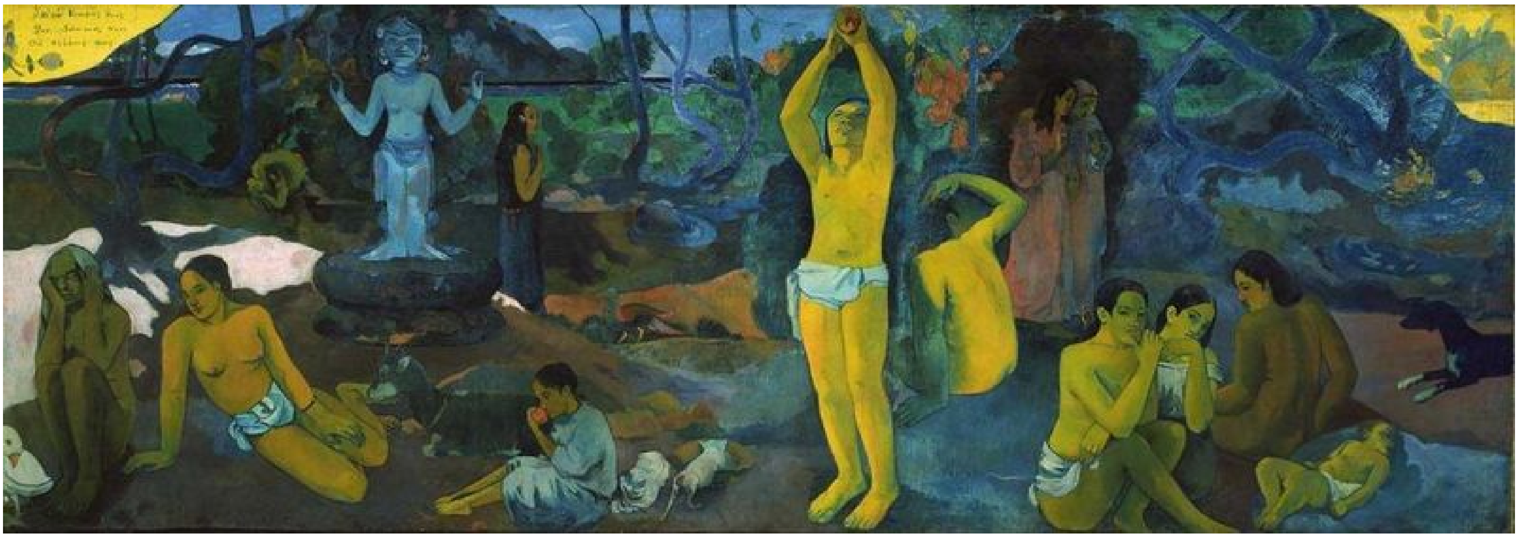

Manifolds: Where Do We Come From? What Are We? Where Are We Going

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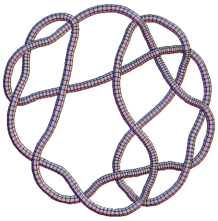


Abstract

Descendants of algebraic kingdoms of high dimensions, enchanted by the magic of Thurston and Donaldson, lost in the whirlpools of the Ricci flow, topologists dream of an ideal land of manifolds – perfect crystals of mathematical structure which would capture our vague mental images of geometric spaces. We browse through the ideas inherited from the past hoping to penetrate through the fog which conceals the future.

1 Ideas and Definitions.

We are fascinated by knots and links. Where does this feeling of beauty and mystery come from? To get a glimpse at the answer let us move by 25 million years in time.

25×10^6 is, roughly, what separates us from orangutans: 12 million years to our common ancestor on the phylogenetic tree and then 12 million years back by another branch of the tree to the present day orangutans.

But are there topologists among orangutans?

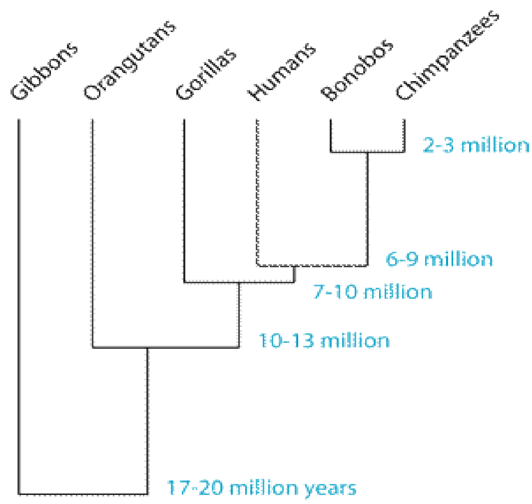
Yes, there definetely are: many orangutans are good in "proving" triviality of elaborated knots, e.g. they fast master the art of untying boats from their mooring when they fancy taking rides downstream in a river, much to annoyance of people making these knots with a different purpose in mind.

A more amazing observation was made by a zoo-psycologist Anne Russon in mid 90's at Wanariset Orangutan Reintroduction Project (see p. 114 in [48]).

"... Kinoi [a juvenile male orangutan], when he was in a possession of a hose, invested every second in making giant hoops, carefully inserting one end of his hose into the other and jumming it in tight. Once he'd made his hoop, he passed various parts of himself back anf forth through it – an arm, his head, his feet, his whole torso – as if completely fascinated with idea of going through the hole."

A play with hoops and knots, where there is no visible goal or any practical gain – be it an ape or a $3D$-topologist – appears fully "non-intelligent" to a practiaclly minded observer. But we, geometers, feel thrilled at seeing an animal whose space perception is so similar to ours.

It is unlikely, however, that Kinoi would formulate his ideas the way we do and that, unlike our students, he could be easily intimidated into accepting



"equivalence classes of atlases" and "ringed spaces" as appropriate definitions of his topological playground. (Despite such display of disobedience, we would enjoy a company of young orangutans; they are charmingly playful creatures, unlike aggressive and reckless chimpanzees – our nearest evolutionary neighbors.)

Apart from topology, orangutans do not rush to accept another human definition, namely that of "tools", as of

"external *detached* objects (to exclude a branch used for climbing a tree) employed for reaching specific goals".

(A use of tools is often taken by zoo-psicoligists for a measure of "intelligence" of an animal.)

Being imaginative arboreal creatures, orangutans prefer a broader definition: For example (see [48]):

• they bunch up leaves to make wipers to clean their bodies without detaching the leaves from a tree;

• they often break branches but deliberately leave them attached to trees when it suits their purposes – these could not have been achieved if orangutans were bound by to the "detached" definition.

Morale. Our best definitions, e.g. that of a manifold, tower as prominent landmarks of our former insights. Yet, we should not be hypnotized by definitions. After all, they are remnants of the past and tend to misguide us when we try to probe the future.

Remark. There is a non-trivial similarity between the neurological structures underlying the behaviour of playful animals and that of working mathematicians (see [18]).

2 Homotopies and Obstructions.

For more than half a century, starting from Poincare, topologists have been laboriously stripping their beloved science of its geometric garments.

"Naked topology", reinforced by homological algebra, reached its to-day breathtakingly high plato with the following

Serre $[S^{n+N} \to S^N]$-Finiteness Therem. (1951) *There are at most finitely many homotopy classes of maps between spheres $S^{n+N} \to S^N$ but for the two exceptions:*

• *equivi-dimensional case: here $\pi_N(S^N) = \mathbb{Z}$ and the homotopy class of a map is determined by its degree.* (Brouwer 1912, Hopf 1926. We define degree in section 4.)

• *Hopf case, where N is even and $n = 2N - 1$. In this case $\pi_{2N-1}(S^N)$ contains a subgroup of finite index isomorphic to $\mathbb{Z}$.*

It follows that

the homotopy groups $\pi_{n+N}(S^N)$ are finite for $N >> n$.

(H. Hopf proved in 1931 that the map $f : S^3 \to S^2 = S^3/\mathbb{T}$, for the group $\mathbb{T} \subset \mathbb{C}$ of the complex numbers with norm one which act on $S^3 \subset \mathbb{C}^2$ by $(z_1, z_2) \mapsto (tz_1, tz_2)$, is non-contractible.

In general, the unit tangent bundle $X = UT(S^{2k}) \to S^{2k}$ has finite homology $H_i(X)$ for $0 < i < 4k - 1$. By Serre's theorem, there exits a map $S^{4k-1} \to X$ of positive degree and the composed map $S^{4k-1} \to X \to S^{2k}$ generates an *infinite* cyclic group of *finite index* in $\pi_{4k-1}(S^{2k})$.

Also one knows that the group $\pi_{n+N}(S^N)$ does not depend on N for $N \geq n$ by the *Freudenthal suspension theorem* of 1928.)

The proof by Serre – a geometer's nightmare – consists in tracking a multitude of linear-algebraic relations between homology and homotopy groups of infinite dimensional spaces of maps between spheres and it tells you next to nothing about the geometry of these maps.

Recall that the set homotopy classes of maps of a sphere S^M to a connected space X makes *a group* denoted $\pi_M(X)$, (π is for Poincare who defined *the fundamental group* π_1) where the definition of the group structure depends on distinguished points $x_0 \in X$ and $s_0 \in S^M$. (The groups π_M defined with different x_0 are mutually isomorphic, and if X is *simply connected*, i.e. $\pi_1(X) = 1$, then they are *canonically* isomorphic.)

This point in S^M may be chosen with the representation of S^M as the one point compactification of the Euclidean space $\mathbb{R}^M$, denoted $\mathbb{R}^M_\bullet$, where this infinity point $\bullet$ is taken for s_0. It is convenient, instead of maps $S^m = \mathbb{R}^m_\bullet \to (X, x_0)$, to deal with maps $f : \mathbb{R}^M \to X$ "with compact supports", where the *support* of an f is the closure of the (open) subset $supp(f) = supp_{x_0}(f) \subset \mathbb{R}^m$ which consists of the points $s \in \mathbb{R}^m$ such that $f(s) \neq x_0$.

A pair of maps $f_1, f_2 : \mathbb{R}^M \to X$ with *disjoint* compact supports obviously defines "the joint map" $f : \mathbb{R}^M \to X$, where the homotopy class of f (obviously) depends only on those of f_1, f_2, provided $supp(f_1)$ lies in the left half space $\{s_1 < 0\} \subset \mathbb{R}^m$ and $supp(f_2) \subset \{s_1 > 0\} \subset \mathbb{R}^M$, where s_1 is a non-zero linear function (coordinate) on $\mathbb{R}^M$.

The composition of the homotopy classes of two maps, denoted $[f_1] \cdot [f_2]$, is defined as the homotopy class of the joint of f_1 moved far to the left with f_2 moved far to the right.

Geometry is sacrificed here for the sake of algebraic convenience: first, we break the symmetry of the sphere S^M and then of $\mathbb{R}^M$ by the choice of s_1.

If $M = 1$, then there are essentially two choices: s_1 and $-s_1$, which correspond to interchanging f_1 with f_2 – nothing wrong with this as the composition is, in general, non-commutative.

In general $M \geq 2$, these $s_1 \neq 0$ are, homotopically speaking, parametrized by the unit sphere $S^{M-1} \subset \mathbb{R}^M$. Since S^{M-1} is connected for $M \geq 2$, the composition is commutative and, accordingly, the composition in π_i for $i \geq 2$ is denoted denoted $[f_1] + [f_2]$. Good for algebra, but the $O(M+1)$-ambiguity seems too great a price for this.

But this is, probably, unavoidable. For example, the best you can do for maps $S^M \to S^M$ in a given non-trivial homotopy class is to make them symmetric (i.e. equivariant) under the action of the maximal torus $\mathbb{T}^k$ in the orthogonal group $O(M+1)$, where $k = M/2$ for even M and $k = (M+1)/2$ for M odd.

And if $n \geq 1$, then, with a few exceptions, there are no apparent symmetric representatives in the homotopy classes of maps $S^{n+N} \to S^N$; yet Serre's theorem does carry a geometric message.

If $n \neq 0, N-1$, then every continuous map $f_0 : S^{n+N} \to S^N$ is homotopic to a map $f_1 : S^{n+N} \to S^N$ of dilation bounded by a constant,

$$dil(f_1) =_{def} \sup_{s_1 \neq s_2 \in S^{n+N}} \frac{dist(f(s_1), f(s_2))}{dist(s_1, s_2)} \leq const(n, N).$$

Dilation Questions. (1) What is the asymptotic behaviour of $const(n, N)$ for $n, N \to \infty$?

For all we know the *Serre dilation constant* $const_S(n, N)$ may be bounded for $n \to \infty$ and, say, for $1 \leq N \leq n-2$, but a bound one can see offhand is that by an exponential tower $(1+c)^{(1+c)^{(1+c)^{\cdots}}}$, of hight N, since each geometric implementation of the homotopy lifting property in a Serre fibrations may bring along an exponential dilation.

(2) Let $f : S^{n+N} \to S^N$ be a contractible map of dilation d, e.g. f equals the m-multiple of another map where m is divisible by the order of $\pi_{n+N}(S^N)$.

What is, roughly, the minimum $D_{min} = D(d, n, N)$ of dilations of maps F of the unit ball $B^{n+N+1} \to S^N$ which are equal to f on $\partial(B^{n+N+1}) = S^{n+N}$?

Of course, this dilation is the most naive invariant measuring the "geometric size of a map". Possibly, an interesting answer to the these questions needs a more imaginative definition of "geometric size/shape" of a map.

Serre's theorem and its decedents underly most of the topology of the high dimensional manifolds. Below are frequently used corollaries which relate homotopy problems concerning general spaces X to the homology groups $H_i(X)$ (see section 4 for definitions) which are much easier to handle.

$[S^{n+N} \to X]$**-Theorems.** Let X be a compact connected triangulated space, or, more generally, a connected space with finitely generated homology groups $H_i(X)$, $i = 1, 2, \ldots$. If the space X is *simply connected*, i.e. $\pi_1(X) = 1$, then its homotopy groups have the following properties.

(1) *Finite Generation.* The groups $\pi_m(X)$ are (Abelian!) *finitely generated* for all $m = 2, 3, \ldots$.

(2) *Sphericity.* If $\pi_i(X) = 0$ for $i = 1, 2, N - 1$, then the (obvious) Hurewicz homomorphism

$$\pi_N(X) \to H_N(X),$$

which assigns, to a map $S^N \to X$, the N-cycle represented by this N-sphere in X, is an *isomorphism.* (This is elementary, Hurewicz 1935.)

(3) $\mathbb{Q}$-*Sphericity.* If the groups $\pi_i(X)$ are finite for $i = 2, N - 1$ (recall that $\pi_1(X) = 1$), then the Hurewicz homomorphism *tensored with rational numbers*,

$$\pi_{N+n}(X) \otimes \mathbb{Q} \to H_{N+n}(X) \otimes \mathbb{Q},$$

is an *isomorphism* for $n = 1, \ldots, N - 2$.

Because of the finite generation property, The $\mathbb{Q}$-sphericity is equivalent to

(3') **Serre m-Sphericity Theorem**. Let the groups $\pi_i(X)$ be finite (e.g. trivial) for $i = 1, 2, \ldots, N - 1$ and $n \leq N - 2$. Then

an m-multiple of every $(N + n)$-cycle in X for some $m \neq 0$ is homologous to an $(N + n)$-sphere continuously mapped to X;

every two homologous spheres $S^{N+n} \to X$ become homotopic when composed with a non-contractible i.e. of degree $m \neq 0$, self-mapping $S^{n+N} \to S^{n+N}$. In more algebraic terms, the elements $s_1, s_2 \in \pi_{n+N}(X)$ represented by these spheres satisfy $ms_1 - ms_2 = 0$.

The following is the dual of the m-Sphericity.

Serre $[\to S^N]_{\mathbb{Q}}$- Theorem. Let X be a compact triangulated space of dimension $n + N$ and let $f, g : X \to S^N$ be continuous maps.

If $n < N - 1$ and the maps are "homologous", i.e. if the homology homomorphisms $f_*, g_* : H_N(X) \to H_N(S^N) = \mathbb{Z}$ are equal, then there exists a continuous self-maping $\sigma : S^N \to S^N$ of non-zero degree such that the composed maps $\sigma \circ f$ and $\sigma \circ f : X \to S^N$ are homotopic.

Moreover, a non-zero multiple of every homomorphism $H_N(X) \to H_N(S^N)$ can be realized by a continuous map $X \to S^N$.

These $\mathbb{Q}$-theorems follow from the Serre finitness theorem for maps between spheres by an elementary argument of *induction by skeletons* and rudimentary *obstruction theory* which run, roughly, as follows.

Let X be a triangulated space, denote by $X_i \subset X$ the *i-skeleton* of X, i.e. the union of all closed i-simplices Δ^i in X, and let Y be a connected space such that $\pi_i(Y) = 0$ for $i = 1, \ldots, n - 1 \geq 1$.

Given a continuous map $f : X \to Y$, let us construct, by induction on $i = 0, 1, ..., n-1$, a map $f_{new} : X \to Y$ which is homotopic to f and which sends X_{n-1} to a point $y_0 \in Y$ as follows.

Assume $f(X_{i-1}) = y_0$. Then the map $\Delta^i \overset{f}{\to} Y$, for each i-simplex Δ^i from X_i, makes an i-sphere in Y, because the boundary $\partial\Delta^i \subset X_{i-1}$ goes to a single point – our to y_0 in Y.

Since $\pi_i(Y) = 0$, this Δ^i in Y can be contracted to y_0 without disturbing its boundary. We do it all i-simpices from X_i and, thus, contract X_i to y_0. (One can not, in general, extend a continuous map from a closed subset $X' \subset X$ to X, but one always can extend a continuous *homotopy* $f'_t : X' \to Y$, $t \in [0,1]$, of a given map $f_0 : X \to Y$, $f_0|X' = f'_0$, to a homotopy $f_t : X \to Y$ for all closed subsets $X' \subset X$, similarly to how one extends $\mathbb{R}$-valued functions from $X' \subset X$ to X.)

The contraction of X to a point in Y can be obstructed on the n-th step, where $\pi_n(Y) \neq 0$, and where each *oriented* n-simplex $\Delta^n \subset X$ mapped to Y with $\partial(\Delta^n) \to y_0$ represents an element $c \in \pi_n(Y)$. (When we switch an orientation in Δ^n, then $c \mapsto -c$.) It is easy to see that the function $c(\Delta^n)$ is (obviously) an *n-cocycle* in X with values in the group $\pi_n(Y)$, i.e. the sum of these $c(\Delta^n)$ over the $(n+2)$ simplices $\Delta^n \subset \partial\Delta^{n+1}$ *equals zero*, for all Δ^{n+1} in the triangulation (if we canonically/correctly choose orientations in all Δ^n).

The cohomology class $[c] \in H^n(X; \pi_n(X))$ of this cocycle does not depend (by an easy argument) on how the $(n-1)$-skeleton was contracted. Moreover, every cocycle c' in the class of $[c]$ can be obtained by a homotopy of the map on X_n which is kept constant on X_{n-2}. (Two A-valued n-cocycles c and c', for an abelian group A, are *in the same cohomology class* if there exits an A-valued function $d(\Delta^{n-1})$ on the oriented simplices $\Delta^{n-1} \subset X_{n-1}$, such that $\sum_{\Delta^{n-1} \subset \Delta^n} d(\Delta^{n-1}) = c(\Delta^n) - c'(\Delta^n)$ for all Δ^n. The set of the cohomology classes of n-cocycles with a natural additive structure is called the *cohomology group* $H^n(X; A)$. It can be shown that $H^n(X; A)$ depends only on X but not an a particular choice of a triangulation of X.)

