.

Let ω *be a* (k,k) *form on* $\mathbb{C}^n$ (i.e. ω *is a* J*-invariant* 2k*-form*), Ω *be the exterior product of* l *positive* (1, 1)*-forms, such that* $k+l = n$, *and let* ω_0 *be a strictly positive* (1,1)*-form. Then the equality*

$$\Omega \wedge \omega_0 \wedge \omega = 0$$

implies that

$$(-1)^k [\Omega \wedge \omega \wedge \omega] \geq 0.$$

2.1.B_3. Let $\omega_1, \omega_2, \ldots, \omega_n$ be diagonal (1,1)-forms,

$$\omega_i = \sum_{j=i}^{n} a_{ij} dx_j \wedge dy_j .$$

Then $[\omega_1 \wedge \ldots \wedge \omega_n]$ equals the *permanent* of the matrix (a_{ij}) that is the sum of the same products of a_{ij} which go into the determinant, but *now* all appearing with the positive sign. Alexandrov's lemma for diagonal forms was recently used to prove the following

Van der Waerden Conjecture. *If* $a_{ij} \geq 0$ *then the permanent* $\mathrm{Per}(a_{ij})$ *is bounded form below in terms of* $p_i = \sum_{j=i}^{n} a_{ij}$ *and* $q_j = \sum_{i=i}^{n} a_{ij}$

as follows

$$\mathrm{Per}\,(a_{ij}) \geq n^{-n}\, n! \prod_{i=1}^{n} p_i q_i .$$

2.2. *Laplace equation.* Let (V,J) be an *almost complex* manifold, that is J is an automorphism of the tangent bundle of V, such that $J^2 = -\mathrm{Id}$. We extend the notions from §1.5. to (V,J) in an obvious way and then for every $(2n-2)$-form Ω on V for $2n = \dim V$, we assign the following differential operator of second order from functions on V to $2n$-forms,

$$f \to \Delta f = d(\Omega \wedge J df).$$

Note that if Ω is closed, then

$$\Delta f = \Omega \wedge dJdf.$$

2.2.A. Call Ω strictly positive if $[\Omega \wedge \omega] > 0$ for all non-vanishing positive (1,1)-forms ω on V. For example, an exterior product of strictly positive (1,1)-forms is strictly positive.

2.2.A_1. *Lemma. If* Ω *is strictly positive then* Δ *is a positive elliptic operator. In particular, if* V *is compact then the kernel of* Δ *consists of constant functions only.*

Proof. If f has compact support, then

$$\int f \Delta f = \int \Omega \wedge J df \wedge df > 0$$

for f $\neq$ const as $Jdf \wedge df$ is a *positive* (1,1)-*form*. This show Δ is positive and hence elliptic.

2.2.A$_2$. **Corollary.** *If* V *is compact and* Ω *is strictly positive then the image of* Δ *consists of all exact forms on* V.

Proof. In fact, the index of Δ is zero since it acts on functions and so

$$\dim \operatorname{Coker} \Delta = \dim \operatorname{Ker} \Delta = 1.$$

Q.E.D.

2.3. *Alexandrov-Fenchel on compact manifolds* V (see 1.6.C$_1$). *Let* (V,J) be a compact *complex* manifold, which signifies local isomorphism between (V,J) and ($\mathbb{C}^n$, $J=\sqrt{-1}$). The relevant property is that dJdf is a (1,1)-form for all functions f on V.

2.3.A. *Let* ω_0 be a closed strictly positive (1,1)-*form on* V and Ω' *the exterior product of* (n-2) *closed strictly positive* (1,1)-*forms. Then the equality*

$$\int_V \Omega' \wedge \omega_0 \wedge \omega = 0 \qquad (1)$$

implies that

$$\int_V \Omega' \wedge \omega \wedge \omega \leq 0 \qquad (*)$$

for all closed (1,1)-forms ω on V, provided V is connected.

Proof. The integral does not change if we replace ω by ω+dJdf for all functions f on V. By 2.2.A$_2$ and connectedness of V, there exists an f_0, such that

$$\Omega' \wedge \omega_0 \wedge dJdf_0 = -\Omega' \wedge \omega_0 \wedge \omega,$$

and so we may assume that instead of (1) *the pointwise relation*

$$\Omega' \wedge \omega_0 \wedge \omega = 0$$

holds true on V, as we can switch to $\omega + dJdf_0$. Then by Alexandrov's lemma 2.1.A.,

$$\Omega' \wedge \omega \wedge \omega \leq 0$$

everywhere on V and (*) follows by integration.

2.3.B. Now $1.6.C_1$ follows from 2.3.A. by the (trivial) argument used in 2.1.B.

2.3.C. *Remark*. The above 2.3.A. remains true for *non-strictly* positive forms if V is *Kähler*, which means the existence of at least one strictly positive closed (1,1)-form on V called a *Kähler form*. In such a case every positive form ω can be approximated by a strictly positive form, that is

$$\omega_\varepsilon = \omega + \varepsilon \text{ (Kähler form)}.$$

2.3.D. *Singular varieties*. Let V be a compact complex space with singularities. Then the above result extends to V with the following convention. A *differential form* on V means a form ω on the non-singular locus $V_{reg} \subset V$, such that the following regularity condition is satisfied. For every complex manifold U and every holomorphic map $\alpha : U \to V$, the form $\alpha^* (\omega)$ on $\alpha^{-1} (V_{reg})$ extends to a smooth form on all of U.

