Probability Versus Topology

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April 21, 2022

BIG QUESTIONS:

1.?Meeting point between probability and algebraic topology.?

2. ?Randomization of Topology and/or algebraisation of probability? (?Random Spaces? ?Categorization of probability?)

3. ?(?Faithful?) Functorial description of (categories of) metric spaces, e.g. Riemannian manifolds and their submanifolds, in algebra-topological or probabilistic terms.?

DEVELOP PERSPECTIVE

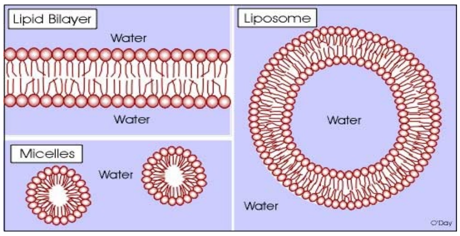

Figure 1:   Geometric measure theory

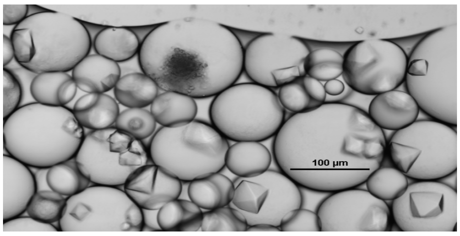

Figure 2: Crystal

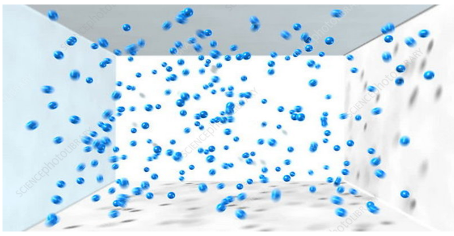

Figure 3: Gaz

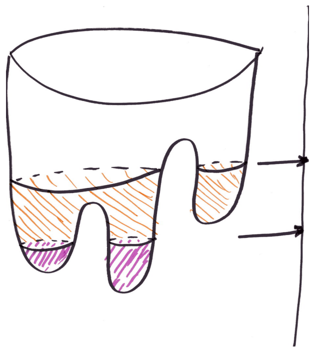

Figure 4: Energy

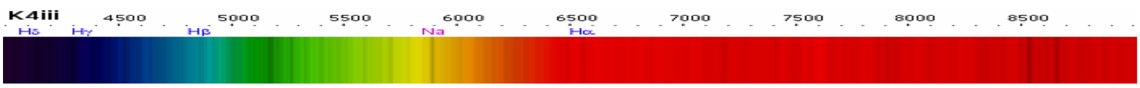

Figure 5: spectrum

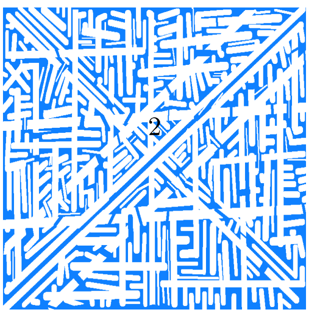

Figure 6: Self avoiding random walk, Flory (1953).

An "ensemble" $\mathcal{A} = \mathcal{A}(X)$ of (finitely or infinitely many) particles in a space X, e.g. in the Euclidean 3-space, is probabilistically characterised by

$$U \mapsto ent_U(\mathcal{A}) = ent(\mathcal{A}_{|U}),\ U \subset X,$$

that assigns the *entropies* of *the U-reductions $\mathcal{A}_{|U}$ of $\mathcal{A}$*, to all bounded open subsets $U \subset X$. (ent_U is "*the logarithm of the number of the states of $\mathcal{E}$ that are effectively observable from U*"),

Replace *"effectively observable number of states"* by

"the number of significant degrees of freedom of ensembles of moving particles"

Packings by r-Balls.

X is a metric space, $\mathcal{P} = \mathcal{P}_{I,r}(X) = \{x_i\} \subset X^I$, such that $dist(x_i, x_j) \geq d_{ij} = 2r$.

Covarinatly functoriality under *expanding* maps $X \to Y$ and

contravariant functoriality under *contracting* maps $f : X \to Y$.

Packings by Tubes motivated by $X = Y \times Z \to Y$. I-tuples of closed subsets $Z_i \in X$, such that mutual distances between them satisfy $dist(Z_i, Z_j) \geq d_{ij}$ and such that $Z_i \subset X$ support given *nonzero* homology classes h_i in X. (T. Richard *On the 2-systole of stretched enough positive scalar curvature metrics on* $S^2 \times S^2$, arXiv:2007.02705v2.)

Packing Energy and Morse Spectrum. $\rho(a) = \min_{x_i \neq x_j} dist(x_i, x_j)$,

$E(a) = \frac{1}{\rho(a)}$ or $E(a) = -\log \rho(a)$.

The homotopy significant (Morse) spectrum of an energy function $E : \mathcal{A} \to \mathbb{R}$, is the set of those values $y \in \mathbb{R}$, where the homotopy type of the sublevel $E^{-1}(-\infty, y]$ undergoes an *irreversible change* d

Guth' Duality between Homology Spectra of Packings and of Cycles.

Example. Relation between the cohomology spectrum of E on $(X)^I$ and $(Y)^I$ for $Y \subset X$ e.g. $X = \mathbb{T}^n$ and $Y = \mathbb{T}^m \subset \mathbb{T}^n$. (Viazovska theorem)

Results.