In particular, if $dim(X) = n$ we, thus, equate the set $[X \to Y]$ of the homotopy classes of maps $X \to Y$ with the cohomology group $H^n(X; \pi_n(X))$. Furthermore, this argument applied to $X = S^n$ shows that $\pi_n(X) = H_n(X)$ and, in general, that

the set of the homotopy calsses of maps $X \to Y$ equals the set of homomorphisms $H_n(X) \to H_n(Y)$, provided $\pi_i(Y) = 0$ for $0 < i < dim(X)$.

Finally, when we use this construction for proving the above $\mathbb{Q}$-theorems where one of the spaces is a sphere, we keep composing our maps with self-mappings of this sphere of suitable degree $m \neq 0$ that kills the obstruction by the Serre finiteness theorem.

The obstruction theory well displays the logic of algebraic topology: the geometric symmetry of X (if there was any) is broken by an arbitrary triangulation and then another kind symmetry, an Abelian algebraic one, emerges on the (co)homology level.

This idea was developed into a full fledged obstruction theory by Eilenberg in 1940 following Pontrygin's 1938 paper.

3 Generic Pullbacks.

A common zero set of n smooth (i.e. infinitely differentiable) functions $f_i : \mathbb{R}^{n+N} \to \mathbb{R}$, $i = 1,...N$, may be very nasty even for $n = 1$ – every closed subset in $\mathbb{R}^{n+N}$ can be represented as a zero of a smooth function. However, if the functions f_i are taken *in general position*, then the common zero set is a smooth n-submanifold in $\mathbb{R}^{n+N}$.

Here and below, "f in general position" or "generic f", where f is an element of a topological space F, e.g. of the space of C^∞-maps with the C^∞-topology, means that what we say about f applies to all f in an *open and dense* subset in F. (Sometimes, one allows not only open dense sets in the definition of genericity but also their *countable* intersections.)

Generic smooth (unlike continuous) objects are as nice as we expect them to be; the proofs of this "niceness" are local-analytic and elementary (at least in the cases we need); everything trivially follows from *Sard's theorem + the implicit function theorem.*

The representation of manifolds with functions generalizes as follows..

Generic Pullback Construction (Pontryagin 1938, Thom 1954). Start with a smooth N-manifold V, e.g. $V = \mathbb{R}^N$ or $V = S^N$, and let $X_0 \subset V$ be a smooth submanifold, e.g. $0 \in \mathbb{R}^N$ or a point $x_0 \in S^N$. Let W be a smooth manifold of dimension M, e.g. $M = n + N$.

if $f : W \to V$ is a generic smooth map, then the pullback $X = f^{-1}(X_0) \subset W$ is a smooth submanifold in W with $codim_W(X) = codim_V(X_0)$, i.e. $M - dim(X) = N - dim(X_0)$.

Moreover, if the manifolds W, V and X_0 are oriented, then X comes with a natural orientation.

Furthermore, if W has boundary then X is a smooth submanifold in W with boundary $\partial(X) \subset \partial(W)$.

Examples. (a) Let $f_0 : W \subset V \supset X_0$ be a smooth, possibly non-generic, embedding. Then a small generic perturbation $f : W \subset V$ of $f_0(W)$ in V makes $W = f(W)$ *transversal* (i.e. nowhere tangent) to X_0 and one sees with the full geometric clarity (with a picture of two planes in the 3-space which intersect at a line) that the intersection $X = W \cap X_0 (= f^{-1}(X_0))$ is a submanifold in V with $codim_V(X) = codim_V(W) + codim_V(X_0)$.

(b) Let $f : S^3 \to S^2$ be a smooth map and $S_1, S_2 \in S^3$ be the pullbacks of two generic points $s_1, s_2 \in S^2$. These S_i are smooth closed curves; they are naturally oriented, granted orientations in S^2 and in S^3.

Let $D_i \subset B^4 = \partial(S^3)$, $i = 1, 2$, be generic smooth oriented surfaces in the ball $B^4 \supset S^3 = \partial(B^4)$ with their oriented boundaries equal S_i and let $h(f)$ denotes the *intersection index* (defined in the next section) between D_i.

Suppose, the map f is homotopic to zero, extend it to a smooth generic map $F : B^4 \to S^2$ and take the F-pullbacks D'_i of s_i.

Let S^4 be the 4-sphere obtained from the two copies of B^4 by identifying the boundaries of the balls and let $C_i = D_i \cup D'_i \subset S^4$.

Since $\partial(D_i) = \partial(D'_i) = S_i$, these C_i are *closed* surfaces; hence, the intersection index between them equals zero (because they are homologous to zero in S^4, see the next section), and since D'_i do not intersect, the intersection index $h(f)$ between D_i is also zero.

It follows that *non-vanishing* of the *Hopf invariant* $h(f)$ implies that f is *non-homotopic to zero.*

For instance, the Hopf map $S^3 \to S^2$ is non-contractable, since every two transversal flat dicks $D_i \subset B^4 \subset \mathbb{C}^2$ bounding equatorial circles $S_i \subset S^3$ intersect at a single point.

The essential point of the seemingly trivial pull-back construction, is that starting from "simple manifolds" $X_0 \subset V$ and W, we produce complicated and more interesting ones by means of "complicated maps" $W \to V$. (It is next to impossible to make an interesting manifold with the "equivalence class of atlases" definition.)

For example, if $V = \mathbb{R}$, and our maps are functions on W, we can generate lots of them by using algebraic and analytic manipulations with functions and then we obtain maps to $\mathbb{R}^N$ by taking N-tuples of functions.

And less obvious (smooth generic) maps, for all kind of V and W, come as smooth generic approximations of continuous maps $W \to V$ delivered by the algebraic topology.

Following Thom (1954) one applies the above to maps into one point compactifications $V_\bullet$ of open manifolds V where one still can speak of generic pullbacks of smooth submanifolds X_0 in $V \subset V_\bullet$ under maps $W \to V_\bullet$

Thom spaces. The Thom space of a *vector bundle* $V \to X_0$ over a compact space X_0 (where the pullbacks of all points $x \in X_0$ are Euclidean spaces $\mathbb{R}^n$) is the one point compactifications $V_\bullet$ of V, where X_0 is canonically embedded into $V \subset V_\bullet$ as the zero section of the bundle (i.e. $v \mapsto 0 \in \mathbb{R}^n_v$).

If $X = X^n \subset W = W^{n+N}$ is a smooth submanifold, then the total space of its *normal bundle* denoted $U^\perp \to X$ is (almost canonically) diffeomorphic to a small (normal) ε-neighbourhood $U(\varepsilon) \subset W$ of X, where every ε-ball $B^N(\varepsilon) = B^N_x(\varepsilon)$ normal to X at $x \in X$ is radially mapped to the fiber $\mathbb{R}^N = \mathbb{R}^N_x$ of $U^\perp \to X$ at x.

Thus the Thom space $U^\perp_\bullet$ is identified with $U(\varepsilon)_\bullet$ and the tautological map $W_\bullet \to U(\varepsilon)_\bullet$, that equals the identity on $U(\varepsilon) \subset W$ and sends the complement $W \setminus U(\varepsilon)$ to $\bullet \in U(\varepsilon)_\bullet$, defines the *Atiyah-Thom map* for all closed smooth submanifold $X \subset W$,

$$A^\perp_\bullet : W_\bullet \to U^\perp_\bullet.$$

Recall that every $\mathbb{R}^N$-bundle over an n-dimensional space with $n < N$, can be induced from the tautological bundle V over the Grassmann manifold $X_0 = Gr_N(\mathbb{R}^{n+N})$ of N-planes (i.e. linear N-subspaces in $\mathbb{R}^{n+N}$) by a continuous map, say $G : X \to X_0 = Gr_N(\mathbb{R}^{n+N})$.

For example, if $W = \mathbb{R}^{n+N}$, one can take *the normal Gauss map* for G that sends $x \in X$ to the N-plane $G(x) \in Gr_N(\mathbb{R}^{n+N}) = X_0$ which is parallel to the normal space of X at x.

Since the Thom space construction is, obviously, functorial, every $U^\perp$-bundle inducing map $X \to X_0 = Gr_N(\mathbb{R}^{n+N})$ for $X = X^n \subset W = W^{n+N}$, defines a map $U^\perp_\bullet \to V_\bullet$ and this, composed with with $A^\perp_\bullet$, gives us the *Thom map*

$$T_\bullet : W_\bullet \to V_\bullet \text{ for the tautological } N\text{-bundle } V \to X_0 = Gr_N(\mathbb{R}^{n+N}).$$

Since all n-manifolds can be (obviously) embedded (by generic smooth maps) into Euclidean spaces $\mathbb{R}^{n+N}$, $N >> n$, every *closed*, i.e. compact without boundary, n-manifold X comes from the generic pullback construction applied to maps

f from $S^{n+N} = \mathbb{R}_\bullet^{n+N}$ to the Thom space $V_\bullet$ of the canonical N-vector bundle $V \to X_0 = Gr_N(\mathbb{R}^{n+N})$,

$$X = f^{-1}(X_0) \text{ for generic } f : S^{n+N} \to V_\bullet \supset X_0 = Gr_N(\mathbb{R}^{n+N}).$$

In a way, Thom has discovered the source of all manifolds in the world and responded to the question "Where are manifolds coming from?" with the following

1954 Answer. *All closed smooth n-manifolds X come as pullbacks of Grassmannians $X_0 = Gr_N(\mathbb{R}^{n+N})$ in the ambient Thom spaces $V_\bullet \supset X_0$ under generic smooth maps $S^{n+N} \to V_\bullet$.*

The manifolds X obtained with the generic pull-back construction come with a grain of salt: generic maps are abundant but it is hard to put your finger on any one of them – we can not say much about topology and geometry of an individual X. (It seems, one can not put *all* manifolds in one basket without some "random string" attached to it.)

But, empowered with Serre's theorem, this construction unravels an amazing structure in the "space of all manifolds" (Before Serre, Pontryagin and following him Rokhlin proceeded in the reverse direction by applying smooth manifolds to the homotipy theory via the Pontryagin construction.)

Selecting an object X, e.g. a submanifold, from a given collection $\mathcal{X}$ of similar objects, where there is *no distinguished* member $X^\star$ among them, is a notoriously difficult problem which had been known since antiquity and can be traced De Cael of Aristotle. It reappeared in 14th century as *Buridan's ass problem* and as *Zermelo's choice problem* at the beginning of 20th century.

A geometer/analyst tries to select an X by first finding/constructing a "value finction" on $\mathcal{X}$ and then by taking the "optimal" X. For example, one may go for n-submanifolds X of *minimal volumes* in an $(n+N)$-manifold W endowed with a *Riemannin metric*. However, a minimal X is usually singular with the only known exception $X^n \subset W^{n+1}$ for $n \leq 6$ (Simons, 1968).

Picking up a "generic" or a "random" X from $\mathcal{X}$ is a geometer's last resort when all "deterministic" options have failed. This is aggravated in topology, since

• on the one hand, there is no known construction delivering *all* manifolds X besides generic pullbacks and their close relatives;

• on the other hand, geometrically interesting manifolds X are not anybody's pullbacks. Often, they are "complicated quotients of simple manifolds", e.g. $X = S/\Gamma$, where S is a symmetric space and Γ is a discrete isometry group acting on S, possibly, with fixed points.

(It is obviouos that every surface X is homeomorphic to such a quotient, and this is also so for compact 3-manifolds by a theorem of Thurston. But if $n \geq 4$, one does not know if every closed smooth manifold X is homeomorphic to such S/Γ.)

Starting from another end, one has *ramified covers* $X \to X_0$ of "simple" manifolds X_0, where one wants the ramification locus $\Sigma_0 \subset X_0$ to be a subvariety with "mild singularities" and with an "interesting" fundamental group of the complement $X_0 \setminus \Sigma_0$, but finding such Σ_0 is difficult (see the discussion following (3) in section 7).

4 Duality and the Signature.

Define *i-cycles* C in a smooth n-manifold X, which represent homology classes $[C] \in H_i(X)$, as "compact oriented i-submanifolds $C \subset X$ with singularities of codimension two".

Such a C contains, by definition, an open dense subset $C_{reg} \subset C$ which *is* a smooth i-submanifold in X and where the complement $C_{sing} \subset C \setminus C_{reg}$ is a "piecewise smooth subset of dimension $\leq i-2$". This may be understood in a most generous sense, e.g. by allowing C_{sing} to be a closed set which is contained in a locally finite union of submanifolds of dimensions $i-2$. (Different concepts of a cycle, i.e. as of "sub-pseudo-manifold", as in section 9, lead to an equivalent definition.)

If X is a closed oriented manifold, then it itself makes an n-cycle which represents what is called the *fundamental class* $[X] \in H_n(X)$. Other n-cycles are $\pm$-combinations of the oriented connected componets of X.

If Y is a manifold with a boundary $X = \partial(Y)$, then *relative* $(i+1)$*-cycles* D in Y are required to be products near X: a small neighbourhood (collar) $U_\varepsilon \subset Y$ of X in Y is the product $U_\varepsilon = X \times [0,\varepsilon]$ and we want $D \cap U = C \times [0,\varepsilon]$. And such a D must be a smooth oriented "submanifold" with codimension ≥ 2 singularities in Y away from X.

Cycles C_1 and C_2 are called *homologous*, written $C_1 \sim C_2$, if there is a relative cycle D in $X \times [0,1]$, such that $\partial(D) = C_1 \times 0 - C_2 \times 1$ where the minus sign signifies the switch of the orientation of C_2.

For example *every contractible cycle* $C \subset X$ *is homologous to zero*, since the cone over C in $Y = X \times [0,1]$ (corresponding to a smooth generic homotopy) is a relative cycle. (There is a little problem here for $codim_X(C) = 1$, which will go away presently.)

Define $H_i(X)$ as the Abelian group with generators $[C]$ for all i-cycles C in X and with the relations $[C_1] - [C_2] = 0$ whenever $C_1 \sim C_2$. Similarly one defines $H_i(X;\mathbb{Q})$, for the field $\mathbb{Q}$ of rational numbers, by generating the vector space over $\mathbb{Q}$ by the cycles with this relation. It is not hard to see that $H_i(X;\mathbb{Q}) = H_i(X) \otimes \mathbb{Q}$ for all compact triangulated spaces X.

Next, define $H_i(X)$ of an arbitrary triangulated n-space X as $H_i(U_\varepsilon(X))$, where $U_\varepsilon(X) \subset \mathbb{R}^{n+N}$, $N >> n$, is a small *regular neighbourhood* of this X imbedded into $\mathbb{R}^{n+N}$ by a generic piecewise smooth map.

Regular Neighbourhoods. Recall (this is fairly obvious) that a small open ε-neighbourhhod $U = U_\varepsilon(X) \subset \mathbb{R}^{n+N}$ of a generically embedded $X \subset \mathbb{R}^{n+N}$ is *regular* in the following sense.

There is a continuous family of maps $R_t : U \to U$, $t \in [0,1]$, such that

• $R_{t=0} = id$(entity) on U;

• $R_t = id|X$ for all t;

• $R_{t_2}(U) \subset R_{t_1}(U)$ for all $0 \leq t_1 \leq t_2 \leq 1$;

• the map R_t is a diffeomorphism of U onto its image for every $t < 1$ and the family R_t is C^∞-continous in t for $t < 1$;

• $R_{t=1}(U) = X$.

Every continous map $f : X_1 \to X_2$, when extended to $U_\varepsilon(X_1) \to U_\varepsilon(X_2)$ and then approximated by a generic map, defines a homomorphism $H_i(X_1) \to H_i(X_2)$. In fact, if $N >> n$, then generic maps send cycles to cycles. (Clearly, $N \geq 2i+1$ is big enough for this, but $N \geq 2$ also suffices.)

This homomorphism *does not depend on the extension and approximation* and it is denoted f_{*i}.

Moreover, f_{*i} is invariant under homotopies represented by generic maps $F : X_1 \times [0,1] \to X_2$, since $F(C,0) \sim F(C,1)$, for $F(C,0) - F(C,1) = \partial(F(C \times [0,1])$.

It follows, by approximation of continous homotopies by smooth generic ones, that f_{*i} is, indeed, correctly defined and is invariant under all continuous homotopies.

A particular role is played by the homomorphism $f_{*n} : H_n(X_1) \to H_n(X_2)$ between closed oriented n-manifolds. If X_1 and X_2 are connected, then f_{*n} is determined by the integer $f_{*n}[X_1] \in \mathbb{Z} = H_n(X_2)$, called *the degree of* f (the definition of which needs only connectedness of X_2).

If f is a smooth map, then the f-pullback $\tilde{U} \subset X_1$ of some small open subset $U \subset X_2$ consists of finitely many connected components $\tilde{U}_i \subset \tilde{U}$, such that the map $f : \tilde{U}_i \to U$ is a diffeomorphism for all $\tilde{U}_i$. Thus, every $\tilde{U}_i$ carries two orientations: one induced from X_1 and the second from X_2 via f.

If the two orientations agree, we assign $+1$ to $\tilde{U}_i$ and -1 otherwise. Then the sum of these ± 1 equals the degree of f which must by obvious by now.

For example l sheeted *covering maps* have degree l.

The homotopy invariance of f_{*i} trivially implies the invariance of H_i under *homotopy equivalences*, and thus, under homeomorphisms, between spaces; other familiar properties of homology also easily follow from our definition, such as the bound on the number of generators of $H_i(X)$ by the number of the i-simplices of the triangulation.

(Recall that a homotopy equivalence between X_1 and X_2 is given by a pair of maps $f_{12} : X_1 \to X_2$ and $f_{21} : X_2 \to X_1$, such that both composed maps $f_{12} \circ f_{21} : X_1 \to X_1$ and $f_{21} \circ f_{12} : X_2 \to X_2$ are homotopic to the identity maps.)

Example. The spheres have $H_0(S^n) = H^n(S^n) = \mathbb{Z}$ (where non-vanishing of the fundamental class $[S^n] \in H^n(S^n)$ and of $m[S^n]$ for $m \neq 0$ will become clear presently) while $H_i(S^n) = 0$ for $0 < i < n$, since the complement to a point in S^n is homeomorphic to $\mathbb{R}^n$ and has zero homologies in positive dimensions.

There is a more subtle geometric property of regular neighbourhoods for $N >> n$ due to B. Mazur (1961) that we shall prove in section 7.

Every homotopy equivalence $U_\varepsilon(X_1) \to U_\varepsilon(X_2)$ *is homotopic to a diffeomorphism.*

If X is a *non-compact* manifold, one may drop "compact" in the definition of cycles. The resulting group is denoted $H_1(X, \partial_\infty)$. If X is compact with boundary, then this group of the interior of X is called the *relative homology group* $H_i(X, \partial(X))$. (The ordinary homology groups of this interior are canonically isomorphic to those of X.)

The intersection of cycles in general position defines a multiplicative structure on the homology of an n-manifold X where this intersection product of $[C_1] \in H_{n-i}(X)$ and $[C_2] \in H_{n-j}(X)$, is denoted

$$[C_1] \cdot [C_2] = [C_1] \cap [C_2] = [C_1 \cap C_2] \in H_{n-(i+j)}(X)$$

(where $[C] \cap [C]$ is defined by intersecting $C \subset X$ with its small generic perturbation $C' \subset X$).