There are two approaches to singular spaces V. First, one can use Hironaka theorem, which claims the existence of a compact complex manifold U of dimension n = dim V which admits a holomorphic map onto V. If V is irreducible (i.e. V_{reg} is connected) then U can be assumed connected and we can apply 2.3.C. if U is Kähler. Such a U exists by Hironaka theorem if, for example, V admits a holomorphic embedding into a Kähler *manifold*. In particular, this applies to projective algebraic varieties.

Another approach consists in extending 2.2.A2 to the *open* manifold V_{reg} by using appropriate boundary conditions. This eventually leads to slightly more general results and will be discussed somewhere else.

2.3.E. *Subvarieties in Kähler manifolds*. A Riemannian metric h on a complex manifold (W,J) is called Kähler, if the associated 2-form ω for

$$\omega(x,y) = h(x,-Jy)$$

is a Kähler form, that is ω a closed (1,1)-form. If V is an n-dimensional complex subvariety in W, then the 2n-dimensional volume of V equals the integral of ω^n over the non-singular part of V,

$$\mathrm{Vol}_h\, V = \int_{V_{reg}} \omega^n.$$

In fact, $\omega^n | V_{reg}$ obviously equals the Riemannian 2n-volume form at every point of V_{reg} and the singular part $V_{sing} = V - V_{reg}$ has zero 2n-dimensional volume, as it is a variety of dimension 2n-2. Now let V be a subvariety in a Cartesian product of Kähler manifolds

$$V \subset W = W_1 \times \ldots \times W_k$$

where $W_i = (W_i, \omega_i)$ and where W is given the product metric. Then the corresponding ω on W is $\tilde{\omega}_1 + \ldots + \tilde{\omega}_k$, where $\tilde{\omega}_i$ denotes the pull-back of ω_i for the projection $W \to W_i$. Thus

$$\mathrm{Vol}\, V = \int_V (\tilde{\omega}_1 + \ldots + \tilde{\omega}_k)^n$$

and so our inequalities can be interpreted in terms of Vol V and

$$\mathrm{Vol}\, V_i = \int_{V_i} \omega_i = \int_V \tilde{\omega}_i$$

where V_i is the projection of V to W_i. In particular, we can reduce inequality (+) in 0.2.A. to 1.6.C_1. since the standard metric in the projective space is Kähler (see below).

2.4. *Toral varieties.* Let us describe the canonical metric on the complex projective space P^N, restricted to $\mathbb{C}^N = P^N \setminus P^{N-1}$. Denote by $\|w\|$ the norm $\left(\sum_{i=1}^{N} w_i \bar{w}_i\right)^{1/2}$ on $\mathbb{C}^N$ and let $\omega = dJd \log(1 + \|w\|^2)$. Then one easily checks that ω smoothly extends to $P^N \supset \mathbb{C}^N$ and is a U(N+1)-invariant Kähler form on P^N, where the associated quadratic form h is called *the Fubini-Study* metric on P^N. Note that h restricts to the standard metric of constant curvature on the real projective space $P^N_{\mathbb{R}} \subset P^N = P^N_{\mathbb{C}}$.

For a covector $y \in (\mathbb{R}^n)^*$, that is a linear function $y : \mathbb{R}^n \to \mathbb{R}$, and all $x = x' + \sqrt{-1}\ x'' \in \mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n$ we set $\langle x,y \rangle = y(x') + \sqrt{-1}\ x''$. Then we take N covectors $y_1, \ldots, y_N$ and consider the $\mathbb{C}$-span of the functions $\varphi_i(x) = \exp\langle x,y_i \rangle$, i=1,...,N. This is the N-dimensional complex linear space, say $\mathbb{C}^N$, and $\mathbb{C}^n$ acts on this $\mathbb{C}^N$ by translation, $\varphi(x) \to \varphi(x+x_0)$ for all $x_0 \in \mathbb{C}^n$. Note that this action is *diagonal* for the basis $\exp\langle x,y_i\rangle$ in $\mathbb{C}^N$.

Next we pull-back the above ω and h on $\mathbb{C}^N$ to the $\mathbb{C}^n$-orbit of $\varphi(x) = \sum_{i=1}^{N} a_i \exp\langle x,y_i\rangle$ and observe the following

Obvious property. If the coefficients a_i are real then the pulled back forms ω and h are $(\sqrt{-1}\ \mathbb{R}^n)$-invariant on $\mathbb{C}^N = \mathbb{R}^n \oplus \sqrt{-1}\ \mathbb{R}^n$ and the induced Riemannian metric h on $\mathbb{R}^n$ equals the Hessian D^2f for the function

$$f(x) = \log (1 + \sum_{i=1}^{N} (a_i \exp\langle x,y_i\rangle^2) = \log (1 + \sum_{i=1}^{N} a_i^2 \exp 2 \langle x,y_i\rangle).$$

