Two Lectures on ???

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1 From Things to Math

La mathématique est l'art de donner le même nom à des choses différentes.
HENRI POINCARÉ

Nothing in Probability Theory makes sense except in the light of ... CHOSES
THEODOSIUS DOBZHANSKY (misquoted).

"CHOSES" ? — Never heard of these.
MATHEMATICIAN.

99% of probability theory are outgrowths of mathematical models of "real life" phenomena.

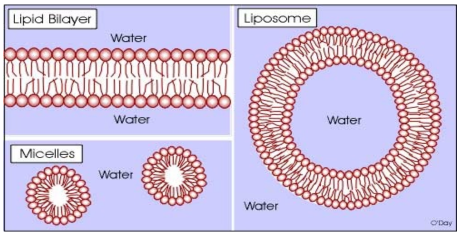


Figure 1: Geometric measure theory

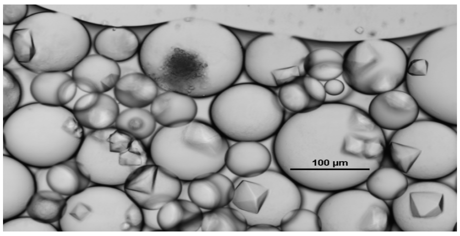


Figure 2: Crystal

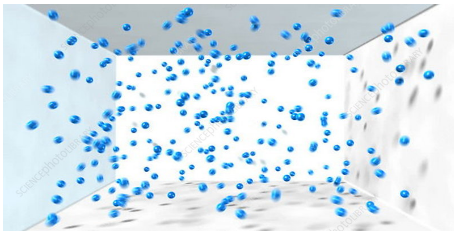


Figure 3: Gaz

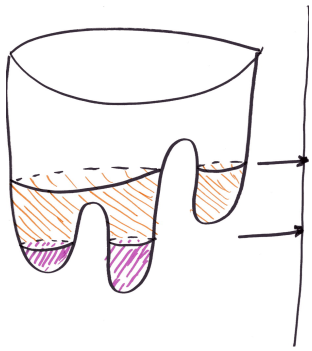


Figure 4: Protein Folding by Percolation

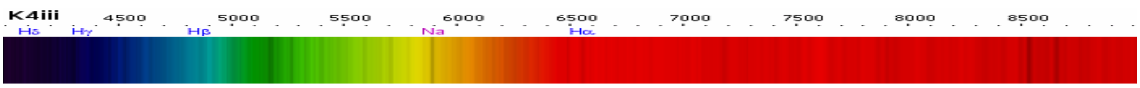


Figure 5: spectrum

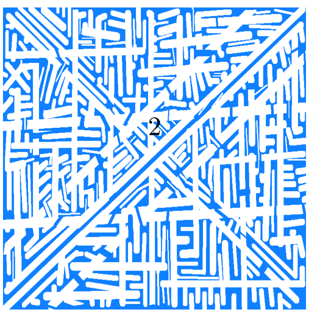


Figure 6: Self avoiding random walk, Flory (1953).

Such a model is a function (functor?) in two variables: the "real life" object ("chose") and the mathematical/psycological/historical) background of the one who models.

(Intuition: Heuristics, Naturality, Functoriality)

Synaptic pruning is elimination of extra synapses during brain development .

Mathematics is the immune system of science

Biologist

A study led by Massachusetts General Hospital (MGH) investigators finds evidence that the process of synaptic pruning, a normal part of brain development during adolescence, is excessive in individuals with schizophrenia.

....

Maxwell-Mendel (1860) Boltzmann, (1890)

... small compound bodies...