It is easy to see that this product is invariant under oriented (i.e. of degrees $+1$) homotopy equivalences between *closed equidimensional* manifolds.

Also notice that the intersection of cycles of *odd* codimensions is *anti-commutative* and if one of the two has *even* codimension it is *commutative.*

Examples. (a) The intersection ring of the complex projective space $\mathbb{C}P^k$ is multiplicatively generated by the homology class of the hyperplane, $[\mathbb{C}P^{k-1}] \in H_{2k-2}(\mathbb{C}P^k)$, with the only relation $[\mathbb{C}P^{k-1}]^{k+1} = 0$ and where, obviously, $[\mathbb{C}P^{k-i}] \cdot [\mathbb{C}P^{k-j}] = [\mathbb{C}P^{k-(i+j)}]$.

The proof is straightforward by observing that $\mathbb{C}P^k \setminus \mathbb{C}P^{k-1}$ is homeomorphic to $\mathbb{R}^{2k}$.

(b) The intersection ring of the n-torus is isomorphic to the exterior algebra on n-generators, i.e. the only realatios between the multiplicative generators $h_i \in H_{n-1}(\mathbb{T}^n)$ are $h_i h_j = -h_j h_i$, where h_i are the homology classes of the n coordinate subtori $\mathbb{T}_i^{n-1} \subset \mathbb{T}^n$.

This follows from the Künneth fomula below, but can be also proved directly with a minor effort.

The intersection ring structure immensely enriches homology. Additively, $H_* = \oplus_i H_i$ is just a graded Abelian group – the most primitive algebraic object (if finitely generated) – fully characterized by simple numerical invariants: the rank and the orders of their cyclic factors.

But the ring structure, say on H_{n-2} of an n-manifold X, for $n = 2d$ defines a symmetric d-form, on $H_{n-2} = H_{n-2}(X)$ which is, when it is simplified by tensoring with $\mathbb{Q}$, is the same as a rational polynomial of degree d in r variables, $r = rank(H_{n-2})$. All number theory in the world can not classify these for $d \geq 3$ (to be certain, for $d \geq 4$).

One can also intersect non-compact cycles, where an intersection of a compact C_1 with a non-compact C_2 is compact; this defines the *intersection pairing*

$$H_{n-i}(X) \otimes H_{n-j}(X, \partial_\infty) \xrightarrow{\cap} H_{n-(i+j)}(X).$$

Finally notice that our 0 cycles C in X are finite sets of points $x \in X$ with the "orientation" signs ± 1 attached to each x in C, where the sum of these ± 1 is called the *index of* C. If X is *connected*, then $ind(C) = 0$ if and only if $[C] = 0$.

Doesn't it look non-sensical? You orient $\mathbb{R}^n$ for $n > 0$ by choosing a basis in it, where you can say when two bases are the same or different orientation-wise. But what is a basis in the 0-dimensional space? Yet, the 0-dim "orientation" is indispensable in the definition of homologous 0-cycles.

Also, the bona-fide concept of the *intersection index* of cycles C_1 and C_2 of complementary dimensions in general position in an *oriented* manifold X (the sum of ± 1, assigned to each intersection point $x \in C_1 \cap C_2$ with the sign depending on whether C_1 and C_2 give the original orientation to X at x or the opposite one) can be defined as the index of the zero cycle $C_1 \cap C_2$.

Euler Class. Let X be an oriented $4k$-manifold and $X \to B$ be a fibration with $\mathbb{R}^{2k}$-fibers. Then, clearly, $H_*(X) = H_*(B)$ and the self-intersection index of $h \in H_{2k}(X)$, regarded as a function on $H_{2k}(B)$ is called the *Euler class* $e(X)$ of the fibration.

In particular, if B is a closed connected oriented manifold, then $e([B])$ is called the the *Euler number* of $X \to B$ also denoted e.

More geometrically, one embeds $B \subset X$ as the zero section, i.e. by $b \mapsto 0_b \in \mathbb{R}_b^k$ and defines e as the self-intersection index of $B \subset X$.

Observe that since the intersection pairing is symmetric on H_{2k} the sign of the Euler number does not depend on the orientation of B, but does depend on the orientation of X.

Also notice that if X is embedded into a larger $4k$-manifold $X' \supset X$ then the self-intersection index of B in X' equals that in X.

If X equals the tangent bundle $T(B)$ then X is canonically oriented (even if B is non-orientable) and the Euler number is non-ambigusly defined.

Poincare Formula. *The Euler number e of the tangent bundle $T(B)$ of every closed oriented $2k$-manifold B satisfies*

$$e = \chi(B) = \sum_{i=0,1,\ldots k} rank(H_i(X;\mathbb{Q})).$$

It is hard the believe this may be true! The single cycle knows something about all the homology of B.

The simplest proof of this formula is, probably, via the Morse theory (known to Poincare) and it hardly can be called "trivial".

The Euler number can be defined for connected non-orientable B as follows. Take the canonical oriented double covering $\tilde{B} \to B$, where each point $\tilde{b} \in \tilde{B}$ over $b \in B$ is represented as b + an orientation of B near b. Let the bundle $\tilde{X} \to \tilde{B}$ be induced from X by the covering map $\tilde{B} \to B$, i.e. this $\tilde{X}$ is the obvious double covering of X corresponding to $\tilde{B} \to B$. Finally, set $e(X) = e(\tilde{X})/2$.

The Poincare formula for non-orientable $2k$-manifolds B follows from the orientable case by the *multiplicativity* of the Euler characteristic χ which is valid for all compact triangulated spaces B,

an l-sheeted covering $\tilde{B} \to B$ has $\chi(\tilde{B}) = l \cdot \chi(B)$.

If the homology is defined via the triangulation, then $\chi(B)$ equals the alternating sum $\sum_i (-1)^i N(\Delta^i)$ of the numbers of i-simplices by straightforward linear algebra and the multiplicativity follows. But it is not so easy with our geometric cycles. (If B is a closed manifold, this also follows from the Poincare formula and the obvious multiplicativity of the Euler number for covering maps.)

Künneth Theorem. *The rational homology of the Cartesian product of two spaces equals the graded tensor product of the homologies of the factors. In fact, the natural homomorphisms*

$$\bigoplus_{i+j=k} H_i(X_1;\mathbb{Q}) \otimes H_j(X_2;\mathbb{Q}) \to H_k(X_1 \times X_2;\mathbb{Q}),\ \ k = 0,1,2,...$$

is an isomorphism. Moreover, if X_1 and X_2 are closed oriented manifolds, this homomorphism is compatible (if you say it right) with the intersection product.

This looks obvious, but the proof is unpleasant in our setting.

Poincare $\mathbb{Q}$-Duality. *Let X be a connected oriented manifold. Then the intersection pairing*

$$H_i(X) \oplus H_{n-i}(X,\partial_\infty) \xrightarrow{\cap} H_0(X) = \mathbb{Z}$$

is faithful: a multiple of a compact i-cycle C is homologous to zero if and only if its intersection index with every non-compact $(n-i)$-cycle in general position equals zero.

Furthermore, if X equals the interior of a compact manifolds with a boundary, then a multiple of a non-compact cycle is homologous to zero if and only *if its intersection index with every compact generic cycle of the complementary dimension equals zero.*

In other words, the intersection index establishes a linear duality between the $\mathbb{Q}$*-vector spaces* $H_i(X)\otimes\mathbb{Q} = H_i(X;\mathbb{Q})$ *and* $H_{n-i}(X,\partial_\infty)\otimes\mathbb{Q} = H_{n-i}(X,\partial_\infty;\mathbb{Q})$.

The non-obvious and hard for us to prove part of the duality is "if", but the obvious "only if" is also powerful, as it allows one to give a *lower* bound on the homology by producing sufficiently many *non-trivially* intersecting cycles of complementary dimensions.

For instance, one immediately sees that $H_n(X^n) \neq 0$ for all closed orienatable n-manifolds X, since the intersection of the fundamental n-cycle $[X] \in H_n(X)$ with a one point 0-cycle c_0 equals $c_0 \neq 0$. It follows that X is *non-contractible*, which is virtually invisible even for $X = S^n$, $n \geq 3$, from inside the category of *all* continuous (rather then smooth generic or piece-wise linear) maps. (The existence of the covering map $\mathbb{R} \to S^1$ implies that S^1 is non-contractible and then one sees that S^2 is also non-contractible with the Hopf fibration $S^3 \to S^2$, where smooth and/or piece-wise linear maps need not be used.)

The $\mathbb{Q}$-duality does not tell you the whole story. For example, the following simple property of *closed n-manifolds X* depends on the full homological duality:

Connectedness/Contractibiliy. *If X is a closed k-connected n-manifold, i.e.* $\pi_i(X) = 0$ *for* $i = 1,...,k$, *then the complement to a point,* $X \setminus \{x_0\}$, *is* $(n-k-1)$*-contractible, i.e. there is a homotopy* f_t *of the identity map* $X\setminus\{x_0\} \to X\setminus\{x_0\}$ *with* $P = f_1(X\setminus\{x_0\})$ *being a smooth triangulated subspace* $P \subset X\setminus\{x_0\}$ *with* $codim(P) \geq k+1$.

For example, if $\pi_i(X) = 0$ *for* $1 \leq i \leq n/2$, *then* X *is homotopy equivalent to* S^n.

Thom Isomorphism. Let $p: V \to X$ be a *fiber-wise oriented* smooth (which is unnecessary) $\mathbb{R}^N$-bundle over X, where $X \subset V$ is embedded as the zero section and let $V_\bullet$ be Thom space of V. Then there are two natural homology homomorphisms.

Intersection $\cap : H_{i+N}(V_\bullet) \to H_i(X)$. This is defined by intersecting generic $(i+N)$-cycles in $V_\bullet$ with X.

Thom Suspension $S_\bullet : H_i(X) \to H_i(V_\bullet)$, where every cycle $C \subset X$ goes to the Thom space of the restriction of V to C, i.e. $C \mapsto (p^{-1}(C))_\bullet \subset V_\bullet$.

These $\cap$ and $S_\bullet$ are mutually reciprocal. Indeed $(\cap \circ S_\bullet)(C) = C$ for all $C \subset X$ and also $(S_\bullet \circ \cap)(C') \sim C'$ for all cycles C' in $V_\bullet$ where the homology is established by the fiberwise radial homotopy of C' in $V_\bullet \supset V$, which fixes $\bullet$ and move each $v \in V$ by $v \mapsto tv$. Clearly, $tC' \to (S_\bullet \circ \cap)(C')$ as $t \to \infty$ for all generic cycles C' in $V_\bullet$.

Thus we arrive at the Thom isomorphism

$$H_i(X) \leftrightarrow H_{i+N}(V_\bullet).$$

Similarly we see that

The Thom space of every $\mathbb{R}^N$*-bundle* $V \to X$ *is* $(N-1)$*-connected, i.e.* $\pi_j(V_\bullet) = 0$ *for* $j = 1,2,...N-1$.

Indeed, a generic j-sphere $S^j \to V_\bullet$ with $j < N$ does not intersect $X \subset V$, where X is embedded into V by the zero section. Therefore, this sphere radially (in the fibers of V) contracts to $\bullet \in V_\bullet$.

Signature. The intersection of (compact) k-cycles in an oriented, possibly disconnected, $2k$-manifold X defines *a bilinear* form on the homology $H_k(X)$. If k is odd, this form is antisymmetric and if k is even it is symmetric.

The signature of the latter, i.e. the number of positive minus the number of negative squares in the diagonalized form is called $sig(X)$, which is well defined if $H_k(X)$ has finite rank, e.g. if X is compact.

Geometrically, a diaganolization of the inetersection form is achived with a *maximal* set of *mutually disjoint* k-cycles in X where each of them has a *non-zero* (positive or negative) self-intersection index.

Examples. (a) $S^{2k} \times S^{2k}$ has zero signature, since the $2k$-homology is generated by the classes of the two coordinate spheres $[s_1 \times S^{2k}]$ and $[S^{2k} \times s_2]$, which both have zero self-intersections.

(b) The complex projective space $\mathbb{C}P^{2m}$ has signature one, since its miiddle homology is generated by the class of the comlex projective subspace $\mathbb{C}P^m \subset \mathbb{C}P^{2m}$ with the self-intersection = 1.

(c) The tangent bundle $T(S^{2k})$ has signature 1, since $H_k(T(S^{2k}))$ is generated by $[S^{2k}]$ with the self-intersection equal the Euler characteristic $\chi(S^{2k}) = 2$.

It is obvious that $sig(mX) = m \cdot sig(X)$, where mX denotes the disjoint union of k copies of X, and that $sig(-X) = -sig(X)$, where "$-$" signifies reversion of orientation. Furthermore

The oriented boundary X of every compact oriented $(4k+1)$-manifold Y has zero signature.(Rokhlin 1952).

(An orientation of a Y induces an orientation of its boundary $\partial(Y)$, if we agree on a choice of directions of the normals to the boundary, either we agree on taking *all* looking inward or *all* outward. There is no apparent rational for preferring one of the two, but we stick to "inward" once and for ever.

Oriented boundaries of non-orienatble manifolds may have non-zero signatures, For example the double covering $\tilde{X} \to X$ with $sig(\tilde{X}) = 2sig(X)$ non-orientably bounds the corresponding 1-ball bundle Y over X.)

Sketch of the Proof. It is obvious that the intersection form vanishes on the kernel $ker_k \subset H_k(X)$ of the inclusion homomorphism $H_k(X) \to H_k(Y)$: if k-cycles C_i, $i = 1, 2$, bound relative $(k+1)$-cycles D_i in Y, then the (zero-dimensional) intersection C_1 with C_2 bounds a relative 1-cycle in Y which makes the index of the intersection zero.

On the other hand, the obvious identity

$$[C \cap D]_Y = [C \cap \partial D]_X$$

and the Poincare duality show that that the spaces $ker_k \subset H_k(X)$ and $H_k(X)/ker_k$ have equal ranks over $\mathbb{Q}$. QED.

It is also easy to see with the Künneth formula that

$$sig(X_1 \times X_2) = sig(X_1) \cdot sig(X_2).$$

Amazingly, the *multiplicativity of the signature* of *closed* manifolds under covering maps can not be seen with comparable clarity.

Multiplicativity Formula *if $\tilde{X} \to X$ is a l-sheeted covering map, then*

$$sign(\tilde{X}) = l \cdot sign(X).$$

We prove this in the next section with the use of the Serre finiteness theorem.

Our presentation of homology is similar to the first approach by Poincare (this is pursued further in [26]), where the essential stumbling block is proving the Poincare duality.

(Thinking in the language of generic cycles is well suited for observing and proving the multitude of obvious little things you come across every moment in topology. Yet, you do not expect to arrive at anything like Serre's finiteness theorem without the computational power of the fully linearized homology theory.)

If you think about it, the Poincare duality is quite amazing: you can say what happens to i-cycles for $i = (0.9)10^{100}$ in a 10^{100}-dimensional manifold X (something like the "classical phase space of the universe") by looking at the dimensions 10^{99} and $10^{99}+1$.

It is unclear (at least to me) what should be a comprehensive formulation and/or a "natural" proof of the Poincare duality which would make transparent, for example, the multiplicativity of the signature and the topological nature of rational Pontryagin classes (which can be derived from the Poincare duality, albeit in a circumvent way [42]) and which would apply to "cycles" of dimensions βN where $N = \infty$ and $0 \le \beta \le 1$ in spaces like these we shall meet in section 11.

5 The Signature and Bordisms.

Let us proof the multiplicativety of the signature by constructing a compact oriented manifold Y with a boundary, such that the *oriented* boundary $\partial(Y)$ equals $m\tilde{X} - mlX$ for some integer $m \neq 0$.

Embed X into $\mathbb{R}^{n+N}$, $N >> n = 2k = dim(X)$ let $\tilde{X} \subset \mathbb{R}^{n+N}$ be an embedding obtained by a small generic perturbation of the covering map $\tilde{X} \to X \subset \mathbb{R}^{n+N}$ and $\tilde{X}' \subset \mathbb{R}^{n+N}$ be the union of l generically perturbed copies of X.

Let $\tilde{A}_\bullet$ and $\tilde{A}'_\bullet$ be the Atiyah-Thom maps from $\mathbb{R}^{n+N}_\bullet$ to the Thom spaces $\tilde{U}_\bullet$ and $U'_\bullet$ of the normal bundles $\tilde{U} \to \tilde{X}$ and $\tilde{U}' \to \tilde{X}'$.

Let $\tilde{P} : \tilde{X} \to X$ and $\tilde{P}' : \tilde{X}' \to X$ be the normal projections. These projections, obviously, induce the normal bundles $\tilde{U}$ and $\tilde{U}'$ of $\tilde{X}$ and $\tilde{X}'$ from the normal bundle $U^\perp \to X$. Let

$$\tilde{P} : \tilde{U}_\bullet \to U^\perp_\bullet \text{ and } \tilde{P}' : \tilde{U}'_\bullet \to U^\perp_\bullet$$

be the corresponding maps between the Thom spaces and let us look at the two maps of the sphere $S^{n+N} = \mathbb{R}^{N+n}_\bullet$ to the Thom space $U^\perp_\bullet$,

$$\tilde{B}_\bullet = \tilde{P} \circ \tilde{A}_\bullet : S^{n+N} \to U^\perp_\bullet, \text{ and } \tilde{B}'_\bullet = \tilde{P}' \circ \tilde{A}'_\bullet : S^{n+N} \to U^\perp_\bullet.$$

Clearly

$[\tilde{\bullet}\tilde{\bullet}']$ $$(\tilde{B}_\bullet)^{-1}(X) = \tilde{X} \text{ and } (\tilde{B}'_\bullet)^{-1}(X) = \tilde{X}'.$$

On the other hand,

$\tilde{B}_\bullet$ sends $[S^{n+N}]$ to the $S_{\bullet n}$-image of $[X]_\sim = \tilde{P}_{*n}[\tilde{X}] \in H_n(X)$

and

$\tilde{B}'_\bullet$ sends $[S^{n+N}]$ to the $S_{\bullet n}$-image of $[X]'_\sim = \tilde{P}'_{*n}[\tilde{X}'] \in H_n(X)$, where $[S^{n+N}] \in H_{n+N}(S^{n+N})$ is the fundamental class of the sphere, where

$$S_{\bullet n} : H_n(X) \to H_{n+N}(U^{\perp}_{\bullet})$$

is the Thom suspension homomorphism, while

$$\tilde{P}_{*n} \text{ and } \tilde{P}'_{*n}$$

are the homology homomorphisms induced by the projections.

Since the projections $\tilde{P} : \tilde{X} \to X$ and $\tilde{P}' : \tilde{X}' \to X$ have equal degrees $(= l)$,

$$[X]'_{\sim} = [X]_{\sim} \in H_n(X); \text{ hence, } \tilde{B}'_{\bullet}[S^{n+N}] = \tilde{B}_{\bullet}[S^{n+N}] \in H_{n+N}(U^{\perp}_{\bullet}),$$

and since $\pi_i(U^{\perp}_{\bullet}) = 0,\ i = 1, ...N-1$,
some non-zero m-multiples of the maps

$$\tilde{B}_{\bullet}, \tilde{B}'_{\bullet} : S^{n+N} \to U^{\perp}_{\bullet}$$

can be joined by a (smooth generic) homotopy $F : S^{n+N} \times [0,1] \to U^{\perp}_{\bullet}$ by Serre's theorem.

