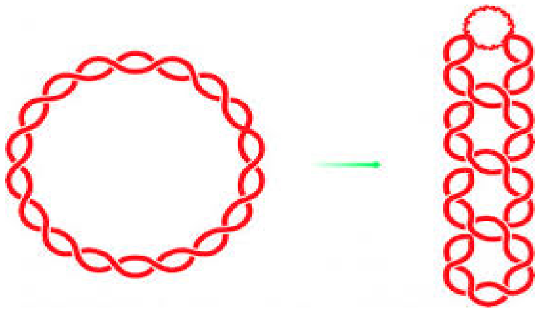

Figure 1: Bacterial DNA: Linking and supercoiling .

Revision I: Cubes → Cubes and Averaging Sets.

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TOPOLOGY OF THE n-CUBE. $\square^n = [-1,1]^n$. (Probabilistic perspective on the cube, *the law of large number and the Shannon inequality* will be in another lecture.)

A. If a *continuous* map between cubes

$$f : \square^n \to \underline{\square}^n,$$

that is an n-tuple of continuous functions $f_i(x_j)\}$, $-1 \le x_j \le 1$, $i, j. = 1, ..., n$,

$$f_i : \square^n \to [-1,1], i = 1, ...n,$$

sends each $(n-1)$-face $\partial_{i\pm} \subset \partial\square^n$ to the corresponding face $\partial_{i\pm}\underline{\square}^n$, i.e.

$$f_i : (.... \pm 1_i ....) \mapsto \pm 1,$$

then f is *onto*, the equation $f(x) = y$ *has a solution for all* $y = (y_1, ...y_n)$.

Equivalent formulation.

B.*If closed subsets* $X_i \subset \square^n$, $i = 1, ..., n$, *separate the pairs of the opposite* $(n-1)$*-faces* $\partial_{i\pm} \subset \partial\square^n$, *then the intersection* $\bigcap_i X_i$ *is non-empty.*

B $\Longrightarrow$ **A** . To solve $f(x) = (y_1, ..., y_n)$ let X_i be the set of $x \in \square^n$, where $f_i(x) = y_i$.

A $\Longrightarrow$ **B**. Given a a closed separating $X_i \subset \square^n$, let $f_i(X)$ let $f_i(x)$ be a function with the zero set X_i and such that $f_i(\mp 1) = \mp 1$.

C. We shall proof A and B. by a homology theoretic argument in lecture??? which, at least in the A-form, applies to face respecting maps between general polyhedral spaces.

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D. Besicovitch-Derrick Distance/Volume Inequality.[1] Let $\tilde{\square}^n$ be the cube with a Riemannin metric g on it, e.g. it is an Euclidean domain homeomorphic to the cube.

If the g*-distances between the opposite* $(n-1)$*-faces of* $\tilde{\square}^n$ *are* $\ge d_i$, $i = 1, ..., n$, *then the* g*-volume of t* $\tilde{\square}^n$ *is bounded from below by the product of* d_i

$$vol(\tilde{\square}^n) \ge d_1 \cdot ... \cdot d_n.$$

(The g-distance between two subsets is the infimum of the g-lengths of the curves jining these subsets.)

[1] *A Volume-diameter inequality for n-cubes*, William R. Derrick, Journal d'Analyse Mathématique volume 22, pages 1-36 (1969)

Proof. Let $\delta_{i\pm}(x)$ be the distances from $x \in \tilde{\Box}^n$ to the pairs of the i-th faces of the cube and let $f_i(x) = \min(\delta_{i,+}x, d_i)$.

The resulting map $f = f_1, ...f_n$, sends $\tilde{\Box}$ *onto* the solid $[0, d_1] \times ... \times [0, d_n]$, since, by the construction, faces go to faces.