Figure 7: 1713

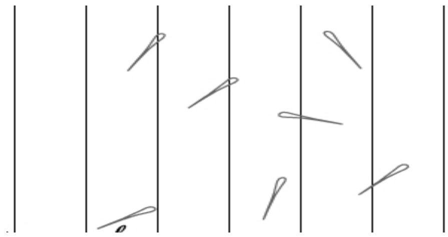


Figure 8: Buffon (1877) → Kolmogorov (1933)

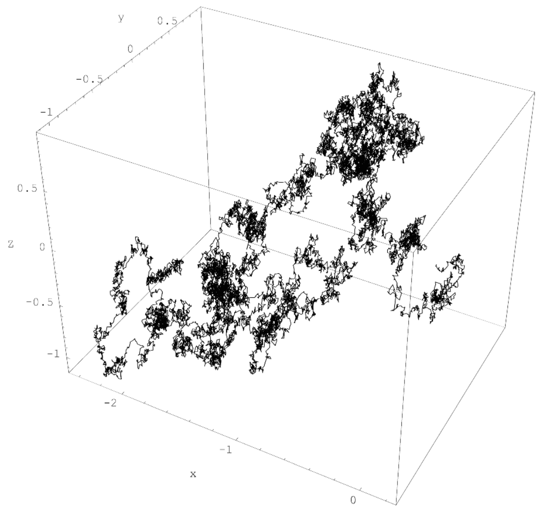


Figure 9: Ingenhousz' (1785) → Einstein&Smoluchowski (1905) → Perrin (1908)

are set in perpetual motion
by the impact of invisible blows.
The movement mounts up
from the atoms
and gradually emerges
to the level of our senses.

—

The true logic of this world is in the calculus of probabilities.

James Clerk Maxwell

Boltzmann equation, functoriality, Enskog-Chapman and the BBGKY hierarchy,

Physical Chemistry of Polymeres. Protein folding gelation and percolation

Flory–Stockmayer theory of the cross-linking and gelation of stepgrowth polymers(1941-46). Broadbent & Hammersley (1957),

Evolution Biology.

Natural Languges.

....since most of the 'normal sentences' of daily life are uttered for the first time in the experience of the speaker-hearer ... they will have had probability zero before this utterance was produced...

...the notion "probability of a sentence" is an entirely useless one, under any known interpretation of this term.

... probabilistic models give no particular insight into some of the basic problems of syntactic structure. NOAM CHOMSKY

Probability of finding 10^{24} air molecules in a definite region of one half of the available volume in a box,[1] is prohibitively small, of order $2^{-10^{24}}$ and the value of this probability may fluctuate with a huge factor, say, $> 2^{10^{24}-10^{24-1/10^{15}}} > 2^{10^6}$.

These numbers, the *values of probabilities* of micro-states of ensembles of particles, are *physically meangless.*

This doesn't bother a physicist. He/she boldly *assumes* that these numbers are all *mutually equal* and derives from this and a few similar *assumtions* physically sound conclusions.

It is symmetry, not any idea of "measure of underminancy", which makes the concept of probability to work so beatifully in physics.

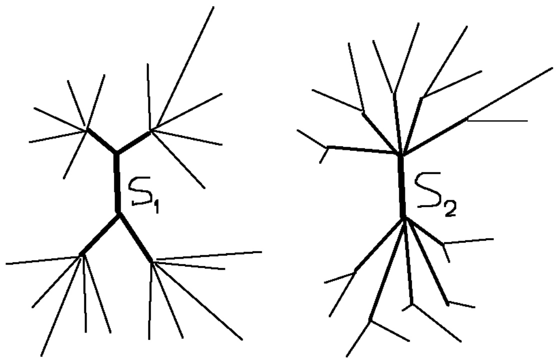
[1]A cubic meter of air contains about $2.5 \cdot 10^{25}$ molecules.

But to be applicable to *heterogeneous structures* of "non-physical worlds", e.g. in the world of languages, the traditional probabilistic formalism must be *limited and modified* in several ways.