Then, because of $[\tilde{\bullet}\tilde{\bullet}']$, the pullback $F^{-1}(X) \subset S^{n+N} \times [0,1]$ *establishes a bordism between* $m\tilde{X} \subset S^{n+N} \times 0$ *and* $m\tilde{X}' = mlX \subset S^{n+N} \times 1$. This implies that

$m \cdot sig(\tilde{X}) = ml \cdot sig(X)$ and since $m \neq 0$ we get $sig(\tilde{X}) = l \cdot sig(X)$. QED.

Bordisms and the Rokhlin-Thom-Hirzebruch Formula. Let us modify our definition of homology of a manifold X by allowing only *non-singular* i-cycles in X, i.e. smooth closed oriented i-submanifolds in X and denote the resulting Abelian group by $\mathcal{B}^o_i(X)$.

If $2i \geq n = dim(X)$ one has a (minor) problem with taking sums of non-singular cycles, since generic i-submanifolds may intersect and their union is unavoidably singular. We assume below that $i < n/2$; otherwise, we replace X by $X \times \mathbb{R}^N$ for $N >> n$, where, observe, $\mathcal{B}^o_i(X \times \mathbb{R}^N)$ does not depend on N for $N >> i$.

Unlike homology, the *bordism groups* $\mathcal{B}^o_i(X)$ may be non-trivial even for a contractible space X, e.g. for $X = \mathbb{R}^{n+N}$. (Every cycle in $\mathbb{R}^n$ equals the boundary of any cone over it but this does not work with manifolds due to the *singularity* at the apex of the cone which is not allowed by the definition of a bordism.) In fact,

if $N >> n$, then the bordism group $\mathcal{B}^o_n = \mathcal{B}^o_n(\mathbb{R}^{n+N})$ is canonically isomorphic to the homotopy group $\pi_{n+N}(V_{\bullet})$, where $V_{\bullet}$ is the Thom space of the tautological oriented $\mathbb{R}^N$-bundle V over the Grassmann manifold $V = Gr^{or}_N(\mathbb{R}^{n+N+1})$ (Thom, 1954).

Proof. Let $X_0 = Gr^{or}_N(\mathbb{R}^{n+N})$ be the Grassmann manifold of *oriented* N-planes and $V \to X_0$ the tautological oriented $\mathbb{R}^N$ bundle over this X_0.

(The space $Gr^{or}_N(\mathbb{R}^{n+N})$ equals the double cover of the space $Gr_N(\mathbb{R}^{n+N})$ of non-oriented N-planes. For example, $Gr^{or}_1(\mathbb{R}^{n+1})$ equals the sphere S^n, while $Gr_1(\mathbb{R}^{n+1})$ is the projective space, that is S^n divided by the $\pm$-nvolution.)

Let $U^{\perp} \to X$ be the oriented normal bundle of X with the orientation induced by those of X and of $\mathbb{R}^N \supset X$ and let $G : X \to X_0$ be the oriented Gauss map which assigns to each $x \in X$ the oriented N-plane $G(x) \in X_0$ parallel to the orineted normal space to X at x.

Since G induces $U^{\perp}$ from V, it defines the Thom map $S^{n+N} = \mathbb{R}^{n+N}_{\bullet} \to V_{\bullet}$ and every bordism $Y \subset S^{n+N} \times [0,1]$ delivers a homotopy $S^{n+N} \times [0,1] \to V_{\bullet}$ between the Thom maps at the two ends $Y \cap S^{n+N} \times 0$ and $Y \cap S^{n+N} \times 1$.

This define a *homomorphism*

$$\tau_{b\pi} : \mathcal{B}^o_n \to \pi_{n+N}(V_{\bullet})$$

since the additive structure in $\mathcal{B}^o_n(\mathbb{R}^{i+N})$ agrees with that in $\pi_{i+N}(V^o_{\bullet})$.

Also note that one needs the extra 1 in $\mathbb{R}^{n+N+1}$, since bordisms Y between manifolds in $\mathbb{R}^{n+N}$ lie in $\mathbb{R}^{n+N+1}$, or, equivalently, in $S^{n+N+1} \times [0,1]$.

On the other hand, the generic pullback construction

$$f \mapsto f^{-1}(X_0) \subset \mathbb{R}^{n+N} \supset \mathbb{R}^{n+N}_{\bullet} = S^{n+N}$$

defines a homomorphism $\tau_{\pi b} : [f] \to [f^{-1}(X_0)]$ from $\pi_{n+N}(V_{\bullet})$ to $\mathcal{B}^o_n$, where, clearly $\tau_{\pi b} \circ \tau_{b\pi}$ and $\tau_{b\pi} \circ \tau_{\pi b}$ are the identity homomorphisms. QED.

Now Serre's $\mathbb{Q}$-sphericity theorem implies the following

Thom Theorem. *The (Abelian) group $\mathcal{B}^o_i$ is finitely generated;*

$\mathcal{B}^o_n \otimes \mathbb{Q}$ is isomorphic to the rational homology group $H_i(X_0;\mathbb{Q}) = H_i(X_0) \otimes \mathbb{Q}$ for $X_0 = Gr^{or}_N(\mathbb{R}^{i+N+1})$.

Indeed, $\pi_i(V^{\bullet}) = 0$ for $N >> n$, hence, by Serre,

$$\pi_{n+N}(V_{\bullet}) \otimes \mathbb{Q} = H_{n+N}(V_{\bullet}) \otimes \mathbb{Q},$$

while

$$H_{n+N}(V_{\bullet}) \otimes \mathbb{Q} = H_n(X_0) \otimes \mathbb{Q}$$

by the Thom isomorphism.

In order to apply this, one has to compute the homology $H_n(Gr^{or}_N(\mathbb{R}^{N+n+j}));\mathbb{Q})$, which, as it is clear from the above, is independent of $N \geq 2n+2$ and of $j > 1$; thus, we pass to

$$Gr^{or} =_{def} \bigcup_{j,N\to\infty} Gr^{or}_N(\mathbb{R}^{N+j}).$$

Let us state the answer in the language of *cohomology*, with the advantage of the multiplicative structure.

(Every cohomology class $c \in H^i(X;\mathbb{Q})$ defines a $\mathbb{Q}$-linear map $H_i(X;\mathbb{Q}) \to \mathbb{Q}$, denoted $h \mapsto c(h)$ which establishes an *isomorphism* between $H^i(X;\mathbb{Q})$ and the $\mathbb{F}$-linear dual of $H_i(X;\mathbb{Q})$.

If X is a closed oriented n-manifold, then the Poincare duality delivers an isomorphism $H_i(X;\mathbb{Q}) \leftrightarrow H^{n-i}(X;\mathbb{Q})$ where the cohomology product corresponds to the intersection product on homology.)

The cohomology ring $H^*(Gr^{or}) \otimes \mathbb{Q}$ is the polynomial ring in some distinguished integer classes, called Pontryagin classes $p_i \in H^{4i}(Gr^{or};\mathbb{Z})$, $i = 1,2,3,...$ [32].

(It would be awkward to express this in the homology language when $N = dim(X) \to \infty$, although the cohomology ring $H^*(X)$ is canonically isomorphic to $H_{N-*}(X)$ by the Poincare duality.)

If X is a smooth oriented n-manifold, its Pontryagin classes $p_i(X) \in H^{4i}(X;\mathbb{Z})$ are defined as the classes induced from p_i by the normal Gauss map $G \to Gr^{or}_N(\mathbb{R}^{N+n}) \subset Gr^{or}$ for an embedding $X \to \mathbb{R}^{n+N}$, $N >> n$.

If Q is a unitary (i.e. a product of powers) monomial in p_i of graded degree $n = 4k$, then the values $Q(p_i)[X]$ is called *the (Pontryagin) Q-number.* Equivalently, these are the values of $Q(p_i) \in H^{4i}(Gr^{or};\mathbb{Z})$ on the image of (the fundamental class) of X in Gr^{or} under the Gauss map.

The Thom theorem now can be reformulated as follows.

Two closed oriented n-manifolds are $\mathbb{Q}$-bordant if and only if they have equal Q-numbers for all monomials Q. In particular $\mathcal{B}_n^o \otimes \mathbb{Q} = 0$, unless n is divisible by 4.

Furthermore, the rank of $\mathcal{B}_n^o \otimes \mathbb{Q}$ equals the nunber of Q-monomials of graded degree n, that are $\prod_i p_i^{n_i}$ with $\sum_i 4n_i = n$.

For example, if $n = 4$, then there is a single such monomial, p_1; if n=8, there two of them: p_2 and p_1^2; if $n = 12$ there three monomials: p_3, p_1p_2 and p_1^3; if $n = 16$ there are five of them, etc.

Thom also observes that, since the top Pontryagin classes p_k of the complex projective spaces do not vanish, $p_k(\mathbb{C}P^{2k}) \neq 0$, (see [32]) the products of these spaces constitute a basis in $\mathcal{B}_n^o \otimes \mathbb{Q}$.

Finally, notice that the bordism groups together make a commuative ring under the Cartesian product of manifolds, denoted $\mathcal{B}_*^o$, and the Thom theorem says that

$\mathcal{B}_*^o \otimes \mathbb{Q}$ *is the polynomial ring over* $\mathbb{Q}$ *in the variables* $[\mathbb{C}P^{2k}]$, $k = 0, 2, 4, ...$

Sice the signature, is additive and also multiplicative under this product it defines a homomorphism $[sig] : \mathcal{B}_*^o \to \mathbb{Z}$ which can be expressed in each degree $4k$ by means of a universal polynomial in the Pontryagin classes, denoted $L_k(p_i)$, by

$$sig(X) = L_k(p_i)[X] \text{ for all closed oriented } 4k\text{-manifolds } X.$$

For example,

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3).$$

Accordingly,

$$sig(X^4) = \frac{1}{3}p_1[X^4], \qquad \text{(Rokhlin 1952)}$$

$$sig(X^8) = \frac{1}{45}(7p_2(X^8) - p_1^2(X^8))[X^8], \qquad \text{(Thom 1954)}$$

and where a concise general formula (see blow) was derived by Hirzebruch who evaluated the coefficients of L_k using the *known* values of p_i and sig for the products of the complex projective spaces and by substituting these products $X = \times_j \mathbb{C}P^{2k_j}$ with $\sum_j 4k_j = n = 4k$, for $X = X^n$ into the formula $sig(X) = L_k[X]$.

Hirzebruch Signature Theorem. Let

$$R(z) = 1 + z/3 + z/45 + ... = \sum_{l \geq 0} \frac{2^{2l}B_{2l}z^l}{(2l)!} = \frac{\sqrt{z}}{\tanh(\sqrt{z})},$$

where B_{2l} are the *Bernully numbers*:

$$B_0 = 1,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ B_8 = -1/30,\ B_{10} = 5/66, ...$$

Write

$$R(z_1)\cdot\ldots\cdot R(z_k) = 1 + P_1(z_j) + \ldots + P_k(z_j) + \ldots$$

where P_j are homogeneous symmetric polynomials of degree j in $z_1, \ldots, z_k$ and rewrite

$$P_k(z_j) = L_k(p_i)$$

where $p_i = p_i(z_1, \ldots, z_k)$ are the elementary symmetric functions in z_j of degree i.

The Hirzebruch theorem say that this L_k is exactly the polynomial which makes the equality $L_k(p_i)[X] = sig(X)$.

A significant aspect of this formula is that the Pontryagin numbers and the signature are *integers* while the Hirzebruch polynomilas L_k have non-trivial denominators. This yields certain *universal divisibility* properties of the Pontryagin numbers (and sometimes of the signatures) for closed smooth orientable $4k$-manifolds.

But despite a significant "integer load" carried by the signature formula, it depends only on the *rational* bordism groups $\mathcal{B}_n^o \otimes \mathbb{Q}$. This point of elementary linear algebra was overlooked by Thom (isn't it incredible?) who derived the signature formula for 8-manifolds from his special and more difficult computation of the true bordism group $\mathcal{B}_8^o$. However, the shape given by Hirzebruch to this formula is something more than just linear algebra.

Is there an implementation of the analysis/arithmetic encoded in the Hirzebruch formula by some infinite dimensional manifolds?

Geometric Questions about Bordisms. Let X be a closed oriented Riemannian n-manifold with locally bounded geometry, which means that every R-ball in X admits a λ-*bi-Lipshitz homeomorphism* onto the Euclidean R-ball.

Suppose X is bordant to zero and consider all compact Riemannian $(n+1)$-manifolds Y extending $X = \partial(Y)$ with its Riemannian tensor and such that the local geometries of Y are bounded by some *constants* $R' << R$ and $l' >> \lambda$ with the obvious precaution near the boundary.

One can show that the infimum of the volumes of these Y is bounded by

$$\inf_Y Vol(Y) \le F(Vol(X)),$$

with the power exponent bound on the function $F = F(V)$. (F also depends on R, λ, R', λ', but this seems non-essential for $R' << R, \lambda' >> \lambda$.)

What is the true asymptotic behaviour of $F(V)$ for $V \to \infty$? It may be linear for all we know.

Is there a better setting of this question with some curvature integrals and/or spectral invariants rather than volumes?

The real cohomology of the Grassmann manifoIds can be analytically represented by invariant differential forms. Is there a compatible analytic/geometric representation of $\mathcal{B}_n^o \otimes \mathbb{R}$? (One may think of a class of n-foliations, for instance, or something more sophisticated than that.)

Combinatorial Pontryagin Classes. Let $L_j(X) = L_j(p_i(X)) \in H^{4j}(X;\mathbb{Q})$ and take a generic smooth map $f : X \to S^{n-4j}$, $n = dim(X)$. Observe that

$Z = f^{-1}(s_0) \subset X$ is a $4j$-dimensional submanifold with *trivial* normal bundle; hence

$$L_j(Z) = L_j(X)|Z \text{ and } sig(Z) = L_j(X)[Z],$$

where $[Z] \in H_{4j}(X)$ is the homology class of Z, where the orientation of Z (needed to define $[Z]$) comes with the orientation in X and an orientation in its (trivial!) normal bundle of Z in X.

Moreover, the class $L_j(X)$ for $4j < n/2$ is uniquely determined by the signatures of all $4j$-submanifolds $Z \subset X$ with trivial normal bundles, since the homology classes of these Z span $H_{4j}(X;\mathbb{Q})$ according to Serre's $[\to S^{n-4l}]$-theorem for $4j < n/2$.

This allows one, following Rokhlin (1957) and Thom (1958), to express the rational Pontryagin classes by these signatures as well: this is possible, since the coefficient at p_j in $L_j = L_j(p_1, ..., p_j)$ is *non-zero* (this can be seen with Hirzebruch's description of his polynomials) and the rational (since L_j are rational rather then integer polynomials) Pontryagin classes p_i of an X can be expressed in terms of $L_j(X) \in H^{4j}(X;\mathbb{Q})$ for $j = 1, ..., i$. (If $4j \geq n/2$ one does this for $X' = X \times S^n$).

Thom and independetly Rochlin-Schwartz observe that this definition of L_j, and hence of rational p_i, applies to *triangulated (not necessarily smooth) manifolds* X, since the pullback of a point $s \in S^{n-4j}$ under a simplicial map is a triangulated topological manifold provided s lies in the *interiour* of an $(n-4)$-simplex in S^{n-4j}. These manifolds satisfy the Poincare duality; hence, the signatures of $4j$-manifolds are invariant under bordisms by $(4j+1)$-manifolds because the Poincare duality is all what is needed for the proof of the bordism invariance of the signature.

In particular,

rational Pontryagin classes of smooth manifolds are invariant under piecewise smooth homeomorphisms between such manifolds.

In fact, the Thom-Rokhlin-Schwartz argument applies to all *rational homology* or $\mathbb{Q}$-*manifolds*, that are compact triangulated n-spaces where the link $L^{n-i-1} \subset X$ of every i-simplex in X has the same rational homology as the sphere S^{n-i-1}. These X satisfy the Poincare duality. In fact the standard combinatorial argument for proving duality does not differentiate between true manifolds and $\mathbb{Q}$-manifolds.

The class of $\mathbb{Q}$-manifolds class is by far wider then that of smooth (or combinatorial) manifolds due to a possibility of having enormous (and beautiful) fundamental groups $\pi_1(L^{n-i-1})$.

Yet, the naturally defined bordism ring of $\mathbb{Q}$-manifolds is only marginally different from $\mathcal{B}^o_*$: the natural homomorphism $\mathcal{B}^o_* \to \mathbb{Q}\mathcal{B}^o_*$ has finite kernel and cokernel in each degree. (This can be easily derived from Serre's theorems.)

Is there a finer, yet workable, notion of bordisms between $\mathbb{Q}$-manifold that would (partially) keep track of $\pi_1(L^{n-i-1})$?

The Thom-Rokhlin-Schwartz combinatorial pull-back argument breaks down in the topological category since there is no good notion of a generic *continuous* map. Yet, S. Novikov (1966) proved that the L-classes and, hence, the rational Pontryagin classes are invariant under *arbitrary homeomorphisms* (see section 10).

6 Exotic Spheres.

In 1956, to everybody's amazement, Milnor found smooth manifolds Σ^7 which were *not diffeomorphic to* S^N; yet, each of them was decomposable into the union of two 7-balls $B_1^7, B_2^7 \subset \Sigma^7$ intersecting over there boundary $S^6 = \partial(B_1^7) = \partial(B_2^7) \subset \Sigma^7$ like the ordinary sphere.

In fact, this decomposition does imply that Σ^7 is "ordinary" in the *topological* category: such a Σ^7 is (obviously) homeomorhic to S^7.

The subtlety resides in the "equality" $\partial(B_1^7) = \partial(B_2^7)$; this identification of the boundaries is far from being the identity map from the point of view of either of the two balls – it does not come from any diffeomorphisms $B_1^7 \leftrightarrow B_2^7$.

This equality can be regarded as a self-diffeomorphism f of the round sphere S^6 – the boundary of standard ball B^7 but this f *does not extend* to a diffeomorphism of B^7; otherwise, Σ^7 whoud be diffeomorphic to S^7. (Yet f radially extends to a *piecewise smooth homeomorphism* of B^7 which yields a piecewise smooth homeomorphism between Σ^7 and S^7.)

It follows, that such an f *can not* be included into a family of diffeomorphisms bringing it to an isometric transformations of S^6. Thus, any geometric "energy minimazing" flow on the diffeomorphism group $diff(S^6)$ either gets stuck or develops singularities. (It seems little, if anything at all, is known about such flows and their singularities.)

Milnor's spheres Σ^7 are rather innocuous spaces – the boundaries of (the total spaces of) certain 4-ball bundles Θ over S^4, i.e. these Σ^7 are certain S^3-bundles over S^4.