Since f_i are distance functions, they are almost everywhere differentiable with unit gradients, i.e. $||df|| = 1$, and, by the (obvious) Hadamard inequality, the Jacobian of f is almost everywhere is ≤ 1. (If you don't like "almost everywhere" approximate f_i by smooth functions with $||grad|| \leq 1 + \varepsilon$ and let $\varepsilon \to 0$.)

Thus $vol(\tilde{\Box}^n \leq vol(\times_i[-.d_i])$. QED.

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E. "Segments and "Cubes" . A compact connected metric space S with two distinguished points $\tilde{0}, \tilde{1} \in S$ is called a "segment", where $\tilde{0}$, and $\tilde{1}$ are regarded as "vertices".

The product $\tilde{\Box}^n = \times_i^n[S_i, \tilde{0}_i, \tilde{1}_i]$, $i = 1, ..., n$, is caled the n-"cube" on the vertex set $\times_i\{"\tilde{0}_i, \tilde{1}_i\}$.

The (K, ν)-face S_ν^K in such a "cube" for $K \subset \{1, ..., N\}$ and $\nu \in \times_{i \notin K}\{\tilde{0}_i, \tilde{1}_i\}$ is

$$S_\nu^K = \bigtimes_{i \in K}(S_i, \tilde{0}_i, \tilde{1}_i) \times \nu \subset \tilde{\Box}^N.$$

F. If a continuous map from an N-"cube" to the true N-cube,

$$f : \tilde{\Box}^n = \bigtimes_1^N[S_i, \tilde{0}_i, \tilde{1}_i] \to [0, 1]^N,$$

sends each face from the "cube" to the the corresponding one in the cube, then the map f is onto.

Proof. Join $\tilde{0}_i$ with $\tilde{1}_i$ by a chain of N_i consecutively mutually ε-close points in S_i, replace S_i by the unit segment $[0, 1]$ divided into $N_i + 1$ equal subsegments and reduce **F** to **A** , where $S_i = [0, 1]$, with $\varepsilon \to 0$.

Speaking formally, let $\sigma_{i,\varepsilon} : \{0, 1, ...N_i\} \to S$, $\varepsilon > 0$ be maps such that $\sigma_i(0) = \tilde{0}_i$, $\sigma_i(N_i) = \tilde{1}_i$, and $dist(j, j+1) \leq \varepsilon)$ for all $i \in \{0, 1, ...N_i\}$ and all i, let

$$\Sigma_\varepsilon = \bigtimes_i^n \sigma_i : \bigtimes_1^n\{0, 1, ...N_i\} \to \tilde{\Box}^n$$

and

$$\Phi_\varepsilon = f \circ \Sigma_\varepsilon : \bigtimes_1^n\{0, 1, ...N_i\} \to [0, 1]^n.$$

Identify the sets $\{0, 1, ...N_i\}$ with the subsets $\{\frac{j}{N_i}\}_{j=1,...,N_i} \subset [0, 1]$ and extend the map Φ_ε to a continuous map $\Psi_\varepsilon : [0, 1]^n \to [0, 1]^n$, which is obtained by consecutive peacewise linear interpolation with conical extension of maps from the boundaries of faces of small cubes. to these faces.

Since the maps Ψ_ε are onto, the maps Φ_ε have ϵ-dense images in $[0, 1]^n$, where $\epsilon \to 0$ for $\varepsilon \to 0$ and the onto property of f follows with $\varepsilon \to 0$.[2]

G. ε-Corollary. If a continuous map

$$f : \tilde{\Box}^n = \bigtimes_1^N[S_i, \tilde{0}_i, \tilde{1}_i] \to \mathbb{R}^n \supset [0, 1]^N,$$

[2]This argument in homological terms proves continuity of *Čech cohomology.*

sends each face from the "cube" ε-close to the corresponding face of $[0,1]^n$, then the image f contains all points in $[0,1]^n$, which lie ε-far from the boundary $\partial[0,1]^n$.