For instance, the concept of *independency* must be reinterpreted for such structures and product formulas, e.g. the chain rule $P(A\&B) = P(A|B) \cdot P(A)$, must be used with moderation:

unrestricted iteration of such formulas leads to an accumulation of errors, which renders results unacceptable, even, where, which is rare in languages, these $P(A|B)$ and $P(A)$ themselves are unambiguously defined.[2]

[2] If each consecutive word in the sentence *... probabilistic models...* is assigned probability $\approx \frac{1}{5}$ – this, albeit inaccurate, is meaningful – then the probability of the whole sentence will come up as *meaningless* $5^{-15} \approx \frac{1}{3 \cdot 10^{10}}$, where a minor perturbation, of $\frac{1}{5}$ to $\frac{1}{4}$ increases the result by a huge (>28) factor. See ?? section for more about it.



Winograd Schema Challenge:

Amazingly (for some), Google, which can't be blamed for understanding the concept of size, tells you with 100% certainty what are the antecedents of *it*, package or bag, in the following two sentences.

This package doesn't fit into my bag because it is too large.

This package doesn't fit into my bag because it is too small.

Indeed, the Google search returns:

>10 000 results for "if the pack-

age is too large" & "doesn't fit",

$<$ 10 results for "if the package is too small" & "doesn't fit",

$>$10 000 results for "if the bag is too small" & "doesn't fit",

$<$10 results for "if the bag is too large" & "doesn't fit",

Unambiguously, "doesn't fit" goes along with *small bags* and *large packages*, that is additionally confirmed by

$>$400 000 results for "fit" & "the bags are large",

$<$ 50 000 results for "fit" & "the bags are small".

How much can you trust Google's numbers? The following examples make you think.

"the package is too large" & "doesn't

fit"   13 000,

"the package is too small" & "doesn't fit"   16 000,

"this package is too large " & "doesn't fit" < 10,

"this package is too small " & "doesn't fit" < 10,

These numbers tell us, that

the basic probabilistic concepts: *frequency, correlation, entropy*, when applied to *recurrent linguistic patterns*, must be

interpreted entirely within the network of (quasi)equivalences between such

"patterns".

Exercise. Should one assign statistical significance to

11001001000011 1111 011 010101000100010 0001011 01 000 11000010001101 00110001001100011 0011000 10100

01011100000000110111 000001110011010001

in a signal that comes from the direction to Alpha Centauri?[3]

2 Randomization of Geometric structures

???

Example 1. Random Homology in Various Models of Percolation.

Example 2. Topology of Imbeddings and Hopeless SAW

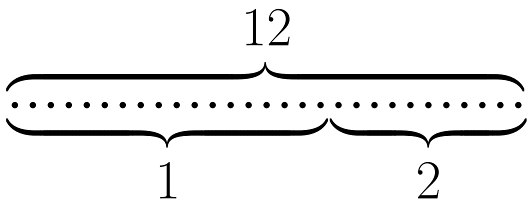
3 Between Imagination and Reality

Entropy: Bernoulli, Boltzmann, Shannon, Von Neumann Grothendieck,

Subadditivity of Entropy.

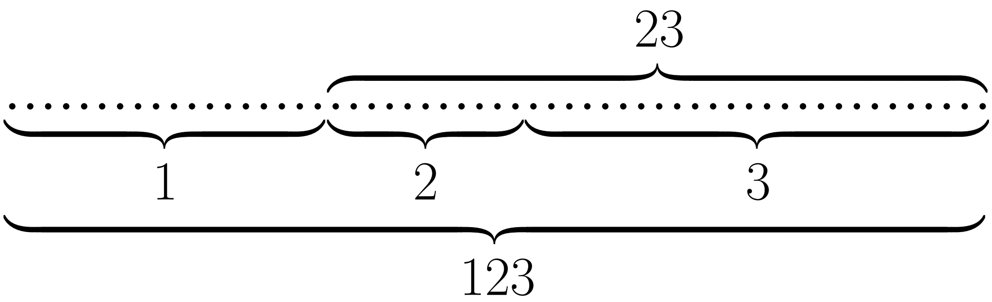
$$ent_{12} \leq ent_1 + ent_2$$

[3] These are the first 128 binary digits of π.



Strong Subadditivity of Entropy.

$$ent_{123} + ent_2 \leq ent_{12} + ent_{23}$$



LOOMIS-WHITNEY THEOREM.