The 4-ball bundles over S^4 are easy to describe: each is determined by two numbers: the *Euler number* e, that is the self-intersection index of $S^4 \subset \Theta$, which assumes all integer values and the Pontryagin number p_1 (i.e. the value of the Pontryagin class $p_1 \in H^4(S^4)$ on $[S^4] \in H_4(S^4)$) which may be an *arbitrary even integer.*

Obviously, all Σ^7 are 2-connected, but $H_3(\Sigma^7)$ may be non-zero (e.g. for the trivial bundle). It is not hard to show that Σ^7 has the same homology as S^7, hence, homotopy equivalent to S^7, if and only if $e = \pm 1$ which means that the selfintersection index of the zero section sphere $S^4 \subset \Theta$ equals ± 1; we stick to $e = 1$ for our candidates for Σ^7.

The basic example of Σ^7 with $e = \pm 1$ (the sign depends on the choice of the orientation in Θ) is the ordinary 7-sphere which comes with the *Hopf fibration* $S^7 \to S^4$, where this S^7 is positioned as the unit sphere in the quaternion plane $\mathbb{H}^2 = \mathbb{R}^8$, where it is freely acted upon by the group $G = S^3$ of the unit quaternions and where S^7/G equals the sphere S^4 representing the quaternion projective line.

(Alternatively, our bundles are classified by the homotopy group $\pi_3(SO(4))$, which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, since the double cover of the special orthogonal group $SO(4)$ equals $S^3 \times S^3$.)

If Σ^7 were diffeomorphic to S^7 one could attach the 8-ball to Θ along the boundary and obtain a smooth closed 8-manifold, say Θ_+ .

Milnor observes that the signature of Θ_+ equals ± 1, since the homology of Θ_+ is represented by a single cycle – the sphere $S^4 \subset \Theta \subset \Theta_+$ the selfintersection number of which equals the Euler number.

Then Milnor invokes the *Thom signature theorem*

$$45sig(X) + p_1^2[X] = 7p_2[X]$$

and concludes that the number $45 + p_1^2$ must be divisible by 7 and, therefore, the boundaries Σ^7 of those Θ which fail this condition, say for $p_1 = 4$, must be exotic.

Finally, using quaternions, Milnor explicitly constructs a Morse function $\Sigma^7 \to \mathbb{R}$ with only two critical points – maximum and minimum on each Σ^7 with $e = 1$; this yields the two ball decomposition.

(Mlnor's topological arguments, which he presents with a meticulous care, became a common knowledge and can be now found in any textbook; his lemmas look apparent to a to-day topology student. The hardest for the modern reader is the final Milnor's lemma claiming that his function $\Sigma^7 \to \mathbb{R}$ is Morse with two critical ponts. Milnor is laconic at this point: "It is easy to verify" is all what he says.)

The 8-manifolds Θ_+ with Milnor's exotic Σ^7 can be triangulated with a single non-smooth point in such a triangulation. Yet, they admit no smooth structures compatible with these triangulations (as defined below), since their combinatorial Pontryagin numbers (defined by Rochlin-Schwartz and Thom) fail the divisibility condition issuing from the Thom formula $sig(X^8) = L_2[X^8]$; in fact, they are not *combinatorially bordant* to smooth manifolds.

Moreover, these Θ_+ are not even *topologically* bordant, and therefore, they are non-homeomorphic to smooth manifolds by (slightly refined) Novikov's topological Pontraygin classes theorem.

Recall that a triangulation S of a smooth n-manifold X is *smooth* and/or the smooth structure of X is *compatible* with S, if S is locally diffeomorphic to a triangulation of $\mathbb{R}^n$ into affine simplices, where "local" means "in a neighbourhood of each closed simplex of S".

Every smooth manifold X can be smoothly triangulated as follows. Smoothly embed $X \subset \mathbb{R}^N$, let T be a standard affine simplicial subdivision of the standard partition of $\mathbb{R}^N$ into ε-cubes and let T' be an affine triangulation of $\mathbb{R}^N$ obtained by *generic* δ-small moves of the vertices of T.

If $\varepsilon \leq \varepsilon_0 = \varepsilon_0(X) > 0$ and if $\delta \lesssim N^{-N}\varepsilon$, then the intersection of X with T' in the r-ball $B_x(r) \subset \mathbb{R}^N$, say for $r = (10\sqrt{n})\varepsilon$, around each point $x \in X$ is diffeomorphic to the intersections of the tangent n-plane $T_x(X) \subset \mathbb{R}^N$ with T' by the implicit function theorem.

Then the intersection of X with T' can be esily refined to a smooth triangulation S of X.

The number of exotic spheres, i.e of mutually non-diffeomorhic manifolds which are homotopic to S^n is not that large, in fact it is finite (e.g. 28 for $n = 7$) for all $n \geq 5$ according to the subsequent work by Milnor and Kervaire-Milnor, where, by *surgery*, it is reduced to the Serre finiteness theorem, and where the final step is furnished with *Smale's h-cobordism theorem* (see section 8).

By Perelman, there is a single smooth structure on the homotopy 3 sphere and the case $n = 4$ remains open.

7 Isotopies and Intersections.

Besides constructing, listing and classifying manifolds X one is concerned with the spaces of maps $X \to Y$.

The space $[X \to Y]_{smth}$ of *all* C^∞ maps carries little geometric load by itself since this space is homotopy equivalent to $[X \to Y]_{cont(inuous)}$.

An analyst may be concerned with completions of $[X \to Y]_{smth}$, e.g. with Sobolev' topologies while a geometer is keen at studieng geometric structures, e.g. Riemannian metrics on this space.

But from a differential topologist point of view the most interesting is the space of *smooth embeddings* $F : X \to Y$ which diffeomorphically send X onto a smooth submanifold $X' = f(X) \subset Y$.

If $dim(Y) > 2dim(X)$ then generic f are embeddings, but, in general, you can not produce them at will so easily. However, given such an embedding $f_0 : X \to Y$, there are plenty of smooth homotopies, called *(smooth) isotopies* f_t, $t \in [0,1]$, of it which remain embeddings for every t and which can obtained with the following

Isotopy Theorem. (Thom, 1954.) Let $Z \subset X$ be a compact smooth submanifold (boundary is allowed) and $f_0 : X \to Y$ is an embedding, where the essential case is where $X \subset Y$ and f_0 is the identity map.

Then every isotopy of Z in Y can be extended to an isotopy of all of X. More generally, the restriction map $R_{|Z} : [X \to Y]_{emb} \to [Z \to Y]_{emb}$ is a fibration; in particular, the isotopy extension property holds for an arbitrary family of embeddings $X \to Y$ parametrized by a compact space.

This is similar to the *homotopy extension property* (mentioned in section 1) for spaces of continous maps $X \to Y$ – the "geometric" cornerstone of the algebraic topology.)

The proof easily reduces with the implicit function theorem to the case, where $X = Y$ and $dim(Z) = dim(W)$.

Since diffeomorphisms are *open* in the space of all smooth maps, one can extend "small" isotopies, those which only slightly move Z, and since diffeomorphisms of Y make a *group*, the required isotopy is obtained as a composition of small diffeomorphisms of Y. (The details are easy.)

Both "open" and "group" are crucial: for example, homotopies by *locally diffeomorphic* maps, say of a disk $B^2 \subset S^2$ to S^2 do not extend to S^2 whenever a map $B^2 \to S^2$ starts overlapping itself. Also it is much harder (yet possible, [7], [25]) to extend *topological* isotopies, since homeomorphisms are, by no means, open in the space of all continuos maps.

For example if $dim(Y) \geq 2dim(Z) + 2$. then a generic smooth homotopy of Z is an isotopy: Z does not, generically, cross itself as it moves in Y (unlike, for example, a circle moving in the 3-space where self-crossings are stable under small perturbations of homotopies). Hence, every generic homotopy of Z exetends to a smooth isotopy of Y.

Mazur Construction Let U_1, U_2 be compact n-manifolds with boundaries and $f_{12} : U_1 \to U_2$ and $f_{21} : U_2 \to U_1$ be embeddings which land in the interiors of their respective target manifolds.

Let W_1 and W_2 be the unions (inductive limits) of the infinite increasing sequences of spaces

$$W_1 = U_1 \subset_{f_{12}} U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} \ldots$$

and

$$W_2 = U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} U_1 \subset_{f_{12}} \ldots$$

Observe that W_1 and W_2 are open manifolds without boundaries and that they are diffeomorphic since dropping the first term in a sequence $U_1 \subset U_2 \subset U_3 \subset \ldots$ does not change the union.

Similarly, both manifolds are diffeomorphic to the unions of the sequences

$$W_{11} = U_1 \subset_{f_{11}} U_1 \subset_{f_{11}} \ldots \text{ and } W_{22} = U_2 \subset_{f_{22}} U_2 \subset_{f_{22 \ldots}}$$

for

$$f_{11} = f_{12} \circ f_{21} : U_1 \to U_1 \text{ and } f_{22} = f_{21} \circ f_{12} : U_2 \to U_2$$

Next, we observe with the isotopy theorem, that if the self-embedding f_{11} is isotopic to the identity map, then W_{11} is diffeomorphic to the interior of U_1 and the same applies to f_{22} (or any self-embedding for this matter).

Thus we conclude with the above, that, for example,

open normal (regular) neighbourhoods U_1^{op} and U_2^{op} of two homotopy equivalent n-manifolds (and triangulated spaces in general) Z_1 and Z_2 in $\mathbb{R}^{n+N}$, $N \geq n+2$, are diffeomorphic.

An everybody guess would be that the "open" condition is a pure technicality and everybody believed so untill Milnor's !961 countersexample to the *Hauptvermutung* – the main conjecture of the combinatorial topology.

Milnor has shown that there are two free isometric actions A_1 and A_2 of the cyclic group $\mathbb{Z}_p$ on the sphere S^3, for every prime $p \geq 7$, such that

the quotient (lens) spaces $Z_1 = S^3/A_1$ and $Z_2 = S^3/A_2$ are homotopy equivalent, but their closed normal neighbourhoods U_1 and U_2 in any $\mathbb{R}^{3+N}$ are not diffeomorphic. (This could not have happened to *simply connected* manifolds Z_i by the h-cobordism theorem.)

Moreover,

the polyhedra P_1 and P_2 obtained by attaching the cones to the boundaries of these manifolds admit no isomorphic simplicial subdivisions.

Yet, the interiors U_i^{op} of these U_i , $i = 1, 2$, are diffeomorphic for $N \geq 5$. In this case,

P_1 and P_2 are homeomorphic as the one point comactifications of two homeomorphic spaces U_1^{op} and U_2^{op}.

It was previously known that these Z_1 and Z_2 are *homotopy equivalent* (J. H. C. Whitehead, 1941); yet, they are *combinatorially non-equivalent* (Reidemeister, 1936) and, hence, by Moise's 1951 positive solution of the the Hauptvermutung for 3-manifolds, *non-homeomorphic.*

There are few direct truly geometric constructions of diffeomorphisms, but those available, are extensively used, e.g. *fiberwise linear diffeomorphisms of vector bundles.* Even the sheer existence of the humble homothety of $\mathbb{R}^n$, $x \mapsto tx$, combined with the isotopy theorem, effortlessly yields, for example, the following

$[B{\to}Y]$**-Lemma.** *The space of embeddings f of the n-ball (or $\mathbb{R}^n$) into an arbitrary $Y = Y^{n+k}$ is homotopy equivalent to the space of tangent n-frames in*

Y; in fact the differential $f \mapsto Df|0$ establishes a homotopy equiavlence between the respective spaces.

For example,

the assignment $f \mapsto J(f)|0$ of the Jacobi matrix at $0 \in B^n$ is a homotopy equivalence of the space of embeddings $f : B \to \mathbb{R}^n$ to the linear group $GL(n)$.

Corollary: Ball Gluing Lemma. Let X_1 and X_2 be $(n+1)$-dimensional manifolds with boundaries Y_1 and Y_2, let $B_1 \subset Y_1$ be a smooth submanifold diffeomorphic to the n-ball and let $f : B_1 \to B_2 \subset Y_2 = \partial(A_2)$ be a diffeomrphism.

If the boundaries Y_i of X_i are connected, the diffeomorphism class of the $(n+1)$-manifold $X_3 = X_1 +_f X_2$ obtained by attaching X_1 to X_2 by f and (obviously canonically) smoothed at the "corner" (or rather the "crease") along the boundary of B_1, does not depend on B_1 and f.

This X_3 is denoted $X_1 \#_\partial X_2$. For example, thus "sum" of balls, $B^{n+1} \#_\partial B^{n+1}$, is again a smooth $(n+1)$-ball.

Connected Sum. The boundary $Y_3 = \partial(X_3)$ can be defined without any reference to $X_i \supset Y_i$, as follows. Glue the manifolds Y_1 an Y_2 by $f : B_1 \to B_2 \subset Y_2$ and then remove the interiors of the balls B_1 and of its f-image B_2.

If the manifolds Y_i (not necessarily anybody's boundaries or even being closed) are connected, then the resulting *connected sum* manifold is denoted $Y_1 \# Y_2$.

Isn't it a waste of glue? You may be wondering why to bother glueing the interiors of the balls if you are going to remove them anyway. Wouldn't it be easier *first* to remove these interiors from both manifolds and *then* glue what remains along the spheres $S_i^{n-1} = \partial(B_i)$?

This *is* easier but also it is also a *wrong* thing to do: the result may depend on the diffeomorphism $S_1^{n-1} \leftrightarrow S_2^{n-1}$, as it happens for $Y_1 = Y_2 = S^7$ in Milnor's example; but the connected sum defined with balls is unique by the $[B \to Y]$-lemma.

Similarly to gluing along balls, the diffeomorphism class of $X_1 +_f X_2$ for an f, defined on an n-submanifold $C_1 \subset X_1$ which is homotopy equivalent to a space of dimension $\leq \frac{n}{2} - 1$, depends only on the homotopy class of $f : C \to X_2$; yet, Milnor's exotic spheres show that you can not replace balls by *anything* you like.

The ball gluing operation may be used many times in succession; thus, for example, one builds "big $(n+1)$balls" from smaller ones, where this lemma in lower dimension may be used for ensuring the ball property of the gluing sites.

All of the above is rather obvious and equally apply to all dimensions. Here is a more interesting construction due to Haefliger (1961) and routed in the *Whitney Lemma* of 1944.

Let Y be a smooth n-manifold and $X', X'' \subset Y$ be smooth closed submanifolds in general position. Denote $\Sigma_0 = X' \cap X'' \subset Y$ and let X be the (abstract) *disjoint union* of X' and X''. (If X' and X'' are connected equividimensional manifolds, one could say that X is a smooth manifold with its two "connected components" X' and X'' being embedded into Y.)

Clearly,

$$dim(\Sigma_0) = n - k' - k'' \text{ for } n = dim(Y),\ n - k' = dim(X') \text{ and } n - k'' = dim(X'').$$

Let $f_t : X \to Y$, $t \in [0,1]$, be a smooth generic homotopy which disengages X' from X'', i.e. $f_1(X')$ does not intersect $f_1(X'')$, and let

$$\tilde{\Sigma} = \{(x', x'', t)\}_{f_t(x')=f_t(x'')} \subset X' \times X'' \times [0,1],$$

i.e. $\tilde{\Sigma}$ consists of the triples (x', x'', t) for which $f_t(x') = f_t(x'')$.

Let $\Sigma \subset X' \cup X''$ be the union $S' \cup S''$, where $S' \subset X'$ equals the projection of $\tilde{\Sigma}$ to the X'-factor of $X' \times X'' \times [0,1]$ and $S'' \subset X''$ is the projection of $\tilde{\Sigma}$ to X''.

Thus, there is a correspondence $x' \leftrightarrow x''$ between the points in $\Sigma = S' \cup S''$, where the two points correspond one to another if $x' \in S'$ meets $x'' \in S''$ at some moment t_* in the course of the homotopy, i.e.

$$f_{t_*}(x') = f_{t_*}(x'') \text{ for some } t_* \in [0,1].$$

Finally, let $W \subset Y$ be the union of the f_t-paths, denoted $[x' *_t x''] \subset Y$, travelled by the points $x' \in S' \subset \Sigma$ and $x'' \in S'' \subset \Sigma$ until they meet at some moment t_*. In other words, $[x' *_t x''] \subset Y$ consists of the union of the points $f_t(x')$ and $f_t(x'')$ over $t \in [0, t_* = t_*(x') = t_*(x'')]$ and

$$W = \bigcup_{x' \in S'} [x' *_t x''] = \bigcup_{x'' \in S''} [x' *_t x''].$$

Clearly,

$$dim(\Sigma) = \dim(\Sigma_0) + 1 = n - k' - k'' + 1 \text{ and } dim(W) = dim(\Sigma) + 1 = n - k' - k'' + 2.$$

To grasp the picture look at X consisting of a round 2-sphere X' (where $k' = 1$) and a round circle X'' (where $k'' = 2$) in the Euclidean 3-space Y, where X and X' intersect at two points x_1, x_2 – our $\Sigma_0 = \{x_1, x_2\}$ in this case.

When X' an X'' move away one from the other by parallel translations in the opposite directions, their intersection points sweep W which equals the intersection of the 3-ball bounded by X' and the flat 2-disc spanned by X''. The boundary Σ of this W consists of two arks $S' \subset X'$ and $S'' \subset X''$, where S' joins x_1 with x_2 in X' and S'' join x_1 with x_2 in X''.

Back to the general case, we want W to be, generically, a smooth submanifold *without double points* as well as without any other singularities, except for the unavoidable corner in its boundary Σ, where S' meet S'' along Σ_0. We need for this

$$2dim(W) = 2(n - k' - k'' + 2) < n = dim(Y) \text{ i.e. } 2k' + 2k'' > n + 4.$$

Also, we want to avoid an intersection of W with X' and with X'' away from $\Sigma = \partial(W)$. If we agree that $k'' \geq k'$, this, generically, needs

$$dim(W) + dim(X) = (n - k' - k'' + 2) + (n - k') < n \text{ i.e. } 2k' + k'' > n + 2.$$

These inequalities imply that $k' \geq k \geq 3$, and the lowest dimension where they are meaningful is the the first Whitney case: $dim(Y) = n = 6$ and $k' = k'' = 3$.

Accordingly, W is called *Whitney's disk*, although it may be non-homeomorphic to B^2 with the present definition of W (due to Haefliger) of W.

Haefliger Lemma (Whitney for $k + k' = n$). Let the dimensions $n - k' = dim(X')$ and $n - k'' = dim(X'')$, where $k'' \geq k'$, of two submanifolds X' and X'' in the ambient n-manifold Y satisfy $2k' + k'' > n + 2$.

Then every homotopy f_t of (the disjoint union of) X' and X'' in Y which disengages X' from X'', can be replaced by a disengaging homotopy f_t^{new} which is an isotopy, on both manifolds, i.e. $f_t^{new}(X')$ and $f^{new}(X'')$ reman smooth without self intersection points in Y for all $t \in [0,1]$ and $f_1^{new}(X')$ does not intersect $f_1^{new}(X'')$.

Proof. Assume f_t is smooth generic and take a small neighbourhood $U_{3\varepsilon} \subset Y$ of W. By genericity, this f_t is an isotopy of X' as well as of X'' *within* $U_{3\varepsilon} \subset Y$: the intersections of $f_t(X')$ and $f_t(X'')$ with $U_{3\varepsilon}$, call them $X'_{3\varepsilon}(t)$ and $X''_{3\varepsilon}(t)$ are smooth submanifolds in $U_{3\varepsilon}$ for all t, which, moreover, do not intersect away from $W \subset U_{3\varepsilon}$.