Proof. Let $dist(z_0, \partial[0,1]^n) > \varepsilon$ and let $\phi_0 : [0,1]^n \to [0,1]^n$ be a continuous map, such that $\phi_0(z) = z$ on the boundary of the cube and in a small neighbourhood of z_0 and which sends the ε-neighbourhoods of the faces of $[0,1]^n$ to these very faces.

Then **F** applies to the composed map $\phi_0 \circ f : \tilde{\Box}^n \to [0,1]^n$ and **G** follows.

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H. The convex hull of a subset $X \subset \mathbb{R}^n$ is the set of all convex combinations

$$z = \sum_{j=1}^{N} p_j x_j,\ \ x_j \in X, p_j \geq 0, \sum_j p_j = 1,$$

where, this is called *Caratheodory theorem,*

if $z = \sum_{j=1}^{N} p_j x_j$, then there exists a subset $K \subset J = \{1, ..., N\}$ of cardinality $n+1$, such that $z = \sum_{k=1}^{n+1} q_k x_k$, for some $q_k \geq 0, \sum_k q_k = 1$.

In fact, the convex polyhedron $conv\{x_j\}$ can be (obviously) subdivided into simplices with vertices in $\{x_j\}$.

A point z in the convex hull of $X \subset \mathbb{R}^n$ is called X-*rational* if it is equal to a convex combination of points from X with *rational weights*,

$$[p_j] \qquad\qquad z = \sum_{j=1}^{N} p_j x_j,\ \ x_j \in X,$$

where $p_i \geq 0$ are rational numbers , such that $\sum_j p_i = 1$.

Equivalently, X-rational points $z \in conv(X)$ are *centers of mass* of finite multisets[3] from X,

$$[1/M] \qquad\qquad z = \frac{1}{M} \sum_{k=1}^{M} x_k,$$

where $[p_j] \implies [1/M]$ for M equal the common denominator of the numbers p_j.

I. SZ Theorem.[4] If a compact subset $X \subset \mathbb{R}^n$ contains $2n$ point $\underline{x}_i, \underline{y}_i \in X$, $i = 1, ..., n$, such that the n vectors $\underline{x}_i - \underline{y}_i \in \mathbb{R}^n$ are *linearly independent* and such that $\underline{x}_i$ and $\underline{y}_i$ lie in *the same connected component of* X for all $i = 1, ..., n$, then all points in the *interior* of the convex hull of X, are X-rational.

Proof. Since rational numbers are dense in $\mathbb{R}$ the X-rational points are dense in the convex hull of X and it suffices to show that the "rational intetrior" of the convex hull $conv(X)$ is *non-empty*: $conv(X)$ contains a ball of positive radius, say $B^n_{\underline{z}}(\underline{\delta}) \subset conv(X)$, $\underline{z} \in X$, $\underline{\delta} > 0$, such that all points in this ball X-rational

In fact, the existence of an X-rational ball $\underline{B} = B^n_{\underline{z}}(\underline{\delta})$ implies the existence of rational δ-balls around all points $z \in conv(S)$, $B = B_z(\delta)$, where δ is bounded from below essentially by the distance from z to the boundary of $conv(X)$, namely

$$\delta \geq \frac{(\underline{\delta} \cdot dist(z, \partial conv(X))}{2diam(X)}.$$

[3] A multiset is an mage of a map $I \to X$, written as $\{\underline{x}_i\} \subset X$, $i \in I$, $\underline{x}_i \in X$.

[4] Seymour, P. D. and Zaskavsky, T., *Averaging set. A generalization of mean values and spherical designs*, Adv. Math. 52 (1984), 213-246.

Indeed, let us extend the straight segment between $\underline{z}$ and z to the boundary of the ball $B_z(d)$, $d = dist(z, \partial conv(X)$, let

$$[z_0, \underline{z}] \subset conv(X)$$

be the extended segment with $z_0 \in \partial B_z(d)$ and with $z \in [z_0, \underline{z}]$, where $||z - z_0|| = d$.