Furthermore, f(x) is a convex function (see 1.4.C_2) and if none of a_i is zero, then the closure of the image $D_f(\mathbb{R}^n) \subset (\mathbb{R}^n)^*$ equals the convex hull of the covectors

$$0, y_1, y_2, \dots y_N \text{ in } (\mathbb{R}^n)^*.$$

In what follows we assume for simplicity's sake that the covectors y_i linearly span $(\mathbb{R}^n)^*$ and we call the system $\{y_1, \dots y_N\}$ *rational* if the *group* A' generated by y_i, that is

$$A' = \left\{ \sum_{i=1}^{N} m_i y_i \right\} m_i \in \mathbb{Z}$$

is discrete in $(\mathbb{R}^n)^*$ Then we denote by $A \subset \sqrt{-1}\ \mathbb{R}^n \subset \mathbb{C}^n = r^n \oplus \sqrt{-1}\ \mathbb{R}^n$ the 2π-*dual* group,

$$x \in A \Leftrightarrow \langle x, a\rangle \in 2\pi \sqrt{-1}\ \mathbb{Z}$$

which is in the rational case is an n-dimensional lattice (isomorphic to $\mathbb{Z}^n$) in $\sqrt{-1}\ \mathbb{R}^n$, whose action on $\mathbb{C}^N$ is *trivial.* Thus our action of $\mathbb{C}^n$ factors through that of the group

$$(\mathbb{C}^\times)^n = \mathbb{R}^n \times T^n = \mathbb{C}^n/A.$$

Then we use the following elementary fact (see 3.3.).

For every linear diagonal action of $(\mathbb{C}^\times)^n$ *on* $\mathbb{C}^N \subset P^N$ *the closure in* P^N *of every* *orbit is an irreducible complex algebraic variety in* $\mathbb{P}^N$.

It is also easy to show in our case that for a *generic* orbit this closure is a ***non-singular*** variety, provided the convex hull Q of the set $\{o, y_1, \dots, y_N\} \subset \mathbb{R}^n$ is a *simple* polyhedron which means at every vertex p of Q exactly n+1 faces of codimension one come together. For example, if N = n then Q is a simplex (which *is* a simple polyhedron) and our variety is isomorphic (by a trivial argument) to $\mathbb{P}^n$.

Now we combine the above observations with the earlier discussion and arrive at the following

Conclusion. The Alexandrov-Fenchel inequality holds true for rational polyhedra Q_1, ..., Q_k where Q is called *rational* if the system of vertices of Q is rational. Then this inequality extends to all convex sets Y as every Y can be (obviously) approximated by rational polyhedra.

Remarks 2.4.A. Instead of the approximation one can prove Alexandrov-Fenchel inequality more directly by extending to $\mathbb{C}^n/A$ the argument we used for compact manifolds V. In fact, this is exactly what Alexandrov does in $[Al]_2$ where he works with supporting functions viewed as functions on S^{n-1} to which he applies the elliptic theory. On the other hand, Alexandrov's proof in $[Al]_1$ first applies to simple polyhedra and then the general case follows by approximation.

2.4.B. The algebraic inequality indicated in $2.1.B_2$ leads by a usual Hodge-theoretic argument to the following

Theorem. *Let* ω_0 *be a Kähler form on a compact variety* V *and* Ω *be the exterior product of* l *Kähler forms. Let* ω *be a closed* (k,k)-form, *where* $k + l = \dim_{\mathbb{C}} V$, *such that the form* $\Omega \wedge \omega_0 \wedge \omega$ *is exact. Then*

$$(-1)^k \int_V \Omega \wedge \omega \wedge \omega \geq 0.$$

I do not know if this result has any significance for convex sets.

2.4.C. *Open questions.* Let V be a complex manifold with an action of a complex Lie group G, such that V/G is compact. Then one asks if the Alexandrov-Fenchel inequality holds true for K-invariant closed positive (1,1)-forms on V, where K is the maximal compact subgroup in G. If $G = (\mathbb{C}^\times)^n$, then we have already proven this in two extreme cases, where $G = V$ or on the contrary, $n = 0$. The general case here looks easy. Next interesting case is where G is semisimple and $V = G$.

Looking into another direction we forget about complex structures and take a C^∞-manifold V with a symmetric affine connection. Then for each (convex) function f on V one has (positive definite) Hessian $h = D^2 f$ and one may speak of the total volume

$$[h^n] \underset{=}{\mathrm{def}} \int_V \mathrm{Det}(h)$$

which is just the total volume of the (Riemannian) manifold (V,h). Similarly one defines the mixed volumes $\left[h_1^{n_1},..,h_k^{n_k}\right]$ and asks about possible inequalities between these. For example, one asks these questions for the hyperbolic space $V = H^n$ with the Levi-Civita connection.

§3. *Miscellany.*

3.1. *Brunn-Minkowski for non-convex sets.* To avoid irrelevant complications we consider the case where X and Y are bounded open sub sets in $\mathbb{R}^n$ with smooth boundaries.

3.1.A. **Main lemma**. *There exists a bijective map* $f : X \to Y$ *with the following two properties*

(i) f *is almost everywhere differentiable and the Jacobian of* f *is a.e. constant.*

(ii) *The Jacobi matrix of* f *(that is the differential* Df *in the standard basis of* $\mathbb{R}^n$*) is triangular with positive eigenvalue almost everywhere on* X.

Proof. The claim is obvious for n=1, where condition (ii) just requires f to be monotone increasing. In fact, for any two absolutely continuous measures μ and υ on $\mathbb{R}^1$ there exists a (essentially) unique monotone increasing map f_1 sending $\mu \to \upsilon$.

Now, for $n \geq 2$, we denote by $p : \mathbb{R}^n \to \mathbb{R}^1$ the projection on the first coordinate line and we apply the above remark to the measures μ and υ which are the push-forwards of the measures of X and Y. (Namely, $\mu(I) = \mathrm{Vol}(p^{-1}(I) \cap X)$ and $\upsilon(I) = \mathrm{Vol}(p^{-1}(I) \cap Y)$ for all I in $\mathbb{R}^1$). Thus we obtain a map f_1 on $\mathbb{R}^1$ and then, by induction on n, we construct maps with properties (i) and (ii) between the intersections of X and Y with the hyperplanes $p^{-1}(t)$,

$$p^{-1}(t) \cap X \to p^{-1}(f_1(t)) \cap Y$$

for all $t \in p(X) \subset \mathbb{R}^1$. The collection of these is the required map $f : X \to Y$.