Among all subsets $Y \subset \mathbb{R}^k$ with given measures of the projections to the k coordinate hyperplanes, the maximal measure is achieved by the rectangular solids (and all subsets obtained from them by measurable transformations of $\mathbb{R}^k$ preserving the coordinate line partitions).

This implies non-sharp isoperimetric inequality that for $k = 3$ reads:

The volume/measure of $Y \subset \mathbb{R}^3$, denoted vol_{123}, is bounded by the areas $area_{ij}$ of the there coordinate planar projections of Y as follows

$$vol_{123}^2 \leq area_{12} \cdot area_{13} \cdot area_{23}.$$

Linearized Loomis-Whitney $3D$*-isoperimetric inequality* for ranks of bilinear forms associated with a 4-linear form $\Phi = \Phi(s_1, s_2, s_3, s_4)$ where we denote $|...| = rank(...)$:

$$|\Phi(s_1, s_2 \otimes s_3 \otimes s_4)|^2 \leq$$

$$|\Phi(s_1 \otimes s_2, s_3 \otimes s_4)| \cdot |\Phi(s_1 \otimes s_3, s_2 \otimes s_4)| \cdot$$

$$\cdot |\Phi(s_1 \otimes s_4, s_2 \otimes s_3)|$$

Grothendieck Semigroup $Gr(\mathcal{P})$, *Bernoulli isomorphism* $Gr(\mathcal{P}) = [1,\infty)^\times$ *and Entropy.*

Functorial representation of infinite probability spaces X by sets of finite partitions of X, that are sets $mor(X \to P)$, for all $P \in \mathcal{P}$ and defining Kolmogorov's dynamical entropy in these terms.

Fisher metric and von Neumann's Unitarization of Entropy.

Hessian $h = Hess(e)$, $e = e(p) = \sum_{i\in I} p_i \log p_i$, on the simplex $\triangle(I)$ is a Riemannian metric on $\triangle(I)$ where the *real moment* map $M_{\mathbb{R}}$: $\{x_i\} \to \{p_i = x_i^2\}$ is, up to 1/4-factor, an *isometry* from the positive "quadrant" of the unit Euclidean sphere onto $(\triangle(I), h)$.

P: positive quadratic forms on

the Euclidean space $\mathbb{R}^n$,

Σ: orthonormal frames $\Sigma = (s_1, ..., s_n)$,
$\underline{P}(\Sigma) = (\underline{p}_1, ..., \underline{p}_n)$, $\underline{p}_i = P(s_i)$,

$$ent_{VN(P)} = ent(P) = \inf_{\Sigma} ent(\underline{P}(\Sigma)).$$

LANFORD-ROBINSON, 1968. *The function $P \mapsto ent(P)$ is concave on the space of density states:*

$$ent\left(\frac{P_1 + P_2}{2}\right) \geq \frac{ent(P_1) + ent(P_2)}{2}.$$

Indeed, the classical entropy is a concave function on the simplex of probability measures on the set I, that is $\{p_i\} \subset \mathbb{R}^I_+, \sum_i p_i = 1$, and infima of familes of concave functions are concave.

Spectral definition/theorem:

$$ent_{VN}(P) = ent_{Shan}(spec((P)).$$

Symmetrization as Reduction and Quantum Superadditivity.

Lieb-Ruskai, 1973.

H and G: compact groups of unitary transformations of a finite dimensional Hilbert space S

P a state (positive semidefinite Hermitian form) on S.