Let $z_0' \subset conv(X)$ be an X-rational point ϵ-close to z_0 for

$$\epsilon \leq \frac{dist(z_0', \underline{z})}{10dist(z, \partial conv(X))}.$$

Now, the ball $B = B_z(\delta)$ for

$$\delta = \frac{\underline{\delta} \cdot dist(z_0', \underline{z})}{2dist(z, \partial conv(X))} - \epsilon$$

is the required X-rational one, since all points in it are are convex combinations $Nz_0' + (1 - N)b, b \in \underline{B}$, for an integer N, such that

$$\left| N - \frac{dist(z_0', \underline{z})}{dist(z, \partial conv(X))} \right| \leq 1.$$

With the above understood, the proof of the theorem reduces to the following.

J. ***Lemma.*** Let $\square^n \subset \mathbb{R}^n$ be the *Minkovski mean* of the straight segments $[\underline{x}_i, \underline{y}_i] \subset \mathbb{R}^n$, that is the set of the averages

$$\frac{1}{n} \sum_i z_i, \;\; z_i \in [\underline{x}_i, \underline{y}_i] \subset conv(X).$$

Then all points in the *interior* of $\square^n$ are *X-rational.* .

Proof. Let us show the existence of subsets, or rather multysets, in the connected components $S_i \subset X$ of $\underline{x}_i \in X$,

$$\{x_{i,j}\} \subset S_i, \; i = 1, ..., n, \; j = 1, ..., N,$$

such that all interior points $z \in int(\square^n)$ are representable as

$$z = \frac{1}{nN} \sum_{i,j} x_{ij}$$

for sufficiently large $N = N(z)$.

Definition of "Chain Segment". Given a "segment" $[S, \tilde{0}, \tilde{1}]$ let the $N * S$-*chain* in the N-"cube" $[S, \tilde{0}, \tilde{1}]^N$ be the union of the N consecutive "edges" E_j in this cube, which join the diagonally opposite "vertices" $(\underbrace{\tilde{0}, ..., \tilde{0}}_N)$ and $(\underbrace{\tilde{1}, ... \tilde{1}}_N)$,

$$E_j = \{\underbrace{\tilde{0}, ..., \tilde{0}}_{j-1}, s, \underbrace{\tilde{1}, ... \tilde{1}}_{N-j}\}_{s \in S} \subset [S, \tilde{0}, \tilde{1}]^N,$$

where this chain $[N * S] = \bigcup_{i=1}^N E_i$ is itself a "segment" with the "vertices" $(\underbrace{\tilde{0}, ..., \tilde{0}}_N)$ and $(\underbrace{\tilde{1}, ... \tilde{1}}_N)$.

Let $\phi : S \to \mathbb{R}^n$ be a continuous map and let

$$N * \phi : [N * S] \to \mathbb{R}^n$$

send $(s_1, ...s_N) \in N * S \subset S^N$ to the *center of mass of the N image points* $\phi(s_j) \in \mathbb{R}^n$, $j = 1, ..., N$,

$$N * \phi : (s_1, ...s_N) \mapsto \frac{1}{N} \sum_1^N \phi(s_j).$$

Clearly, the "division points" from the chain, that are

$$\{\underbrace{\tilde{0}, ..., \tilde{0}}_{j}, \underbrace{\tilde{1}, ...\tilde{1}}_{N-j}\},$$

lands in the segment $[\phi(\tilde{0}), [\phi(\tilde{1})] \subset \mathbb{R}^n$, such that

- these points divide this segment into N equal subsegments,
- the image of the j-th copy of S in $N * S$ goes to the δ-neighbourhood

of the j-th subsegment in $[\phi(\tilde{0}), [\phi(\tilde{1})]$, where δ is small when N is much greater than the diameter of the ϕ-image of S in $\mathbb{R}^n$:

$$\delta \leq \frac{diam(\phi(S))}{\sqrt{N}}.$$

Next let $S_i \subset X$ be the common connected components of $\underline{x}_i, \underline{y}_i \in X$, where we set $\tilde{0}_i = \underline{x}_i$ and $\tilde{1}_i = \underline{y}$, and let $N * S_i \subset S_i^N$ be their chain "segments" $N * S_i$.