3.1.A$_1$. *Remark.* The map f is itself "triangular" in the obvious sense with respect to the (sequence of) partitions of $\mathbb{R}^n$ into subspaces of dimensions (n-1), (n-2),...,1, parallel to the standard flag of linear subspaces in $\mathbb{R}^n$. It is also clear that the map f satisfying (i) and (ii) is a.e. unique.

3.1.B. *The proof of the inequality*

$$(\mathrm{Vol}\,(X+Y))^{1/n} \geq (\mathrm{Vol}\,X)^{1/n} + (\mathrm{Vol}\,Y)^{1/n}.$$

Observe (using induction on n as earlier) that the map $g = \mathrm{Id}+f : X \to \mathbb{R}^n$ for the above f is a.e. injective and the image, say $Z \subset \mathbb{R}^n$, of this map (obviously) is contained in X+Y.

The volume of Z equals the integral of the Jacobian of g,

$$\mathrm{Vol}\,Z = \int_X \mathrm{Det}\,(1+Df)$$

and because of (ii) this Jacobian satisfies

$$(\mathrm{Det}\,(1+Df))^{1/n} \geq 1+(\mathrm{Det}\,Df)^{1/n}.$$

Next we see with (i) that

$$\mathrm{Det}\,Df = C = \mathrm{Vol}\,Y/\mathrm{Vol}\,X,$$

and so

$$(\mathrm{Vol}\,Z)^{1/n} \geq \left[\int_X \left(1 + C^{1/n}\right)^n\right]^{1/n} = (\mathrm{Vol}\,X)^{1/n} + (\mathrm{Vol}\,Y)^{1/n}.$$

Q.E.D.

3.1.C. *Remarks.* The above argument is standard (compare [MS]) and it applies to all solvable Lie groups G in place of $\mathbb{R}^n$. This yields the inequality

$$(\mathrm{Vol}\,(X+Y))^{1/k} \geq (\mathrm{Vol}\,X)^{1/k} + (\mathrm{Vol}\,Y)^{1/k} \qquad (*)$$

for the codimension k of the maximal compact subgroup $K \subset G$.

Note that (*) is sharp *only* for Abelian groups G and the best bound

$$\mathrm{Vol}\,(X_1+X_2) \geq B_G(V_1,V_2),$$

for $V_i = \mathrm{Vol}\,X_i$, $i = 1,2$, is unknown for non-Abelian (solvable and unsolvable) groups G. (Some information for $G = 0(n+1)$ and $0(n,1)$ is provided by the classical isoperimetric inequality for S^n and H^n which concerns $0(n)$-invariant subsets in $0(n+1)$ and $0(n,1)$.

3.1.D. Here is a Kähler version of 3.1.A.

Let ω *be a Kähler form on a compact manifold* V *and* Ω *be a positive* 2n*-form for* $n = \dim_{\mathbb{C}} V$, *such that* $\Omega - \omega^n$ *is exact. Then there exists a function* f *on* V, *such that*

$$(\omega + dJdf)^n = \Omega.$$

This is the celebrated Calabi conjecture solved by Yau. Using this result one obtains an alternative proof of the Brunn-Minkowski inequality on Kähler manifolds.

3.2. *Gradient actions and the moment map.* Consider a manifold V with a bilinear form g which is viewed as a homomorphism between tangent and cotangent bundles of V,

$$g : T(V) \to T^*(V).$$

A vector field ∂ on V is called g-*gradient* (or *Hamiltonian*) if

$$g(\partial) = df$$

for some function f on V, called the *potential* (or Hamiltonian) of f.

Next, an action of a Lie algebra L on V is called gradient (Hamiltonian) if all vector fields ∂ on V constituting the action are gradient. Then the potentials of these fields form a map of V into the linear space of linear functions on L.

$$f : V \rightarrow L',$$

such that

$$d(f(\partial)) = g(\partial) \qquad (*)$$

for all fields ∂ in L.

Remarks on terminology. The words "gradient" and "potential" are customary used if g is a symmetric form while "Hamiltonian' refers to an antisymmetric form. In the first case the above f is called the *gradient* map and in the second the *moment* map.

3.2.A. *Example.* Consider the standard action of the Abelian algebra $L = \mathbb{R}^n$ on $\mathbb{R}^n$ and let this action be gradient for some symmetric form g on $\mathbb{R}^n$. Then (*) implies that $dg(\partial) = 0$, which is equivalent to the *symmetry* of the (full) differential Dg defined by

$$Dg(\partial_1, \partial_2, \partial_3) = \partial_1\, g(\partial_2, \partial_3),$$

for all parallel fields ∂_i $i = 1,2,3$ on $\mathbb{R}^n$. Now we observe that the symmetry of Dg that is $\partial_1\, g(\partial_2, \partial_3) = \partial_2\, g(\partial_1, \partial_2)$, implies that g equals the Hessian D^2P of some function P on V,

$$g(\partial_1, \partial_2) = \partial_1(\partial_2\, P).$$

In this case, the gradient map of the action equals the gradient (or Legendre) map of P,

$$v \ \rightarrow d_v\, P \in L', \text{ for all } v \in R^n.$$

In particular, we see with Legendre theorem (see 1.2.) that if g is positive definite, then the image of the gradient map is *convex* .