If the actions of H and G commute,

then the von Neumann entropies of the G- and H-averages of P satisfy

$$ent(G * (H * P)) - ent(G * P) \leq ent(H * P) - ent(P).$$

On Algebraic Inequalities. Besides "unitarization" some Shannon inequalities admit linearization, where the first non-trivial instance of this is the following

linearized Loomis-Whitney $3D$*-isoperimetric inequality* for ranks of bilinear forms associated with a 4-linear form $\Phi = \Phi(s_1, s_2, s_3, s_4)$ where we denote $|...| = rank(...)$:

$$|\Phi(s_1, s_2 \otimes s_3 \otimes s_4)|^2 \leq$$

$$|\Phi(s_1 \otimes s_2, s_3 \otimes s_4)| \cdot |\Phi(s_1 \otimes s_3, s_2 \otimes s_4)| \cdot$$

$$\cdot |\Phi(s_1 \otimes s_4, s_2 \otimes s_3)|$$

4 Homology Instead of Probability

- Non-Parametric Packings.

Classically, one is concerned with *maximally dense* packings of spaces X by *disjoint balls*, rather than with the homotopy properties of families of moving balls in X.

The densest packing of $\mathbb{R}^n$ are known only for $n = 1, 2, 3, 8, 24$,where the densest packing are $\mathbb{Z}^n$-*periodic* (The case $n = 1$ is obvious, the case $n = 2$ is due to Lagrange, who proved that the optimal packing is the hexagonal one) and the case of $n = 3$, conjectured by Kepler was resolved by Thomas Hales.

In 2016 Maryna Viazovska proved that the E_8 *root lattice* is the densest sphere packing for $n = 8$ with the optimality of the *Leech lattice* in dimension 24 to follow (Cohn, Kumar, Miller, Radchenko and Viazovska)

Problem: Relate packing of $\mathbb{R}^n$ and of $\mathbb{R}^{n+m}$.

———————————————–

• • An "ensemble" $\mathcal{A} = \mathcal{A}(X)$ of (finitely or infinitely many) particles in a space X, e.g. in the Euclidean 3-space, is probabilistically characterised by

$$U \mapsto ent_U(\mathcal{A}) = ent(\mathcal{A}_{|U}),\ U \subset X,$$

that assigns the *entropies* of *the* U-*reductions* $\mathcal{A}_{|U}$ *of* $\mathcal{A}$, to all bounded open subsets $U \subset X$. (ent_U is "*the logarithm of the number of the states of* $\mathcal{E}$ *that are effectively observable from* U"),

Replace "*effectively observable number of states*" by

"*the number of significant degrees of freedom of ensembles of moving particles*"

Packings by r-Balls.

X is a metric space, $\mathcal{P} = \mathcal{P}_{I,r}(X) = \{x_i\} \subset X^I$, such that $dist(x_i, x_j) \geq d_{ij} = 2r$.

Covarinatly functoriality under *expanding* maps $X \to Y$ and

contravariant functoriality under *contracting* maps $f : X \to Y$.

Packings by Tubes motivated by $X = Y \times Z \to Y$. I-tuples of closed subsets $Z_i \in X$, such that mutual distances between them satisfy $dist(Z_i, Z_j) \geq d_{ij}$ and such that $Z_i \subset X$ support given *nonzero* homology classes h_i in X. (T. Richard *On the 2-systole of stretched enough positive scalar curvature metrics on* $S^2 \times S^2$, arXiv:2007.02705v2.)

Packing Energy and Morse Spectrum. $\rho(a) = \min_{x_i \neq x_j} dist(x_i, x_j)$,

$E(a) = \frac{1}{\rho(a)}$ or $E(a) = -\log \rho(a)$.

The homotopy significant (Morse) spectrum of an energy function $E : \mathcal{A} \to \mathbb{R}$, is the set of those values

$y \in \mathbb{R}$, where the homotopy type of the sublevel $E^{-1}(-\infty, y]$ undergoes an *irreversible change* d

Guth' Duality between Homology Spectra of Packings and of Cycles.

Example. Relation between the cohomology spectrum of E on $(X)^I$ and $(Y)^I$ for $Y \subset X$ e.g. $X = \mathbb{T}^n$ and $Y = \mathbb{T}^m \subset \mathbb{T}^n$. (Viazovska theorem)

Results.