Map $\times_1^n [N * S_i] \to conv(X)$ by

$$\Phi_N : s_{i,j} \mapsto \frac{1}{n} \sum_{i=1}^n \frac{1}{N} \sum_{j=1}^N s_{i,j}$$

where, by the above, with ϕ_i being the imbeddings $S_i \hookrightarrow \mathbb{R}^n$,

the map Φ_N sends each face of the n-"cube" $\times_1^n [N * S_i]$ to the δ-neighbourhood of the corresponding face in $\square^n$, for

$$\delta \leq \frac{\sum_1^n diam(\phi_i(S_i))}{\sqrt{N}}.$$

Finally, the ε-Corollary **G** applies and the proof follows.

K. From Multysets to Sets. The X-rationality of an z gives you a multyset in X with the enter of mass z but the above proof allows disengagement of multiple points by small perturbations. Therefore,

all points in the interior of $conv(X)$ are representable by centers of mass of (true) fintie subsets in X.

L. The original formulation of **I** reads:

Let X be a compact connected[5] space with a probability (total mass one) Borel measure dx, which is strictly positive on non-empty open subsets in X

[5]In the Seymour- Zaslavsky paper X is assumed path connected but not necessarily compact.

and let $f_i(x)$, $i = 1,...,n$, be continuous functions on X. Then there exists a finite subset $\Sigma \in X$ such that

$$\frac{1}{card(\Sigma)}\sum_{\sigma\in\Sigma} f_i(\sigma) = \int_X f_i(x)dx$$

for all $i = 1,...,n$.

Reduction $\boldsymbol{L} \implies \boldsymbol{K}$. Map $X \to \mathbb{R}^n$ by $x \mapsto 1(x),...,f(n)x)$, observe that the vector

$$z = \left(\int_X f_1(x)dx, ..., \int_X f_n(x)dx\right) \in \mathbb{R}^n$$

is the interior of $conv(X)$ due to positivity of dx. Then the subset $\Sigma \subset X$ with the center of mass z does the job.

M. *Exercises.* (a) Reduce the SZ-theorem for *no-compact path connected* X to the compact case.

(b) Let S_i be the images of C^1-maps $\phi_i : S_i \to \mathbb{R}^n$ of smooth connected manifolds S_i and show that the linear independence of $\underline{x}_i - \underline{y}_i$ implies that *the mages of the differentials* $d\phi_i : T(S_i) \to \mathbb{R}^n$ *at some points* $s_i \in S_i$ *span* $\mathbb{R}^n$.

Then prove lemma **J** in this case by applying the implicit function theorem.

N, Question Let $S_i \subset \mathbb{R}^n$, $n = 1,...n$, be compact connected subsets (e.g. the images of $[0,1]$ under continuous maps) which contain pairs of points $x_i, y_i \in S_i$ with linearly independent $x_i - y_i$. Is then the interior of the Minkovski mean (or the sum if you wish) non-empty.

(Seems easy but I couldn't figure it out.)

O. Hilbert's Rationality. Hilbert in his solution of the Waring problem[6] uses and proves **I** in the case, where points with rational coordinates are dense in X and where this is done for images of spheres S^l in $\mathbb{R}^n$ under polynomial maps with rational coefficients. Thus, this is small step in Hilbert's proof, he constsructs what is now-a-days called spherical designs $\Sigma \subset S^l$, where all points $\sigma \in \Sigma$ are rational.

[6]For all $p = 2,3,...$, there exists a constant $N = N(p)$, such that every positive integer x is the sum $x\sum_1^M y_i^p$ for positive integers y_i and $M \leq N$.