A similar convexity property is satisfied by a gradient action of an Abelian algebra L on every *compact connected* Riemannian manifold. In this case, the action integrates to that of the Abelian group, that is $\mathbb{R}^n$, and the gradient map $f : V' \rightarrow L'$ obviously satisfies the following two properties.

(i) *Monotonicity of on the orbits* (compare §1). Let $l \in L\backslash\{0\}$ and let $l_t(v) \in V$ for $t \in \Gamma$ be the orbit of some $v \in V$ under the one parameter group l_t corresponding to l. Then the function $\alpha_l(t,v) = <f(l_t(v)), l>$ is monotone increasing in t, where $<l', l>$ stands for $l'(l)$. In particular, $\alpha_l(t,v)$ converges for $t \to +\infty$.

(ii) The limit $\alpha_l(v) = \lim_{t \to +\infty} \alpha_l(t,v)$ is constant on every $\mathbb{R}^n$-orbit in V for all $l \in L$. These two properties easily yield the convexity of the image f(V) in $L' = \mathbb{R}^n$ (see $[At]_1$ and $[At]^2$).

Example. The standard (diagonal) action of $(R_+^x)^n = \mathbb{R}^n/\mathbb{R}$ on the real projective space P_R^n (as well as on $P_{\mathbb{C}}^n \subset P_{\mathbb{R}}^n$) is gradient for the standard metric in $P_{\mathbb{R}}^n$ and f sends $P_{\mathbb{R}}^n$ onto a simplex in $L' = \mathbb{R}^n$ (compare 2.4.).

3.2.B. *Remark*. The above action can be described as follows. Take n+1 points $y_0,...y_n$ in L' and let P_R^n be identified with the projectivisation of the space $\mathbb{R}^{n+1}$ of the maps $\varphi:\{y_0,...,y_n\} \to \mathbb{R}$.
Then $\mathbb{R}^n$ acts on this $\mathbb{R}^{n+1}$ and hence on P^n by

$$\varphi(y) \to \varphi(y) \exp<x,y>$$

for all $x \in \mathbb{R}^n$ and $f(x) \in L' = (\mathbb{R}^n)'$ equals the center of gravity of the points $\{y_i\}_{i=0,...,n}$ with masses $\varphi^2(y_i) \exp<x,y_i>$ attached to them. This agrees with the discussion in 1.4.C_2 which applies to an arbitrary probability measure μ with compact support in $L' = \mathbb{R}^n$, and where $\mathbb{R}^n$ acts on μ by

$$\mu \to x(\mu) = \mu'/\mu'(L')$$

for $\mu' = \mu \exp<x, >$ for all $x \in \mathbb{R}^n$. Then x is sent to $f(x) \in L'$ which is the center of gravity of $x(\mu)$ (which is the same as the center of μ'). We have seen in §1 that the map $x \to f(x)$ sends $\mathbb{R}^n$ onto the interior of the convex hull of the support of μ, provided this support spans L'. It is also not hard to see that f continuously extends to the closure of the $\mathbb{R}^n$-orbit $\{x(\mu)\}_{x\in \mathbb{R}^n}$ in the space of probability measures on L' with the weak topology. One can show (we leave this to the reader) that f *homeomorphically* maps this closure onto the convex hull of the support of μ.

To grasp the geometry of the map $x \to f(x)$ it is useful to replace exp by the step function. Now, to every half-space $H \subset L'$ which intersects the support of μwe assign the measure

$$H(\mu) = \chi(H)\, \mu/\mu\, (H),$$

where χ is the characteristic function of H. Here one sees more clearly the structure of the weak closure of the measures $H(\mu)$ as well as of the map $H \to$ (center of gravity of $H(\mu)$). As for the above exp-case, here the picture is especially simple if the convex hull of the support of μ is *strictly* convex. (This never happens for finite measures, but these are easy anyway).

3.2.C. *Convex maps*. The convexity of the image is often accompanied by the following stronger property.

Definition. A continuous map $f : V \to \mathbb{R}^n$ is called *convex* if it satisfies the following three condition

(i) the image $f(V) \subset \mathbb{R}^n$ is convex;
(ii) the pull-back $f^{-1}(x) \in V$ is connected or empty;
(iii) for every open subset $U \subset V$ the image $f(U)$ is an *open* subset of $f(V)$ with the topology induced from $\mathbb{R}^n \supset f(V)$.

3.2.C_1. *Remarks*. (a) The conditions (i) and (ii) are equivalent to the following

(i') For every convex subset $X \subset \mathbb{R}^n$ the pull-back $f^{-1}(x) \subset V$ is connected or empty.

(b) Property (iii) is satisfied by every *embedding* $V \to \mathbb{R}^n$. In this case (i) and (ii) just say that the image is convex.

Next, call f *locally convex* if every point in V admits a neighborhood $U \subset V$ such that f restricts to a *convex* map $U \to \mathbb{R}^n$.

Now we invoke the following well known

3.2.C_2. **Lemma.** *If* V *is a compact connected space then every locally convex map* $V \to \mathbb{R}^n$ *is convex.*

One sees the idea by looking at the case where $V \hookrightarrow \mathbb{R}^n$ is an embedding and the convexity of $V \subset \mathbb{R}^n$ is obtained by showing that the shortest curve in V between any two points necessarily is a straight segment since the local convexity does not allow such a curve to touch the boundary of V from inside.