Hence, by the Thom isotopy theorem, there exists an isotopy F_t of $Y \smallsetminus U_\varepsilon$ which equals f_t on $U_{2\varepsilon} \smallsetminus U_\varepsilon$ and which is constant in t on $Y \smallsetminus U_{3\varepsilon}$.

Since f_t and F_t within $U_{3\varepsilon}$ are equal on the overlap $U_{2\varepsilon} \smallsetminus U_\varepsilon$ of their definition domains, they make together a homotopy of X' and X'' which, obviuoulsy, satisfies our requirements.

There are several immedaite generalizations/applications of this theorem.

(1) One may allow self-intersections Σ_0 *within* connected components of X, where the necessary homotopy condiition for removing Σ_0 (which was expressed with the disengaging f_t in the present case) is formulated in terms of maps $\tilde{f} : X \times X \to Y \times Y$ commuting with the involutions $(x_1, x_2) \leftrightarrow (x_2, x_1)$ in $X \times X$ and $(y_1, y_2) \leftrightarrow (y_2, y_1)$ in $Y \times Y$ and having the pullbacks $\tilde{f}^{-1}(Y_{diag})$ of the diagonal $Y_{diag} \subset Y \times Y$ equal $X_{diag} \subset X \times X$, [20].

(2) One can apply all of the above to p parametric families of maps $X \to Y$, by paying the price of the extra p in the excess of $dim(Y)$ over $dim(X)$, [20].

If $p = 1$, this yield an isotopy classifiaction of embeddings $X \to Y$ for $3k > n+3$ by homotopies of the above symmetric maps $X \times X \to Y \times Y$, which shows, for example, that there are *no knots* for these dimensions (Haefliger, 1961).

if $3k > n + 3$, then every smooth embedding $S^{n-k} \to \mathbb{R}^n$ is smoothly isotopic to the standard $S^{n-k} \subset \mathbb{R}^n$.

But if $3k = n + 3$ and $k = 2l + 1$ is odd then there are

infinitely many isotopy of classes of embeddings $S^{4l-1} \to \mathbb{R}^{6l}$, (Haefliger 1962).

Non-triviality of such a knot $S^{4l-1} \to \mathbb{R}^{6l}$ is detected by showing that a map $f_0 : B^{4l} \to \mathbb{R}^{6l} \times \mathbb{R}_+$ extending $S^{4l-1} = \partial(B^{4l})$ can not be turned into an embedding, keeping it transversal to $\mathbb{R}^{6l} = \mathbb{R}^{6l} \times 0$ and with its boundary equal our knot $S^{4l-1} \subset \mathbb{R}^{6l}$.

The Whitney-Haefliger W for f_0 has dimension $6l + 1 - 2(2l + 1) + 2 = 2l + 1$ and, generically, it transversally intersects B^{4l} at several points.

The resulting (properly defined) intersection index of W with B is non-zero (otherwise one could eliminate these points by Whitney) and it does not depend on f_0. In fact, it equals the linking invariant of Haefliger.

(3) In view of the above, one must be careful if one wants to relax the dimension constrain by an inductive application of the Whitney-Haefliger disengaging procedure, since obstructions/invariants for removal "higher" intersections

which come on the way may be not so apparent.

But this is possible, at least on the $\mathbb{Q}$-level, where one has a comprehensive algebraic control of self-intersections of all multiplicities for maps of codimension $k \geq 3$ and where he answers are the simplest in the combinatorial category. For example,

there is no combinatorial knots of codimension $k \geq 3$ (Zeeman, 1963).

The essential mechanism of knotting $X = X^n \subset Y = Y^{n+2}$ depends on the fundamental group Γ of the complement $U = Y \subset X$. The group Γ may look a nuisance when you want to untangle a knot, especially a surface X^2 in a 4-manifold, but these $\Gamma = \Gamma(X)$ for various $X \subset Y$ form beautifully intricate patterns which are poorly understood.

For example, the groups $\Gamma = \pi_1(U)$ capture *the étale cohomology of algebraic manifolds* and the *Novikov-Pontryagin classes of topological manifolds* (see section 9). Possibly, the groups $\Gamma(X^2)$ for surfaces $X^2 \subset Y^4$ have much to tell about the smooth topology of 4-manifolds.

There are few systematic ways of constructing "simple" $X \subset Y$, e.g. immersed submanifolds, with "interesting" (e.g. far from being free) fundamental groups of their complements.

Offhand suggestions are pullbacks of divisors X_0 in complex algebraic manifolds Y_0 under generic maps $Y \to Y_0$ and immersed subvarieties X^n in cubically subdivided Y^{n+2}, where X^n are made of n-sub-cubes $\square^n$ *inside* the cubes $\square^{n+2} \subset Y^{n+2}$ and where these interior $\square^n \subset \square^{n+2}$ are parallel to the n-faces of $\square^{n+2}$.

It remains equally unclear what is the possible topology of self-intersections of immersions $X^n \to Y^{n+2}$, say for $S^3 \to S^5$, where the self-intersection makes a link in S^3, and for $S^4 \to S^6$ where this is an immersed surface in S^4.

(4) One can control the position of the image of $f^{new}(X) \subset Y$, e.g. by making it to land in a given open subset $W_0 \subset W$, if there is no homotopy obstruction to this.

The above generalizes and simplifies in the combinatorial or "piecewise smooth" category, e.g. for "unknotting spheres", where the basic construction is as follows

Engulfing. Let X be a piecewise smooth polyhedron in a smooth manifold Y.

If $n - k = dim(X) \leq dim(Y) - 3$ *and if* $\pi_i(Y) = 0$ *for* $i = 1, ...dim(Y)$, *then there exists a smooth isotopy* F_t *of* Y *which eventually (for* $t = 1$*) moves* X *to a given (small) neighbourhood* $B_\circ$ *of a point in* Y.

Sketch of the Proof. Start with a generic f_t. This f_t does the job away from a certain W which has $dim(W) \leq n - 2k + 2$. This is $< dim(X)$ under the above assumption and the proof proceeds by induction on $dim(X)$.

This is called "engulfing" since $B_\circ$, when moved by the time reversed isotopy, engulfs X; engulfing was invented by Stallings in his approach to the *Poincare Conjecture* in the combinatorial category, which goes, roughly, as follows.

Let Y be a smooth n-manifold. Then, with a simple use of two mutually dual smooth triangilations of Y, one can decompose Y, for each i, into the union of *regular neighbourhoods* U_1 and U_2 of smooth subpolyhedra X_1 and X_2 in Y of dimensions i and $n - i - 1$ (similarly to the *handle body decomposition* of a 3-manifold into the union of two thickened graphs in it), where, recall, a neighbourhood U of an $X \subset Y$ is regular if there exists an isotopy $f_t : U \to U$ which brings all of U arbitrarily close to X.

Now let Y be a homotopy sphere of dimension $n \geq 7$, say $n = 7$, and let $i = 3$ Then X_1 and X_2, and hence U_1 and U_2, can be engulfed by (diffeomorphic imags of) balls, say by $B_1 \supset U_1$ and $B_2 \supset U_2$ with their centers denoted $0_1 \in B_1$ and $0_2 \in B_2$.

By moving the 6-sphere $\partial(B_1) \subset B_2$ by the radial isotopy in B_2 toward 0_2, one represents $Y \setminus 0_2$ by the union of an increasing sequence of isotopic copies of the ball B_1. This implies (with the isotopy theorem) that $Y \setminus 0_2$ is diffeomorphic to $\mathbb{R}^7$, hence, Y is homeomorphic to S^7.

(A refined generalization of this argument delivers the Poincare conjecture in the combinatorial and topological categories for $n \geq 5$. See [47] for an account of techniques for proving various "Poincare conjectures" and for references to the source papers.)

8 Handles and h-Cobordosms.

The original approach of Smale to the Poincare conjecture depends on *handle decompositions* of manifolds – counterparts to cell decompositions in the homotopy theory.

Such decompositions are more flexible, and by far more abundant than triangualtions and they are better suited for a match with algebraic objects such as homology. For example, one can sometimes realize a basis in homology by suitably chosen cells or handles which is not even possible to formulate properly for triangulations.

Recall that an *i-handle of dimension n* is the ball B^n decomposed into the product $B^n = B^i \times B^{n-i}(\varepsilon)$ where one think of such a handle as an ε-thickening of the unit i-ball and where

$$A(\varepsilon) = S^i \times B^{n-1}(\varepsilon) \subset S^{n-1} = \partial B^n$$

is seen as an ε-neighbourhood of its *axial $(i-1)$-sphere* $S^{i-1} \times 0$ – *an equatotial i-sphere in S^{n-1}*.

If X is an n-manifold with boundary Y and $f : A(\varepsilon) \to Y$ a smooth embedding, one can attach B^n to X by f and the resulting manifold (with the "corner" along $\partial A(\varepsilon)$ made smooth) is denoted $X +_f B^n$ or $X +_{S^{i-1}} B^n$, where the latter subscript refers to the f-image of the axial sphere in Y.

The effect of this on the boundary, i.e. modification

$$\partial(X) = Y \rightsquigarrow_f Y' = \partial(X +_{S^{i-1}} B^n)$$

does not depend on X but only on Y and f. It is called an *i-surgery of Y at the sphere $f(S^{i-1} \times 0) \subset Y$*.

The manifold $X = Y \times [0,1] +_{S^{i-1}} B^n$, where B^n is attached to $Y \times 1$, makes a bordism between $Y = Y \times 0$ and Y' which equals the surgically modified $Y \times 1$-component of the boundary of X. If the manifold Y is oriented, so is X, unless $i = 1$ and the two ends of the 1-handle $B^1 \times B^{n-1}(\varepsilon)$ are attached to the same connected componet of Y with opposite orientations.

When we attach an i-handle to an X along a *zero-homologous* sphere $S^{i-1} \subset Y$, we create a new i-cycle in $X +_{S^{i-1}} B^n$; when we attach an $(i+1)$-handle along an i-sphere in X which is *non-homologous to zero*, we "kill" an i-cycle.

These creations/annihilations of homology may cancel each other and a handle decomposition of an X may have by far more handles (balls) than the number of independent homology classes in $H_*(X)$.

Smale's argument proceeds in two steps.

(1) The overall algebraic cancelation is decomposed into "elementary steps" by "reshaffling" handles (in spirit of Whitehead's theory of the *simple homotopy type*);

(2) each elementary step is implemented geometrically as in the example below (which does not elucidates the case $n = 6$).

Cancelling a 3*-handle by a* 4*-handle.* Let $X = S^3 \times B^4(\varepsilon_0)$ and let us attach the 4-handle $B^7 = B^4 \times B^3(\varepsilon)$, $\varepsilon << \varepsilon_0$, to the (normal) ε-neigbourhood $A_\sim$ of some sphere

$$S^3_\sim \subset Y = \partial(X) = S^3 \times S^3(\varepsilon_0) \text{ for } S^3(\varepsilon_0) = \partial B^4(\varepsilon_0).$$

by some diffeomorphism of $A(\varepsilon) \subset \partial(B^7)$ onto $A_\sim$.

If $S^3_\sim = S^3 \times b_0$, $b_0 \in S^3(\varepsilon_0)$, is the standard sphere, then the resulting $X_\sim = X +_{S^3_\sim} B^7$ is obviously diffeomorphic to B^7: adding $S^3 \times B^4(\varepsilon_0)$ to B^7 amounts to "bulging" the ball B^7 over the ε-neighbourhood $A(\varepsilon)$ of the axial 3-sphere on its boundary.

Another way to see it is by observing that this addition of $S^3 \times B^4(\varepsilon_0)$ to B^7 can be decomposed into gluing two balls in succession to B^7 as follows.

Take a ball $B^3(\delta) \subset S^3$ around some point $s_0 \in S^3$ and decompose $X = S^3 \times B^4(\varepsilon_0)$ into the union of two balls that are

$$B^7_\delta = B^3(\delta) \times B^4(\varepsilon_0)$$

and

$$B^7_{1-\delta} = B^3(1-\delta) \times B^4(\varepsilon_0) \text{ for } B^3(1-\delta) =_{def} S^3 \smallsetminus B^3(\delta).$$

Clearly, the attachment loci of $B^7_{1-\delta}$ to X and of B^7_δ to $X + B^7_{1-\delta}$ are diffeomorphic (after smoothing the corners) to the 6-ball.

Let us modify the sphere $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0) = \partial(X)$ by replacing the original standard embedding of the 3-ball

$$B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) \subset \partial(X)$$

by another one, say,

$$f_\sim : B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) = \partial(X),$$

such that $f_\sim$ equals the original embedding near the boundary of $\partial(B^3(1-\delta)) = \partial(B^3(\delta)) = S^2(\delta)$.

Then the same "ball after ball" argument applies, since the first gluing site where $B^7_{1-\delta}$ is being attached to X, albeit "wiggled", remains diffeomorphic to B^6 by the isotopy theorem, while the second one does not change at all. So we conclude:

whenever $S^3_\sim \subset S^3 \times S^3(\varepsilon_0)$ *transversally intersect* $s_0 \times S^3(\varepsilon_0)$, $s_0 \in S^3$, *at a single point, the manifold* $X_\sim = X +_{S^3_\sim} B^7$ *is diffeomorphic to* B^7.

This shows, for instance, that a Milnor's Σ^7 (section 6) minus a small ball is diffeomorphic to B^7 if $e = \pm 1$.

Finally, by Whitney's lemma, every embedding $S^3 \to S^3 \times S^3(\varepsilon_0) \subset S^3 \times B^4(\varepsilon_0)$ which is *homologous in* $S^3 \times B^4(\varepsilon_0)$ to the standard $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0)$, can be isotoped to another one which meets $s_0 \times S^3(\varepsilon_0)$ transversally at a single point. Hence,

the handles do cancel one another: if a sphere

$$S^3_{\sim} \subset S^3 \times S^3(\varepsilon_0) = \partial(X) \subset X = S^3 \times B^4(\varepsilon_0),$$

is homologous in X to

$$S^3 \times b_0 \subset X = S^3 \times B^4(\varepsilon_0),\ b_0 \in B^4(\varepsilon),$$

then the manifold $X +_{S^3_{\sim}} B^7$ is diffeomorhic to the 7-ball.

The handles shaffling/cancellation techniques do not solve the existence problem for diffeomorphisms $Y \leftrightarrow Y'$ but rather reduce it to the existense of *h-cobordisms* between manifolds, where a compact manifold X with two boundary components Y and Y' is called an h-cobordism (between Y and Y') if the inclusion $Y \subset X$ is a homotopy equivalence.

Smale h-Cobordism Theorem. *If an h-cobordism has $dim(X) \geq 6$ and $\pi_1(X) = 1$ then X is diffeomorphic to $Y \times [0,1]$, by a diffeomorphism keeping $Y = Y \times 0 \subset X$ fixed. In particular, h-cobordant simply connected manifolds of dimensions ≥ 5 are diffeomorphic.*

Notice that the Poncare conjecture for the homotopy spheres Σ^n, $n \geq 6$, follows by applying this to Σ^n minus two small open balls, while the case $m = 1$ is solved by Smale with a construction of an h-cobordism between Σ^5 and S^5.

Also Smale's handle techniques deliver the following geometric version of the Poincare connectedness/contractibilty correspondence (see section 4).

Let X be a closed n-manifold, $n \geq 5$, with $\pi_i(X) = 0$, $i = 1, ..., k$. Then X contains a $(n-k-1)$-dimensional smooth sub-polyhedron $P \subset X$, such that the complement of the open regular neighbourhood $U_\varepsilon(P) \subset X$ of P is diffeomorphic to the n-ball, (where the boundary $\partial(U_\varepsilon)$ is the $(n-1)$-sphere "ε-collapsed" onto $P = P^{n-k-1}$).

If $n = 5$ and if the *normal bundle of X embedded into some $\mathbb{R}^{5+N}$ is trivial*, i.e. if the normal Gauss map of X to the Grassmanian $Gr(\mathbb{R}^{5+N})$ is contractible, then Smale proves, assuming $\pi_1(X) = 1$, that

one can choose $P = P^3 \subset X = X^5$ that equals the union of a smooth topological segment $s = [0,1] \subset X$ and several spheres S^2_i and S^3_i, where each S^3_i meets s at one point, and also transversally intersects S^2_i at a single point and where there are no other intersections between s, S^2_i and S^3_i.

In other words,

(Smale 1965) *X is diffeomorphic to the connected sum of several copies of $S^2 \times S^3$.*

The triviality of the bundle in this theorem is needed to ensure that all embedded 2-sheres in X have trivial normal bundles, i.e. their normal neighbouthoods split into $S^2 \times \mathbb{R}^3$ which comes handy when you play with handles.

If one drops this triviality condition, one has

Classifiaction of Simply Connected 5-Manifolds. (Barden 1966) *There is a finite list of explicitly constructed 5-manifolds X_i, such that every closed simply connected manifold X is diffeomorphic to the connected sum of X_i.*

This is possible, in view of the above Smale theorem, since all simply connected 5-manifolds X have "almost trivial" normal bundles e.g. their only possible Pontryagin class $p_1 \in H^4(X)$ is zero. Indeed $\pi_1(X) = 1$ implies that $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] = 0$ and then $H^4(X) = H_1(X) = 0$ by the Poincare duality.

When you encounter bordisms, the generecity sling launches you to the stratosphere of algebraic topology so fast that you barely discern the geometric string attached to it.

Smale's cells and handles, on the contrary, feel like slippery amebas which merge and disengage as they reptate in the swamp of unruly geometry, where n-dimensional cells continuously collapse to lower dimensional ones and keep squeezing through paper-thin crevices. Yet, their motion is governed, for all we know, by the rules dictated by some *algebraic K-theory* (theories?)

This motion hardly can be controlled by any traditional geometric flow. First of all, the "simply connected" condition can not be encoded in geometry ([34], [15] [35] and also breaking the symmetry by dividing a manifold into handles along with "genericity" poorly fare in geometry.

Yet, some generalized "Ricci flow with partial collapse and surgeries" in the "space of (generic, random?) amebas" might split away whatever it fails to untangle and bring fresh geometry into the picture.

9 Manifolds under Surgery.

The Atyiah-Thom construction and Serre's theory allows one to produce "arbitararily large" manifolds X for the *m-domination* $X_1 \succ_m X_2$, $m > 0$, meaning that there is a map $f : X_1 \to X_2$ of degree m.

Every such f between closed connected oriented manifolds induces a *surjective* homomomorphisms $f_{*i} : H_i(X_1; \mathbb{Q}) \to H_i(X_1; \mathbb{Q})$ for all $i = 0, 1, ..., n$, or equivalently, an injective cohomology homomorphism $f^{*i} : H^i(X_2; \mathbb{Q}) \to H^i(X_2; \mathbb{Q})$.

Indeed, by the Poincare $\mathbb{Q}$-duality, the cup-product pairing $H^i(X_2; \mathbb{Q}) \otimes H^{n-i}(X_2; \mathbb{Q}) \to \mathbb{Q} = H^n(X_2; \mathbb{Q})$ is faithful; therefore, if f^{*i} vanishes, then so does f^{*n}. But the latter amounts to multiplication by $m = deg(f)$,

$$H^n(X_2; \mathbb{Q}) = \mathbb{Q} \to_{\cdot d} \mathbb{Q} = H^n(X_1; \mathbb{Q}).$$

If $m = 1$, then (by the full cohomological Poincare duality) the above remains true for all coefficient fields $\mathbb{F}$; moreover, the induced homomorphism $\pi_i(X_1) \to \pi_i(X_2)$ is surjective as it is seen by looking at the lift of $f : X_1 \to X_2$ to the induced map from the covering $\tilde{X}_1 \to X_1$ induced by the universal covering $\tilde{X}_2 \to X_2$ to $\tilde{X}_2$. (A map of degree $m > 1$ sends $\pi_1(X_1)$ to a subgroup in $\pi_1(X_2)$ of a finite index dividing m.)