1. *δ-Waist Inequality.* $f : S^n \to \mathbb{R}^{n-k}$ a continuous map. $\exists p \in \mathbb{R}^{n-k}$, such that $vol(U_\delta(f^{-1}(p) \geq vol(U_\delta(S^k)$ for all $\delta > 0$

2 **Guth' Steenrod Spectrum Theorem.** X be the space of m-dimensional submanifolds x in the n-ball V and $F(x) = vol_m(x)$.

Then the volume spectrum of this

F satisfies

$\lambda_i \leq const \cdot i^{\frac{1}{m+1}}$

and

$\lambda_i \geq const(\varepsilon) \cdot i^{\frac{1}{m+1}-\varepsilon}$ *for all* $\varepsilon > 0$.

Weyl law for the codimension 1 volume spectrum (Liokumovich-Marques-Neves)

If $m = n - 1$ then $\lambda_i \sim const_n \cdot i^{\frac{1}{n}}$.

Question 1. How much of the geometry of a space X, say with a metric or symplectic geometry, can be seen in the homotopies of spaces of packings of X by such U_i?

Question 2. Is there a good category of "abstract packing-like objects", that are not, a priori, associated to actual packings of geometric spaces?

• *Homotopy and Cohomotopy Energy Spectra.*

• *Homotopy Dimension, Cell Numbers and Cohomology Valued Measures.*

• *Infinite Packings and Equivariant Topology of Infinite Dimensional Spaces Acted upon by Non-compact Groups.*

• *Bi-Parametric Pairing between Spaces of Packings and Spaces of Cycles.*

• *Non-spherical Packings, Spaces of Partitions and Bounds on Waists.*

• *Symplecting Packings.*

• *Parametric coverings.*

Homology Measures

$$\Psi \supset D \mapsto \mu(D) = \mu^*(D;\Pi) =$$
$$\mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^* = H^*(\Psi;\Pi),$$

where Π is an Abelian (homology coefficient) group, e.g. a field $\mathbb{F}$, and $\mathbf{0}^{\smallsetminus *}(D;\Pi)$ is the *kernel* of the cohomology restriction homomorphism for the complement $\Psi \smallsetminus D \subset \Psi$,

$$H^*(\Psi;\Pi) \to H^*(\Psi \smallsetminus D;\Pi).$$

The set function

$$\mu^* : \{subsets \subset \Psi\} \to \{subgroups \subset H^*\}$$

is additive for the sum-of-subsets in H^* *and super-multiplicative*[4] *for the the* $\smile$*-product of ideals in the case where* Π *is a commutative ring*:

[4] This, similarly to *Shannon's subadditivity inequality*, implies the existence of "*thermodynamic limits*" of *Morse Entropies*, see [?].

[$\cup +$]
$\mu^*(D_1 \cup D_2) = \mu^*(D_i) + \mu^*(D_2)$
for *disjoint* open subsets D_1 and D_2 in Ψ, and
[$\cap \smile$]
$\mu^*(D_1 \cap D_2) \supset \mu^*(D_1) \smile \mu^*(D_2)$
for all open $D_1, D_2 \subset \Psi$.

Homology spectra on spaces of infinitely many particles in non-compact manifolds

Infinite dimensional space Ψ,
action of an infinite group Υ on Ψ.

Example. Υ is a countable group call it Γ, e.g. $\Gamma = \mathbb{Z}^n$, and $\Psi = B^\Gamma$ is the space of maps $\Gamma \to B$.

H^* is a graded algebra (over some field) acted upon by a countable amenable group Γ.

Exhaust Γ by finite *Følner subsets* $\Delta_i \subset \Gamma$, $i = 1, 2, ...$, and, given a finite dimensional graded subalgebra $K = K^* \subset H^*$, let $P_{i,K}(t)$ denote the Poincare polynomial of the graded subalgebra in H^* generated by the γ-transforms $\gamma^{-1}(K) \subset H^*$ for all $\gamma \in \Delta_i$.

Define *polynomial entropy* of the action of Γ on H^* as follows.

$$Poly.ent(H^* : \Gamma) = \sup_K \lim_{i\to\infty} \frac{1}{card(\Delta_i)} \log P_{i,K}$$

(Permutation Symmetries and Equivariant Homology)

?*Energy* $\rightsquigarrow$ *Boltzmann distribution*?