3.2.C_3 *Remark.* The Lemma remains valid for maps into spaces more general than $\mathbb{R}^n$ which possess a good notion of convexity. For example, one can replace $\mathbb{R}^n$ by a complete simply connected Riemannian manifold with non-positive curvature (This is an exercise to the reader).

3.2.D. **Convexity of the moment maps.** *Let* ω *be a nonsingular antisymmetric* 2-*form on* V *and let the Abelian Lie algebra* $L = \mathbb{R}^n$ *act on* V, *such that the following four conditions are satisfied*

(i) The action preserves ω.

(ii) *The action is Hamiltonian (i.e.* ω*-gradient).*

(iii) *If a vector field* $\partial \in L$ *on* V *vanishes at some point* $v \in V$, *then the differential of* ∂ *on the tangent space* T_v (V), say $D_\partial : T_v \to T_v$ *is an operator diagonalizable over* $\mathbb{C}$ *with purely imaginary eigenvalues.*

(iii)' *The zero set of* ∂ *is a smooth submanifold in* V *whose dimension at* v *equals that of the zero eigenspace of* D_∂.

If V *is compact and connected, then the moment map (i.e. the* ω*-gradient map)* $f : V \to L'$ *is convex.*

This is proven by checking the local convexity of f, where the local goemetry of the action is seen by looking at *linear* Hamiltonian actions on $V = \mathbb{R}^{2m}$ with $\omega = \sum_{i=1}^{m} dx_i \wedge dy_i$. Details of the proof are left to the reader.

3.2.D_1. *Remarks* (a). The most (if not the only) interesting case of the above convexity (due to Atiyah $[At]_1$ and Guillemin-Sternberg [G-S] is where ω is a *closed* (and hence symplectic) form and where the action of L integrates to an action of a compact torus T^n on V (or to a non-compact subgroup in such a torus). Note that for *closed* forms ω condition (i), that is $\partial\omega = 0$ for $\partial \in L$, is equivalent to $d(\omega(\partial)) = 0$, and so (ii) $\Rightarrow$ (i). The opposite implication (i) $\Rightarrow$ (ii) also holds true in many cases, for example if V is simply connected or at least $H^1(V,\mathbb{R}) = 0$. Also note that (iii) and (iii)' are obviously satisfied for actions coming from (compact !) tori T^n.

(b) If ω is a Kähler form on a complex manifold (V,J) and the action of L preserves J, then JL also acts on V and preserves J. It is also clear that the action of L is ω-gradient if and only if the action of JL is h-gradient for the quadratic form $h(\partial_1, \partial_2) = \omega$

$(\partial_1, J\partial_2)$.

(c) If (V,ω) is a symplectic manifold with a Hamiltonian T^n-action then the moment map $f : V \to \mathbb{R}^n$ pushes forward the measure (associated to) ω^m, for 2m=dim V, to a measure on the image f(V) with *piecewise* polynomial density, provided the moment map is *proper* (see [D-H] and $[At]_2$). In the special case of n=m this polynomial is constant and so the volume of (V,ω^m) equals, up to a universal constant, to that of the image of the moment map. For the standard action of T^1 on the sphere S^2 this result goes back to Archimedes (this remark I owe to Michael Atiyah) who proved that the orthogonal projection $S^2 \to \mathbb{R}^n$ sends the spherical measure to 2π(Lebesgue measure) on the segment $[-1,1] \subset \mathbb{R}$.

3.3. *Recollection on algebraic sets.* Here we give basic definitions and state some elementary properties of algebraic varieties.

3.3.A. A subset $V \subset \mathbb{C}^N$ is called (complex) *algebraic* if there exists a complex polynomial map $p : \mathbb{C}^N \to \mathbb{C}^m$ for some m, such that

$$V = p^{-1}(0).$$

A point $v_o \in V$ is called *regular* if there exists a polynomial map $p_o : \mathbb{C}^N \to \mathbb{C}^{m_o}$, of rank m_o at v_o, such that $p_o^{-1}(0)$ equals V in a small neighborhood in $\mathbb{C}^N$ around v_o. The set of regular points $Vr_{eg} \subset V$ obviously is a smooth manifold of (real) dimension 2n at v_o for $n = N\text{-}m_o$, where n is called the *complex dimension* of V. Then an easy argument shows that the singular part

$$V_{sing} = V \setminus V_{reg}$$

is an algebraic set and

$$\dim_{\mathbb{C}} V_{sing} < \dim_{\mathbb{C}} V \underset{\text{def}}{=} \dim_{\mathbb{C}} V_{reg} .$$

One says that V is *reducible* if it is a union $V = V_1 \cup V_2$, where V_1 and V_2 are algebraic subsets both different from V. Otherwise V is called *irreducible*. It is easy to see that V is irreducible if and only if V_{reg} is connected.

3.3.B. Let us observe an obvious correspondence between *cones* in $\mathbb{C}^{N+1}$ and subsets in P^N, where a subset in $\mathbb{C}^n$ is called a cone if it is a union of complex lines through the

origin. Call a subset $V \subset P^N$ *algebraic* if the corresponding cone $CV \subset \mathbb{C}^{N+1}$ is algebraic. Then one extends the above definitions from $\mathbb{C}^{N+1}$ to P^N in an obvious way (Warning : V in P^N is non-singular if the cone CV is non-singular *away from the origin*. In fact if some cone is non-singular at the origin, then it is a linear subspace).
Similarly one defines algebraic subsets in products of projective spaces $P^{N_1} x ... x P^{N_k}$ using *k-cones* in $\mathbb{C}^{N_1+1} x ... x \mathbb{C}^{N_k+1}$ which are unions of Cartesian products of lines in $\mathbb{C}^{N_i+1}$. It is not hard to show that the projection of an (irreducible) algebraic subset from $P^{N_1} x ... x P^{N_k}$ to P^{N_1} an (irreducible) algebraic subset P^{N_1}.