Let us construct manifolds starting from *pseudo-manifolds*, where a compact oriented n-dimensional pseudo-manifold is a triangulated n-space X_0, such that

• every simplex of dimension $< n$ in X_0 lies in the boundary of an n-simplex,

• The complement to the union of the $(n-2)$-simplices in X_0 is an orieneted manifold.

Pseudo-manifolds are infinitely easier to construct and to recognize than manifolds: essentially, these are simplicial complexes with exactly two n-simplices adjacent to every $(n-1)$-simplex.

There is no comparably simple characteriziation of triangulated n-manifolds X where the links $L^{n-i-1} = L_{\Delta^i} \subset X$ of the i-simplices must be topological $(n-i-1)$-spheres. But even deciding if $\pi_1(L^{n-i-1}) = 1$ is *an unsolvable problem* except for a couple of low dimensions.

Acoordingly, it is very hard to produce manifolds by combinatorial constructions; yet, one can "dominate" any pesudo-manifold by a manifold, where, observe, the notion of degree perfectly applies to oriented pseudo-manifolds.

Let X_0 be a connected oriented n-pseudomanifold. Then there exists a smooth closed connected oriented manifold X and a continuous map $f : X \to X_0$ of degree $m > 0$.

Moreover, given an oriented $\mathbb{R}^N$-bundle $V_0 \to X_0$, $N \geq 1$, one can find an m-dominating X, which also admits a smooth embedding $X \subset \mathbb{R}^{n+N}$, such that our $f : X \to X_0$ of degree $m > 0$ induces the normal bundle of X from V_0.

Proof. Since that the first $N-1$ homotopy groups of the Thom space of $V_\bullet$ of V_0 vanish (see section 5), Serre's m-sphericity theorem delivers a map $f_\bullet : S^{n+N} \to V_\bullet$ a non-zero degree m, provided $N > n$. Then the "generic pullback" X of $X_0 \subset V_0$ (see section 3) does the job as it was done in section 5 for Thom's bordisms.

In general, if $1 \leq N \leq n$, the m-sphericity of the fundamental class $[V_\bullet] \in H_{n+N}(V_\bullet)$ is proven with the *Sullivan's minimal models* – the ultimate algebraic embodiment of the $\mathbb{Q}$-homotopy theory (see theorem 24.5 in [12]).

Surgery and the Browder-Novikov Theorem (1962 [6],[36]). Let X_0 be a smooth closed simply connected oriented n-manifold, $n \geq 5$, and $V_0 \to X_0$ be a *stable* vector bundle where "stable" means that $N = rank(V) >> n$. We want to modify the smooth structure of X_0 keeping its homotopy type unchanged but with its original normal bundle in $\mathbb{R}^{n+N}$ replaced by V_0.

There is an obvious algebraic-topological obstruction to this highlighted by Atiyah in [1] which we call *$[V_\bullet]$-sphericity* and which means that there exists a *degree one*, map $f_\bullet$ of S^{n+N} to the Thom space $V_\bullet$ of V_0, i.e. $f_\bullet$ sends the generator $[S^{n+N}] \in H_{n+N}(S^{n+N}) = \mathbb{Z}$ (for some orientation of the sphere S^{n+N} to the fundamental class of the Thom space, $[V_\bullet \in H_{n+N}(V_\bullet) = \mathbb{Z}$, which is distinguished by the orientation in X. (One has to be pedantic with orientations to keep track of possible/impossible algebraic cancellations.)

However, this obstruction is "$\mathbb{Q}$-nonessential", [1] : the set of the vector bundles admitting such an $f_\bullet$ constitutes a coset of a subgroup of *finite index* in Atiyah's (reduced) *K-group* by Serre's finiteness theorem.

Recall that $K(X)$ is the Abelian group formally generated by the isomorphism classes of vector bundles V over X, where $[V_1] + [V_2] =_{def} 0$ whenever the *Whitney sum* $V_1 \oplus V_2$ is isomorphic to a trivial bundle.

The *Whitney sum* of an $\mathbb{R}^{n_1}$-bundle $V_1 \to X$ with an $\mathbb{R}^{n_2}$-bundle $V_2 \to X$, is the $\mathbb{R}^{n_1+n_2}$-bundle over X. which equals the fiber-wise Caresian product of the two bundles.

For example the Whitney sum of the tangent bundle of a smooth submanifold $X^n \subset W^{n+N}$ and of its normal bundle in W equals the tangent bundle of W restricted to X. Thus, it is *trivial* for $W = \mathbb{R}^{n+N}$, i.e. it isomorphic to $\mathbb{R}^{n+N} \times$

$X \to X$, since the tangent bundle of $\mathbb{R}^{n+N}$ is, obviously, trivial.

Granted an $f_\bullet : S^{n+N} \to V_\bullet$ of degree 1, we take the "generic pullback" X of X_0,

$$X \subset \mathbb{R}^{n+N} \subset \mathbb{R}_\bullet^{n+N} = S^{n+N},$$

and denote by $f : X \to X_0$ the restriction of $f_\bullet$ to X, where, recall, f induces the normal bundle of X from V_0. .

The map $f : X_1 \to X_0$, which is clearly *onto*, is far from being injective – it may have uncontrollably complicated folds. In fact, it is not even a homotopy equivalence – the homology homomorphism induced by f

$$f_{*i} : H_i(X_1) \to H_i(X_0),$$

is, as we know, surjective and it may (and usually does) have non-trivial kernels $ker_i \subset H_i(X_1)$. However, these kernels can be "killed" by a "surgical implementation" of the obstruction theory as follows.

Assume $ker_i = 0$ for $i = 0, 1, ..., k-1$, invoke Hurewicz' theorem and realize the cycles in ker_k by k-spheres mapped to X_1, where, observe, the f-mages of these spheres are contractible in X_0 by a relative version of the (elementary) Hurewicz theorem.

Furthermore, if $k < n/2$, then these spheres $S^k \subset X_1$ are generically *embedded* (no self-intersections) and have *trivial* normal bundles in X_1, since, essentially, they come from $V \to X_1$ via contractible maps. Thus, small neighbourhoods (ε-annuli) $A = A_\varepsilon$ of these spheres in X_1 split: $A = S^k \times B_\varepsilon^{n-k} \subset X_1$.

It follows, that the corresponding sperical cycles can be killed by $(k+1)$-surgery (where X_1 now plays the role of Y in the definition of the surgery); moreover, it is not hard to arrange a map of the resulting manifold to X_0 with the same properties as f.

If $n = dim(X_0)$ is odd, this works up to $k = (n-1)/2$ and makes *all* ker_i, including $i > k$, equal zero by the Poincare duality.

Since

a continuous map between *simply connected* spaces which induces an *isomorphism on homology* is a *homotopy equivalence* by the (elementary) *Whitehead theorem*,

the resulting manifold X is a *homotopy equivalent* to X_0 via our surgically modified map f, call it $f_{srg} : X \to X_0$.

Besides, by the construction of f_{srg}, this map induces the normal bundle of X from $V \to X_0$. Thus we conclude,

the Atiyah $[V_\bullet]$*-sphericity is the only condition for realizing a stable vector bundle* $V_0 \to X_0$ *by the normal bundle of a smooth manifold* X *in the homotopy class of a given odd dimensional simply connected manifold* X_0.

If n is even, we need to kill k-spheres for $k = n/2$, where an extra obstruction arises. For example, if k is even, the surgery does not change the signature; therefore, the Pontryagin classes of the bundle V *must satisfy* the Rokhlin-Thom-Hirzebruch formula to start with.

(There is an additional constrain for the tangent bundle $T(X)$ – the equality between the Euler characteristic $\chi(X) = \sum_{i=0,...,n}(-1)^i rank_{\mathbb{Q}}(H_i(X))$ and the Euler number $e(T(X))$ that is the self-intersection index of $X \subset T(X)$.)

On the other hand the equality $L(V)[X_0] = sig(X_0)$ (obviously) implies that $sig(X) = sig(X_0)$. It follows that

the intersection form on $ker_k \subset H_k(X)$ *has zero signature*,

since all $h \in ker_k$ has zero intersection indices with the pullbacks of k-cycles from X_0.

Then, assuming $ker_i = 0$ for $i < k$ and $n \neq 4$, one can use Whitney's lemma and realize a basis in $ker_k \subset H_k(X_1)$ by $2m$ *embedded* spheres $S^k_{2j-1}, S^k_{2j} \subset X_1$, $i = 1, ...m$, which have zero self-intersection indices, one point crossings between S^k_{2j-1} and S^k_{2j} and no other intersections between these spheres.

Since the spheres $S^k \subset X$ with $[S^k] \in ker_k$ have trivial *stable* normal bundles $U^\perp$ (i.e. their Whitney sums with trivial 1-bundles, $U^\perp \oplus \mathbb{R}$, are trivial), the normal bundle $U^\perp = U^\perp(S^k)$ of such a sphere S^k is trivial if and only if the Euler number$e(U^\perp)$ vanishes (this is easy) where $e(U^\perp(S^k))$ is conveniently equals the self-intersection index of S^k in X. ($e(U^\perp(S^k))$ equals, by definition, the self-intersecion of $S^k \subset U^\perp(S^k)$ which is the same as the self-intersection of this sphere in X.)

Then it easy to see that the $(k+1)$-surgeries applied to the spheres S^k_{2j}, $j = 1, ..., m$, kill all of ker_k and make $X \to X_0$ a homotopy equivalence.

There several points to check (and to correct) in the above argument, but everything fits amazingly well in the lap of the linear algebra (with a few subtleties for odd k).

Notice, that our starting X_0 does not need to be a manifold, but rather a *Poincare (Browder) n-space*, i.e. a finite cell complex satisfying the Poincare duality: $H_i(X_0, \mathbb{F}) = H^{n-i}(X_0, \mathbb{F})$ for all coefficient fields (and rings) $\mathbb{F}$, where these "equalities" must be coherent in an obvious sense for different $\mathbb{F}$.

Also, besides the *existence* of smooth n-manifolds X, the above *surgery argument* applied to a bordism Y between homotopy equivalent manifolds X_1 and X_2 under suitable conditions on the normal bundle of Y, delievers an h-cobordism. This with the h-cobordism theorem, leads to an algebraic *classification* of smooth structures on simply connected manifolds of dimension $n \geq 5$. (see [36]).

Then the Serre finiteness theorem implies that

there are at most finitely many smooth closed simply connected n-manifolds X in a given a homotopy class and with given Pontryagin classes $p_k \in H^{4k}(X)$.

Summimg up, the question "What are manifolds?" has the following

1962 Answer. *Smooth closed simply connected n-manifolds for $n \geq 5$, up to a "finite correction term", are "just" simply connected Poincare n-spaces X with distinguished cohomology classes $p_i \in H^{4i}(X)$, such that $L_k(p_i)[X] = sig(X)$ if $n = 4k$.*

This is a fantastic answer to the "manifold problem" undreamed of 10 years earlier. Yet,

• Poincare spaces are not classifiable. Even the candidates for the cohomology rings are not classifiable over $\mathbb{Q}$.

Are there special "interesting" classes of manifolds and/or coarser than $diff$ classifications? (Something mediating between bordisms and h-cobordisms maybe?)

• The $\pi_1 = 1$ is very restrictive. The surgery theory extends to manifolds with an arbitrary fundamental group Γ and, modulo *the Novikov conjecture* – a non-simply connected counterpart to the relation $L_k(p_i)[X] = sig(X)$ (see next section) – delivers a comparably exhaustive answer in terms of the "Poincare complexes over (the group ring of) Γ" (see [58]).

But this does not tells you much about "topologically interesting" Γ, e.g. fundamental groups of n-manifold X with the universal covering $\mathbb{R}^n$ (see [8] [9] about it).

10 Elliptic Wings and Parabolic Flows.

The geometric texture in the topology we have seen so far was all on the side of the "entropy"; topologists were finding gentle routes in the rugged landscape of all possibilities, you do not have to sweat climbing up steep energy gradients on these routs. And there was no essential new analysis in this texture for about 50 years since Poincare.

Analysis came back with a bang in 1963 when Atiyah and Singer discovered the index theorem.

The underlying idea is simple: the "difference" between dimensions of two spaces, say Φ and Ψ, can be defined and be *finite* even if the spaces themselves are *infinite dimensional*, provided the spaces come with a linear (sometimes non-linear) *Freholm* operator $D : \Phi \to \Psi$. This means, there exists an operator $E : \Psi \to \Phi$ such that $(1 - D \circ E) : \Psi \to \Psi$ and $(1 - E \circ D) : \Phi \to \Phi$ are compact operators. (In the non-linear case, the definition(s) is local and more elaborate.)

If D is Fredholm, then the spaces $ker(D)$ and $coker(D) = \Psi/D(\Phi)$ are finite dimensional and the *index* $ind(D) = dim(ker(D)) - dim(coker(D))$ is (by a simple argument) is a *homotopy invariant* of D in the space of *Fredholm* operators.

If, and this is a "big IF", you can associate such a D to a geometric or topological object X, this index will serve as an invariant of X.

It was known since long that *elliptic* differential operators, e.g. the ordinary Laplace operator, are Fredholm under suitable (boundary) conditions but most of these "natural" operators are self-adjoint and always have zero indices: they are of no use in topology.

"Interesting" elliptic differential operators D are scares: the ellipticity condition is a tricky inequality (or, rather, non-equality) between the coefficients of D. In fact, all such (linear) operators currently in use descend from a single one: *the Atiyah-Singer-Dirac operator on spinors.*

Atiyah and Singer have computed the indices of their geometric operators in terms of traditional topological invariants, and thus discovered new properties of the latter.

For example, they expressed the signature of a closed smooth Riemannian manifold X as an index of such an operator D_{sig} acting on differential forms on X. Since the *parametrix* operator E for an elliptic operator D can be obtained by piecing together local parametrices, the very existence of D_{sig} implies the multiplicativity of the signature.

The elliptic theory of Atiyah and Singer and their many followers, unlike the classical theory of PDE, is functorial in nature as it deals with many interconnected operators at the same time in coherent manner.

Thus smooth structures on potential manifolds (Poincare complexes) define a functor from the homotopy category to the category of "Fredholm diagrams" (e.g. operators – one arrow diagrams); one is tempted to forget manifolds and study such functors per se. For example, a closed smooth manifold represents a homology class in Atiyah's K-theory – the index of D_{sig}, twisted with vector bundles over X with connections in them.

Interestingly enough, one of the first topological applications of the index theory, which equally applies to all dimensions be they big or small, was the solution (Massey, 1969) of the *Whitney* $4D$*-conjecture* of 1941 which, in a simplified form, says the following.

The number $N(Y)$ *of possible normal bundles of a closed connected non-orientable surface* Y *embedded into the Euclidean space* $\mathbb{R}^4$ *equals* $|\chi(Y)-1|+1$, *where* χ *denotes the Euler characteristic.Equivalently, there are* $|\chi(Y)-1|=1$ *possible homeomorphisms types of small normal neighbourhoods of* Y *in* $\mathbb{R}^4$.

If Y is an orienetable surface then $N(Y)=1$, since a small neigbourhood of such a $Y\subset\mathbb{R}^4$ is homeomorphic to $Y\times\mathbb{R}^2$ by an elementary argument.

If Y is non-orienatble, Whitney has shown that $N(Y)\geq|\chi(Y)-1|+1$ by constructing $N=|\chi(Y)-1|+1$ embeddings of each Y to $\mathbb{R}^4$ with different normal bundles and then conjectured that one could not do better.

Outline of Massey's Proof. Take the (unique in this case) ramified double covering X of $S^4\supset\mathbb{R}^4\supset Y$ branched at Y with the natural involution $I:X\to X$. Express the *signature of* I, that is the quadratic form on $H_2(X)$ defined by the intersection of cycles C and $I(C)$ in X, in terms of the *Euler number* $e^\perp$ of the normal bundle of $Y\subset\mathbb{R}^4$ as $sig=e^\perp/2$ (with suitable orientation and sign conventions) by applying the *Atiyah-Singer equivariant signature theorem.* Show that $rank(H_2(X))=2-\chi(Y)$ and thus establish the bound $|e^\perp/2|\leq 2-\chi(Y)$ in the agreement with Whitney's conjecture.

(The experience of the high dimensional topology would suggest that $N(Y)=\infty$. Now-a-days, multiple constrains on topology of embeddings of surfaces into 4-manifolds are derived with Donaldson's theory.)

Non-simply Connected Analytic Geometry. The Browder-Novikov theory implies that, besides the Euler-Poincare formula, there is a *single* "$\mathbb{Q}$-essential (i.e. non-torsion) homotopy constrain" on tangent bundles of closed *simply connected* $4k$-manifolds– the Rokhlin-Thom-Hirzebruch signature relation.

But in 1966, Sergey Novikov, in the course of his proof of the *topological* invariance of the of the *rational Pontryagin classes*, i.e of the homology homomorphism $H_*(X^n;\mathbb{Q})\to H_*(Gr_N(\mathbb{R}^{n+N});\mathbb{Q})$ induced by the normal Gauss map, found the following new relation for *non-simply connected* manifolds X.

Let $f:X^n\to Y^{n-4k}$ be a smooth map. Then the signature of the $4k$-dimensional pullback manifold $Z=f^{-1}(y)$ of a generic point, $sig[f]=sig(Z)$, does not depend on the point and/or on f within a given homotopy class $[f]$ by the generic pull-back theorem and the cobordism invariance of the signature, but it may change under a homotopy equivalence $h:X_1\to X_2$.

By an elaborated (and, at the first sight, circular) *surgery* + *algebraic* K*-theory* argument, Novikov proves that

$$\textit{if } Y \textit{ is a } k\textit{-torus, then } sig[f\circ h]=sig[f],$$

where the simplest case of the projection $X\times\mathbb{T}^{n-4k}\to\mathbb{T}^{n-4k}$ is (almost all) what is needed for the topological invariance of the Pontryagin classes.

Novikov conjectured (among other things) that a similar result holds for an arbitrary closed manifold Y with *contractible* universal covering. (This would imply, in particular, that if an oriented manifold Y' is orientably homotopy equivalent to such a Y, then it is bordant to Y.) Mishchenko (1974) proved this for manifolds Y admitting metrics of *non-positive curvature* with a use of an index theorem for operators on infinite dimensional bundles, thus linking the Novikov conjecture to geometry.

(Hyperbolic groups also enter Sullivan's existence/uniquenss theorem of *Lipschitz structures* on topological manifolds of dimensions ≥ 5.