1. δ-*Waist Inequality.* $f : S^n \to \mathbb{R}^{n-k}$ a continuous map. $\exists p \in \mathbb{R}^{n-k}$, such that $vol(U_\delta(f^{-1}(p) \geq vol(U_\delta(S^k)$ for all $\delta > 0$

2 **Guth' Steenrod Spectrum Theorem.** X be the space of m-dimensional submanifolds x in the n-ball V and $F(x) = vol_m(x)$.

Then the volume spectrum of this F satisfies

$\lambda_i \leq const \cdot i^{\frac{1}{m+1}}$

and

$\lambda_i \geq const(\varepsilon) \cdot i^{\frac{1}{m+1}-\varepsilon}$ *for all* $\varepsilon > 0$.

Weyl law for the codimension 1 volume spectrum (Liokumovich-Marques-Neves)

If $m = n - 1$ then $\lambda_i \sim const_n \cdot i^{\frac{1}{n}}$.

Question 1. How much of the geometry of a space X, say with a metric or symplectic geometry, can be seen in the homotopies of spaces of packings of X by such U_i?

Question 2. Is there a good category of "abstract packing-like objects", that are not, a priori, associated to actual packings of geometric spaces?

- *Classical (Non-parametic) Sphere Packings.*
- *Homotopy and Cohomotopy Energy Spec-*

tra.

- *Homotopy Dimension, Cell Numbers and Cohomology Valued Measures.*
- *Infinite Packings and Equivariant Topology of Infinite Dimensional Spaces Acted upon by Non-compact Groups.*
- *Bi-Parametric Pairing between Spaces of Packings and Spaces of Cycles.*
- *Non-spherical Packings, Spaces of Partitions and Bounds on Waists.*
- *Symplecting Packings.*
- *Parametric coverings.*

Homology Measures

(*Morse Spectra, Homology Measures and Parametric Packing Problems*)

$$\Psi \supset D \mapsto \mu(D) = \mu^*(D;\Pi) =$$
$$\mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^* = H^*(\Psi;\Pi),$$

where Π is an Abelian (homology coefficient) group, e.g. a field $\mathbb{F}$, and $\mathbf{0}^{\smallsetminus *}(D;\Pi)$ is the *kernel* of the cohomology restriction homomorphism for the complement $\Psi \smallsetminus D \subset \Psi$,

$$H^*(\Psi;\Pi) \to H^*(\Psi \smallsetminus D;\Pi).$$

The set function

$\mu^* : \{subsets \subset \Psi\} \to \{subgroups \subset H^*\}$

is additive for the sum-of-subsets in H^* *and super-multiplicative*[1] *for the the* $\smile$*-product of ideals in the case where* Π *is a commutative ring:*

$[\cup +]$ $\mu^*(D_1 \cup D_2) = \mu^*(D_i) + \mu^*(D_2)$

for *disjoint* open subsets D_1 and D_2 in Ψ, and

$[\cap \smile]$ $\mu^*(D_1 \cap D_2) \supset \mu^*(D_1) \smile \mu^*(D_2)$

for all open $D_1, D_2 \subset \Psi$.

Homology spectra on spaces of infinitely many particles in non-compact manifolds

Infinite dimensional space Ψ,

action of an infinite group Υ on Ψ.

Example. Υ is a countable group call it Γ, e.g. $\Gamma = \mathbb{Z}^n$, and $\Psi = B^\Gamma$ is the space of maps $\Gamma \to B$.

[1] This, similarly to *Shannon's subadditivity inequality*, implies the existence of *"thermodynamic limits"* of *Morse Entropies*, see [?].

H^* is a graded algebra (over some field) acted upon by a countable amenable group Γ.

Exhaust Γ by finite *Følner subsets* $\Delta_i \subset \Gamma$, $i = 1, 2, ...$, and, given a finite dimensional graded subalgebra $K = K^* \subset H^*$, let $P_{i,K}(t)$ denote the Poincare polynomial of the graded subalgebra in H^* generated by the γ-transforms $\gamma^{-1}(K) \subset H^*$ for all $\gamma \in \Delta_i$.

Define *polynomial entropy* of the action of Γ on H^* as follows.

$$Poly.ent(H^* : \Gamma) = \sup_K \lim_{i \to \infty} \frac{1}{card(\Delta_i)} \log P_{i,K}(t).$$

(Permutation Symmetries and Equivariant Homology)

?*Energy* $\rightsquigarrow$ *Boltzmann distribution*?

Sup_ϑ-Spectra, Scalar Curvature and Spaces of Symplectic Packings.

Θ is a set of metrics ϑ on a topological space X

"Inv"(X, ϑ) is an invariant,

sup_ϑ"Inv" is *the supremum* of the invariants "$Inv : (X, \vartheta)$ over all $\vartheta \in \Theta$.

Example 1. $X = S^3$ and ϑ are metrics with $Sc(\vartheta) \geq 6$.

Then $sup_\vartheta waist_2 = 4\pi$.(Marques-Neves)

Example 2. $X = (X, \omega)$ a symplectic manifold of dimension $n = 2m$ and ϑ are ω-adapted metrics

Question. Which part of the (suitably factorized/coarsened) homotopy/homology area spectra of (X, ϑ) remains finite after taking suprema over ϑ?

If $k = 2$, then upper bounds in some cases are obtained with pairing ball pacing with "psedoholomorphic curves" defined here as oriented surfaces $Y \subset X = (X, \omega, \vartheta)$, such that $area_\vartheta(Y) = \int_Y \omega$.