3.3.C. For an n-dimensional subset $V \subset P^N$, define deg V as the maximal (possible) number of points in the intersections of V with (N-n) -dimensional projective subspaces in P^N. If V is algebraic, then the above number of intersection points is constant and equals the degree for *almost all* (N-n)-dimensional subspaces. Namely, the subset of exceptional subspaces (where the number of intersection points less than the degree) is algebraic in the *Grassmann* $Gr_{N-n}P^N$ of all (N-n)-subspaces, where one naturally define the notion of an algebraic subset.
If V is irreducible, one knows that the top-dimensional homology group $H^{2n}(V)$ is free cyclic and the index of the image of $H^{2n}(V)$ in $H^{2n}(P^N)$ (which is also free cyclic) equals deg V. Then in the reducible case one sees that there are no more than deg V of top-dimensional irreducible components in V as their degrees add up to deg V.

3.3.D. The space P^N carries a natural complex structure and one constructs many Kahler forms on P^N as follows. Let μ be a smooth positive measure on the (dual) projective space of hyperplanes in P^N, that is $Gr_{N-1}P^N$. Then (by an easy argument) there exists a unique smooth 2-form $\omega = \omega_\mu$ on P^N such that for all oriented surfaces $S \subset P^N$ the integral $\int_S \omega$ equals the μ-*average* of the *algebraic* (i.e. counted with the sign defined by the orientations) intersection number of S with the hyperplanes $H \in Gr = Gr_{N-1}P^N$,

$$\int_{Gr} (S \cap H) d\mu.$$

It is not hard to show that such an ω is Kähler and if μ is normalized by $\mu(Gr) = 1$, then

$$\int_V \omega^n \overset{\text{def}}{=} \int_{V_{reg}} \omega^n = \deg V,$$

for all algebraic subsets $V \subset P^N$ which agrees with the previous homological definition of deg V. Another easy fact (valid for all Kähler forms) is

$$\int_V \omega^n = \mathrm{Vol}V$$

where "Vol" stands for the 2n-dimensional volume for the Riemannian metric h associated to ω, that is $h(\partial_1,\partial_2) = \omega(\partial_1, J\partial_2)$.
Note that $h = h_\mu$ is U(N+1)-invariant on P^N if and only if the measure μ on $Gr_{N-1}P^N$ is U(N+1)-invariant.

3.3.E. Let $V \subset P^{N_1} \times P^{N_2}$ be an *irreducible* subset of $\dim_{\mathbb{C}} V = 2$ and recall Hodge's inequality

$$\left(\int_V \omega_1 \wedge \omega_2\right)^2 \geq \int_V \omega_1^2 \int_V \omega_2^2 \qquad (*)$$

where ω_1 and ω_2 are the pull-backs of the standard Kähler forms on P^{N_1} and P^{N_2}. This inequality imposes a restriction on the homology class

$$[V] \in H^4(P^{N_1} \times P^{N_2}) = \mathbb{Z}^3,$$

which can be equivalently expressed in terms of the intersection numbers of V with pull-backs to $P^{N_1} \times P^{N_2}$ of hyperplanes in P^{N_1} and P^{N_2},

$$(V \cap H_1 \cap H_2)^2 \geq (V \cap H_1^2) \times (V \cap H_2^2), \qquad (**)$$

where H_i^2 for i = 1,2 stands for the intersection of H_i and the hypersurface H'_i obtained from H_i by a generic (holomorphic) motion of $P^{N_1} \times P^{N_2}$.

Next consider $V \subset P^{N_1} \times P^{N_2} \times Q$ where

$$Q = P^{N_3} \times \ldots \times P^{N_k}$$

and let $H \subset P^{N_1} \times P^{N_2} \times Q$ be the intersection of the pull-backs of projective subspaces in $P^{N_3}, P^{N_4}, \ldots, P^{N_k}$, in general positions, such that the codimensions of these subspaces add up to n-2 (i.e. $\mathrm{codim}_{\mathbb{C}} H = n-2$) for $n = \dim_{\mathbb{C}} V$. Now *Teissier-Hovanski theorem* can be expressed by the following inequality

$$(V \cap H \cap H_1 \cap H_2)^2 \geq (V \cap H \cap H_1^2) \times (V \cap H \cap H_2^2) \qquad (***)$$

for all *irreducible* V.

Proof. By *Bertini theorem* the intersection $V \cap H$ is an *irreducible* variety for generic H and thus (***) follows from (**) applied to the projection of $V \cap H$ to $P^{N_1} \times P^{N_2}$.

Note that (***) applied to Abelian varieties V yields Alexandrov's lemma (see 2.1.) and that Alexandrov's proof in $[Al]_1$ is similar in spirit to the above use of Bertini's theorem.

3.3.E'. In order to clarify the relation between (***) and the corresponding Kahlerian inequality (see 1.6.C.) we recall the following fundamental theorem of Lefschetz-Kodaira.