A bi-Lipschitz homeomorphism may look very nasty. Take, for instance, infinitely many disjont round balls $B_1, B_2, \ldots$ in $\mathbb{R}^n$ of radii $\to 0$, take a diffeomorphism f of B_1 fixing the boundary $\partial(B_1)$ an take the scaled copy of f in each B_i. The reslting homeomorpism, fixed away from these balls, becomes quite complicated whenever the balls accumulate at some closed subset, e.g. a hypersurface in $\mathbb{R}^n$. Yet, one can extend the signature index theorem and some of the Donaldson theory to this unfriedly bi-Lipschitz, and even to quasi-conformal, enviroment.)

The Novikov conjecture remains unsolved. It can be reformulated in purely group theoretic terms, but the most significant progress which has been achieved so far depends on geometry and on the index theory.

In a somewhat similar vein, Atiyah (1974) introduced L_2-cohomology on non-compact manifolds $\tilde{X}$ with cocompact discrete group actions and proved the *L_2-index theorem.* For example, he has shown that

if a compact Riemannian $4k$-manifolds has non-zero signature, then the universal covering $\tilde{X}$ admits a non-zero square summable harmonic $2k$-form.

This L_2-index theorem was extended to *foliatated spaces with transversal measures* (and eventually without such a measure) by Alain Connes, where the two basic manifolds' attributes– the smooth structure and the measure – are separated: the *smooth structures in the leaves* allow differential operators while the *transversal measures* underly integration and where the two cooperate in the "non-commutaive world" of Alain Connes.

(Possibly there is a similar non-linear analysis on foliation, where solutions of, e.g. parabolic Hamilton-Ricci for $3D$ and of elliptic Yang-Mills/Seiberg-Witten for $4D$, equations fast, e.g. L_2, decay on each leaf where "decay" for non-linear objects may refer to a decay of distances between pairs of objects.)

Linear operators are difficult to delinearize keeping them topologically interesting. The two exceptions are the *Cauchy-Riemann operator* and the signature operator in dimension 4. The former is used by Thurston (starting from late 70s) in his $3D$-geometrization theory and the latter, in the form of the *Yang-Mills equations*, begot Donaldson's $4D$-theory (1983) and the Seiberg-Witten theory (1994).

The logic of Donaldson's approach has some similarity to that of the index theorem. Yet, his operator $D : \Phi \to \Psi$ is *non-linear* Fredholm and instead of the index he studies the bordism-like invariants of (finite dimensional!) pullbacks $D^{-1}(\psi) \subset \Phi$ of suitably generic ψ.

These invariants for the Yang-Mills and Seiberg-Witten equations unravel an incredible richness of the smooth $4D$-topological structures which remain invisible from the perspectives of pure topology" and/or of linear analysis.

The non-linear Ricci flow equation of Richard Hamilton, the parabolic relative of Einstein, does not have any built-in topological intricacy; it is similar to the plain heat equation associated to the ordinary Laplace operator. Its potential role is not in exhibiting new structures but, on the contrary, in showing that these do not exist by ironing out all geometric bumps of Riemannian metrics. This potential was realized in dimension 3 by Perelman in 2003:

The Ricci flow on Riemannian 3*-manifolds, when manually redirected at its singularities, eventually brings every closed Riemannian* 3*-manifold to a canonical geometric form predicted by Thurston.*

There is hardly anything in common between the proofs of Smale and Perelman of the Poincare conjecture. Why the statements look so similar? Is it the same "Poincare conjecture" they have proved? Probably, the answer is "no".

To get a perspective let us look at another, seemingly remote, fragment of mathematics – the theory of algebraic equations, where the numbers $2, 3$ and 4 also play an exceptional role.

If topology followed a contorted path 2→5...→4→3, algebra was going straight 1→2→3→4→5... and it certainly did not stop at this point.

Thus, by comparison, the Smale-Browder-Novikov theorems correspond to non-solvability of equations of degree ≥ 5 while the present day $3D$- and $4D$-theories are brethren of the magnificent formulas solving the equations of degree 3 and 4.

What does, in topology, correspond to the Galois theory, class field theory, the modularity theorem... ?

Is there, in truth, anything in common between this algebra/arthmetic and geometry?

It seems so, at least on the surface of things, since the reason for the particularity of the numbers 2, 3, 4 in both cases arises from the same formula:

$$4 =_3 2 + 2 :$$

a 4 element set has exactly 3 partitions into two 2-element subsets and where, observe $3 < 4$. No number $n \geq 5$ admits a similar class of decompositions.

In algebra, the formula $4 =_3 2 + 2$ implies that the alternating group $A(4)$ admits an epimorphism onto $A(3)$, while the higher groups $A(n)$ are simple non-Abelian.

In geometry, this transforms into the splitting of the Lie algebra $so(4)$ into $so(3) \oplus so(3)$. This leads to the splitting of the space of the 2-forms into self-dual and anti-self-dual ones which underlies the Yang-Mills and Seiberg-Witten equations in dimension 4.

In dimension 2, the group $SO(2)$ "unfolds" into the geometry of Riemann surfaces and then, when extended to $homeo(S^1)$, brings to light the conformal field theory.

In dimension 3, Perelman's proof is grounded in the infinitesimal $O(3)$-symmetry of Riemannian metrics on 3-manifolds (which is broken in Thurston's theory and even more so in the high dimensional topology based on surgery) and depends on the irreducibility of the space of traceless curvature tensors.

It seems, the geometric topology has a long way to go in conquering high dimensions with all their symmetries.

11 Crystals, Liposomes and Drosophila.

Many geometric ides were nurtured in the cradle of manifolds; we want to follow these ideas in a larger and yet unexplored world of more general "spaces".

Several exciting new routes were recently opened to us by the high energy and statistical physics, e.g. coming from around the string theory and non-commutative geometry – somebody else may comment on these, not myself. But there are a few other directions where geometric spaces may be going.

Infinite Cartesian Products and Related Spaces. A crystal is a collection of identical molecules $mol_\gamma = mol_0$ positioned at certain sites γ which are the elements of a discrete (crystallographic) group Γ.

If the space of states of each molecule is depicted by some "manifold" M, and the molecules do not interact, then the *space X of states* of our "crystal" equals the the Cartesian power $M^\Gamma = \times_{\gamma\in\Gamma} M_\gamma$.

If there are inter-molecular constrains, X will be a subspace of M^Γ; furthermore, X may be a quotient space of such a subspace under some equivalence relation, where, e.g. two states are regarded equivalent if they are indistinguishable by a certain class of "measurements".

We look for mathematical counterparts to the following physical problem. Which properties of an individual molecule can be determined by a given class of measurement of the whole crystal?

Abstractly speaking, we start with some category $\mathcal{M}$ of "spaces" M with Cartesian (direct) products, e.g. a category of finite sets, of smooth manifolds or of algebraic manifolds over some field. Given a countable group Γ, we enlarge this category as follows.

Γ-*Power Category* $\Gamma^{\mathcal{M}}$. The objects $X \in \Gamma^{\mathcal{M}}$ are projective limits of finite Cartesian powers M^Δ for $M \in \mathcal{M}$ and finite subsets $\Delta \subset \Gamma$. Every such X is naturally acted upon by Γ and the admissible morphisms in our Γ-category are Γ-*equivariant* projective limits of morphisms in $\mathcal{M}$.

Thus each morphism, $F : X = M^\Gamma \to Y = N^\Gamma$ is defined by a single morphism in $\mathcal{M}$, say by $f : M^\Delta \to N = N$ where $\Delta \subset \Gamma$ is a *finite* (sub)set.

Namely, if we think of $x \in X$ and $y \in Y$ as M- and N-valued valued functions $x(\gamma)$ and $y(\gamma)$ on Γ then the value $y(\gamma) = F(x)(\gamma) \in N$ is evaluated as follows:

translate $\Delta \subset \Gamma$ to $\gamma\Delta \subset \Gamma$ by γ, restrict $x(\gamma)$ to $\gamma\Delta$ and apply f to this restriction $x|\gamma\Delta \in M^{\gamma\Delta} = M^\Delta$.

In particular, every morphism $f : M \to N$ in $\mathcal{M}$ tautologically defines a morphism in $\mathcal{M}^\Gamma$, denoted $f^\Gamma : M^\Gamma \to N^\Gamma$, but $\mathcal{M}^\Gamma$ has many other morphisms in it.

Which concepts, constructions, properties of morpisms and objects, etc. from $\mathcal{M}$ "survive" in $\Gamma^{\mathcal{M}}$ for a given group Γ? In particular, what happens to topological invariants which are multiplicative under Cartesian products, such as the Euler characteristic and the signature?

For instance, let M and N be manifolds. Suppose M admits *no topological embedding* into N (e.g. $M = S^1$, $N = [0,1]$ or $M = \mathbb{R}P^2$, $N = S^3$). When does M^Γ admit an injective morphism to N^Γ in the category $\mathcal{M}^\Gamma$?

(One may meaningfully reiterate these questions for *continuos* Γ-equivariant maps between Γ-Cartesian products, since *not all* continuos Γ-equivariant maps lie in $\mathcal{M}^\Gamma$.)

Conversely, let $M \to N$ be a map of non-zero degree. When is the corresponding map $f^\Gamma : M^\Gamma \to N^\Gamma$ equivariantlty homotopic to a *non-sujective* map?

Γ-*Subvarieties*. Add new objects to $\mathcal{M}^\Gamma$ defined by equivariant systems of equations in $X = M^\Gamma$, e.g. as follows.

Let M be an algebraic variety over some field $\mathbb{F}$ and $\Sigma \subset M \times M$ a subvariety, say, a generic algebraic hypersurface of bi-degree (p,q) in $\mathbb{C}P^n \times \mathbb{C}P^n$.

Then every directed graph $G = (V, E)$ on the vertex set V defines a subvariety, in M^V, say $\Sigma(G) \subset M^V$ which consists of those M-valued functions $x(v)$, $v \in V$, where $(x(v_1), x(v_2)) \in \Sigma$ whenever the vertices v_1 and v_2 are joined by a directed edge $e \in E$ in G. (If $\Sigma \subset M \times M$ is symmetric for $(m_1, m_2) \leftrightarrow (m_2, m_1)$, one does not need directions in the edges.)

Notice that even if Σ is non-singular, $\Sigma(G)$ may be singular. (I doubt, this ever happens for generic hypersurfaces in $\mathbb{C}P^n \times \mathbb{C}P^n$.) On the other hand, if we have a "sufficiently ample" *family* of subvarietis Σ in $M \times M$ (e.g. of (p,q)-hypersurfaces in $\mathbb{C}P^n \times \mathbb{C}P^n$) and, for each $e \in E$, we take a generic representative $\Sigma_{gen} = \Sigma_{gen}(e) \subset M \times M$ from this family, then the resulting generic subvariety in $M \times M$, call it $\Sigma_{gen}(G)$ is non-singular and, if $\mathbb{F} = \mathbb{C}$, its topology does not depend on the choices of $\Sigma_{gen}(e)$.

We are manly interested in $\Sigma(G)$ and $\Sigma_{gen}(G)$ for infinite graphs G with a cofinite action of a group Γ, i.e. where the quotient graph G/Γ is finite. In particular, we want to understand "infinite dimensional (co)homology" of these spaces, say for $\mathbb{F} = \mathbb{C}$ and the "cardinalities" of their points for finite fields $\mathbb{F}$ (see [4] for some results and references). Here are test questions.

Let Σ be a hypersurface of bi-degree (p,q) in $\mathbb{C}P^n \times \mathbb{C}P^n$ and $\Gamma = \mathbb{Z}$. Let $P_k(s)$ denote the Poincare polynomial of $\Sigma_{gen}(G/k\mathbb{Z})$, $k = 1, 2, ...$ and let

$$P(s,t) = \sum_{k=1}^{\infty} t^k P(s) = \sum_{k,i} t^k s^i rank(H_i(\Sigma_{gen}(G/k\mathbb{Z})).$$

Observe that the function $P(s,t)$ depends only on n, and (p,q).

Is $P(s,t)$ meromorphic in the two complex variables s and t? Does it satisfy some "nice" functional equation?

Similarly, if $\mathbb{F} = \mathbb{F}_p$, we ask the same question for the generating function in two variables counting the $\mathbb{F}_{p^l}$-points of $\Sigma(G/k\mathbb{Z})$.

Γ-*Quotients.* These are defined with equivalence relations $R \subset X \times X$ where R are subobjects in our category.

The transitivity of (an *equivalence* relation) R, and it is being a *finitary* defined sub-object are hard to satisfy simultaneously. Yet, *hyperbolic dynamical systems* provide encouraging examples at least for the category $\mathcal{M}$ of *finite sets.*

If $\mathcal{M}$ is the category of finite sets then subobjects in $\mathcal{M}^\Gamma$, defined with subsets $\Sigma \subset M \times M$ are called *Markov* Γ-*shifts.* These are studied, mainly for $\Gamma = \mathbb{Z}$, in the context of *symbolic dynamics* [27], [5].

Γ-*Markov quotients* Z of Markov shifts are defined with equivalence relations $R = R(\Sigma') \subset Y \times Y$ which are Markov subshifts. (These are called *hyperbolic* and/or *finitely presented* dynamical systems [13], [14].)

If $\Gamma = \mathbb{Z}$, then the counterpart of the above $P(s,t)$, now a function only in t, is, essentially, what is called the ζ-*function* of the dynamical system which counts the number of periodic orbits. It is shown in [13] with a use of *(Sinai-Bowen) Markov partitions* that this function is rational in t for all $\mathbb{Z}$-Markov quotient systems.

The local topology of Markov quotient (unlike that of shift spaces which are Cantor sets) may be quite intricate, but some are *topological manifolds.*

For instance, classical *Anosov systems* on *infra-nilmanifolds* V and/or *expanding endomorphisms* of V are representable as a $\mathbb{Z}$- Markov quotient via Markov partitions [22].

Another example is where Γ is the fundamental group of a closed n-manifold V of negative curvature. The *ideal boundary* $Z = \partial_\infty(\Gamma)$ is a topological $(n-1)$-sphere with a Γ-action which admits a Γ-Markov quotient presentation [14].

Since the topological S^{n-1}-bundle $S \to V$ associated to the universal covering, regarded as the principle Γ bundle, is, obviously, isomorphic to the unit tangent bundle $UT(V) \to V$, the Markov presentation of $Z = S^{n-1}$ defines the topological Pontryagin classes p_i of V in terms of Γ.

Using this, one can reduce the homotopy invariance of the Pontryagin classes p_i of V to the ε-*topological* invariance.

Recall that an ε-*homeomorphism* is given by a pair of maps $f_{12} : V_1 \to V_2$ and $f_{21} : V_2 \to V_1$, such that the composed maps $f_{11} : V_1 \to V_1$ and $f_{22} : V_2 \to V_2$ are ε-close to the respective identity maps for some metrics in V_1, V_2 and a small $\varepsilon > 0$ depending on these metrics.

Most known proofs, starting from Novikov's, of invariance of p_i under homeomorphisms equally apply to ε-homeomorphisms.

This, in turn, implies the *homotopy* invariance of p_i if the homotopy can be "rescaled" to an ε-homotopy.

For example, if V is a *nil-manifold* $\tilde{V}/\Gamma$, (where $\tilde{V}$ is a nilpotent Lie group homeomorphic to $\mathbb{R}^n$) with an *expanding endomorphism* $E : V \to V$ (such a V is a $\mathbb{Z}$-Markov quotient of a shift), then a large negative power $\tilde{E}^{-N} : \tilde{V} \to \tilde{V}$ of the lift $\tilde{E} : \tilde{V} \to \tilde{V}$ brings any homotopy close to identity. Then the ε-topological invariance of p_i implies the homotopy invariance for these V. (The case of $V = \mathbb{R}^n/\mathbb{Z}^n$ and $\tilde{E} : \tilde{v} \to 2\tilde{v}$ is used by Kirby in his topological *torus trick.*)

A similar reasoning yields the homotopy invariance of p_i for many (manifolds with fundamental) groups Γ, e.g. for *hyperbolic groups.*

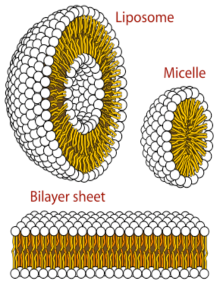
Questions. Can one effectively describe the local and global topology of Γ-Markov quotinets Z in combinatorial terms? Can one, for a given (e.g. hyperbolic) group Γ, "classify" those Γ-Markov quotients Z which are *topological manifolds* or, more generally, *locally contractible* spaces?

For example, can one describe the classical Anosov systems Z in terms of the combinatorics of their $\mathbb{Z}$-Markov quotient representations? How restrictive is the assumption that Z is a topological manifold? Is the topology of the local dynamics at the periodic points in Z essential?

Liposomes and *Micelles* are surfaces of *membranes* surrounded by water which are assembled of rod-like (phospholipid) molecules oriented normally to the surface of the membrane with *hydrophilic* "heads" facing the exterior and the interior of a cell while the *hydrophobic* "tails" are buried inside the membrane.

These surfaces satisfy certain partial differential equations of rather general nature (see [17]). If we heat the water, membranes dissolve: their constituent molecules become (almost) randomly distributed in the water; yet, if we cool the solution, the surfaces and the equations they satisfy re-emerge.

Question. Is there a (quasi)-canonical way of associatiing statistical ensembles $\mathcal{S}$ to geometric system S of PDE, such that the equations emerge at low



temperatures T and also can be read from the properties of high temperature states of $\mathcal{S}$ by some "analytic continuation" in T?

The architectures of liposomes and micelles in an ambient space, say W, which are composed of "somethings" normal to their surfaces $X \subset W$, are reminiscent of Thom-Atiyah representation of submanifolds with their normal bundles by generic maps $f_\bullet : W \to V_\bullet$, where $V_\bullet$ is the Thom space of a vector bundle V_0 over some space X_0 and where manifolds $X = f_\bullet^{-1}(X_0) \subset W$ come with their normal bundles induced from the bundle V_0.

The space of these "generic maps" $f_\bullet$ looks as an intermediate between an individual "deterministic" liposome X and its high temperature randomization. Can one make this precise?

Poincare-Sturtevant Functors. All what the brain knows about the geometry of the space is a flow S_{in} of electric impulses delivered to it by our sensory organs. All what an alien browsing through our mathematical manuscripts would directly perceive, is a flow of symbols on the paper, say G_{out}.

Is there a natural functorial-like transformation $\mathcal{P}$ from sensory inputs to mathematical outputs, a map between "spaces of flows" $\mathcal{P} : \mathcal{S} \to \mathcal{G}$ such that $\mathcal{P}(S_{in})$"="G_{out}?

It is not even easy to properly state this problem as we neither know what our "spaces of flows" are, nor what the meaning of the equality "=" is.

Yet, it is an essentially mathematical problem a solution of which (in a weaker form) is indicated by Poincare in [39]. Besides, we all witness the solution of this problem by our brains.

An easier problem of this kind presents itself in the classical genetics.

What can be concluded about the geometry of a genome of an organism by observing the phenotypes of various representatives of the same species (with no molecular biology available)?

This problem was solved in 1913, long before the advent of the molecular biology and discovery of DNA, by 19 year old Alfred Sturtevant (then a student in T. H. Morgan's lab) who reconstructed *the linear structure* on the set of genes on a chromosome of Drosophila melanogaster from *samples of a probability measure* on the space of gene linkages.

Here mathematics is more apparent: geometry of a space X is represented by something like a measure on the set of subsets in X; yet, I do not know how to formulate clear-cut mathematical questions in either case (compare [16], [18]).

Who knows where manifolds are going?

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