Let ω *be a Kahler form on compact complex* n*-dimensional manifold* V, *such that* $\int_S \omega$ *is an integer for all closed oriented (real) surfaces* S *in* V. *Then there exists a holomorphic embedding* $I : V \hookrightarrow P^N$ *for* $N = 2n+1$, *such that the image* I(V) *is an algebraic subvariety and the homology class of* $[V \cap H] \subset H^{n-2}(V)$ *for a generic hyperplane* $H \subset P^N$ *is the Poincaré dual of the class* $[\omega] \in H_2(V)$.

That is

$$V \cap H \cap S = \int_S \omega$$

for all closed oriented surfaces S *in* V.

3.3.F. Consider a finite subset of multi-indices $I \subset Z^n_+$ and let $x^I = x_1^{i_1} \ldots x_n^{i_n}$ for $x = (x_1, \ldots, x_n)$ and $I = (i_1, \ldots, i_n)$ be the monomials for all $I \in I$, which are viewed as $\mathbb{C}$-valued functions on $\mathbb{C}^n$. All these together define a polynomial map, say $\alpha : \mathbb{C}^n \to \mathbb{C}^{N+1}$ where N+1 is the number of elements in *I*. By multiplying each entry x_i of x^I by a non-zero complex number we obtain a natural action of $(\mathbb{C}^\times)^n$ on monomials as well as a monomorphism

$$\beta = \beta_I : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^{N+1}$$

such that the map α is equivariant with respect to β for the standard (diagonal) actions of $(\mathbb{C}^\times)^n$ on $\mathbb{C}^n$ and of $(\mathbb{C}^\times)^{N+1}$ on $\mathbb{C}^{N+1}$.

Next we take away zero from the image $\alpha(\mathbb{C}^n) \subset \mathbb{C}^{N+1}$ and project $\alpha(\mathbb{C}^n)\backslash\{0\}$ from $\mathbb{C}^{N+1}\backslash\{0\}$ to P^N. One knows (for *all* polynomial maps α) that the topological closure V of the image of this projection is an irreducible subvariety in P^N and it is quite easy to see that $\dim_{\mathbb{C}}$ V equals the dimension of the *convex hull* $I \subset \mathbb{R}^n$ of $\bar{I} \subset Z^n_+ \subset \mathbb{R}^n$.

The previous equivariance discussion shows that V is a toral variety and the moment map restricted to $V_0 \subset V_{reg} \subset V$, where the pertinent action of $T^n \subset (\mathbb{C}^\times)^n$ is free, is a fibration of V_0 over the interior of the convex hull $\bar{I}$ assuming $\dim \bar{I} = n$. Then by Archimedes theorem $\mathrm{Vol}\, V = \int_V \omega^n = n!\,\mathrm{Vol}\,\bar{I}$ and so deg V also equals $n!\mathrm{Vol}\,\bar{I}$.

3.3.F1. Recall that deg V equals the number of intersection points of V with n hyperplanes in general positions,

$$V \cap H_1 \cap H_2 \cap \ldots \cap H_n \qquad (+)$$

and observe that every hyperplane H_i is given by $l_i = 0$ for a linear function l_i on $\mathbb{C}^{N+1}$ whose zero set is the cone over H_i. Then we observe that the functions $P_i = l_i \circ \alpha$, $i = 1,\ldots,n$, are polynomials on $\mathbb{C}^n$ which are linear combinations of monomials x^I and that the intersection points in (+) correspond to common zeros of p_i. Thus we arrive at *Kushnirenko's theorem* equating the number of common zeros of p_i (which are generic linear combinations of x^I for $I \in I$) to $n!\mathrm{Vol}\, I$.

3.3.F. Let $V_{\mathbb{R}}$ be the real locus of the above V. Now the intersection number in (+) varies as we vary the real hyperplanes H_i, but yet the *average* number of points in (+) equals the (properly normalized) volume of $V_{\mathbb{R}}$ by the standard integral geometry. An explicit formula for length of $V_{\mathbb{R}}$ for n = 1 and $I = \{0,1,\ldots,d\} \subset \mathbb{Z}_+$ can be extracted from formulas in Example 2 of Ch.1 in [Ka] and a similar computation can be made for corresponding subsets $I \subset Z^n_+$. But in general one does not know the behaviour of deg and Vol for $V_{\mathbb{R}}$ (compare [Ho]). It is worth noticing that the imaginary part of V, (that is the T^n-orbit instead of the $(R^\times_+)^n$-orbit) has simpler Riemannian geometry as the action of T^n is isometric on P^N. In particular, the average number of zeros is easier to compute for *trigonometric* polynomials.

3.3.F_1. The above map α on $\mathbb{R}^n$ has many amusing properties besides sheer size. Take for example the curve

$$(t, t^2, t^3, \ldots, t^{2m}) \subset \mathbb{R}^{2m} \text{ for } t \in R$$

(which is accidentally called the *moment curve*) let F be a finite subset in this curve and $\bar{F}$ the convex hull of F. Then (by an easy argument) for every subset $F' \subset F$ containing m points or less, the convex hull $\bar{F}'$ lies in the *boundary* of $\bar{F}$. For example, if $m = 2$, then $\bar{F}$ is 4-dimensional and the segment between any two vertices in $\bar{F}$ is an *edge* in F. There is no such convex polyhedron in $\mathbb{R}^3$ apart from the simplex and with $\bar{F}$ one arrives at a counter-example for $n = 4$ (This example has been appearing in literature every other year since the last century).

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