Sup$_\vartheta$*-Spectra, Scalar Curvature and Spaces of Symplectic Packings.*

Θ is a set of metrics ϑ on a topological space X

"Inv"(X,ϑ) is an invariant,

sup_ϑ"Inv" is *the supremum* of the invariants "$Inv : (X,\vartheta)$ over all $\vartheta \in \Theta$.

Example 1. $X = S^3$ and ϑ are metrics with $Sc(\vartheta) \geq 6$.

Then $sup_\vartheta waist_2 = 4\pi$.(Marques-Neves)

Example 2. $X = (X,\omega)$ a symplectic manifold of dimension $n = 2m$ and ϑ are ω-adapted metrics

Question. Which part of the (suitably factorized/coarsened) homotopy/homology area spectra of (X,ϑ) remains finite after taking suprema over ϑ?

If $k = 2$, then upper bounds in

some cases are obtained with pairing ball pacing with "psedoholomorphic curves" defined here as oriented surfaces $Y \subset X = (X, \omega, \vartheta)$, such that $area_{\vartheta}(Y) = \int_Y \omega$.

5 References

Maxwell, James (1860), "V. Illustrations of the dynamical theory of gases.—Part I. On the motions and collisions of perfectly elastic spheres", Philosophical Magazine, 19 (124): 19–32.

1891 percoltion

Paul J . Flory (1941) Molecular Size Distribution in Three Dimensional Polymers. I. Gelation'

Paul J . Flory (1974) Spatial Configuration of Macromolecular Chains

(Nobel Lecture)

Analogy and Difference between Gelation and Percolation Process Kazumi Suematsu

· Percolation Processes: Lower Bounds for the Critical Probability

JM Hammersley · 1957 ·

JÜRG FRÖHLICH C. E. PFISTER THOMAS SPENCER On the Statistical Mechanics of Surfaces

From Boltzmann to Euler: Hilbert's 6 th problem revisited Marshall Slemrod

Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits Stanislav SMIRNOV

Smirnov's Lecture `https://www.youtube.com/watch?v=6hCSSCV7lvQ`

HOMOLOGICAL PERCOLATION ON A TORUS: PLAQUETTES AND

PERMUTOHEDRA PAUL DUNCAN, MATTHEW KAHLE, AND BENJAMIN SCHWEINHART —

——————————- ——————————

———— NG:

In a Search for a Structure, Part11: On Entropy.

Geometry, Topology and Spectra of Non-Linear Spaces of Maps - Wolfgang Pauli Lectures

Bernoulli Lecture: Alternative Probabilities

Mathematics of life spaces

It is symmetry, not any idea of "measure of underminancy", which makes the concept of probability to work so beatifully in physics.

But to be applicable to *heterogeneous structures* of "non-physical worlds", e.g. in the world of languages, the traditional probabilistic formalism must be *limited and modified* in several ways.

For instance, the concept of *independency* must be reinterpreted for such structures and product formulas, e.g. the chain rule $P(A\&B) = P(A|B) \cdot P(A)$, must be used with moderation:

unrestricted iteration of such formulas leads to an accumulation of errors, which renders results unacceptable, even, where, which is rare in languages, these $P(A|B)$ and $P(A)$ themselves are unambiguously defined.[39]

Because of this, we shall be careful in saying *probabale/impobable* when it comes to sentences or even typical words in a languge; instead, we shall often use the concept of *feasibility*, where we start doing this prior to a (quasi)formal definition coming later on.

...combinatory play seems to be the essential feature in productive thought.
ALBERT EINSTEIN IN A LETTER TO JACQUES HADAMARD

Our models of language will be, at least overtly, *predominantly combinatorial*[40] rather than stochastic with no large numbers (and their reciproacals) in them. Yet, determination of adequate combinatorics will *rely on systematic counting observable linguistic patterns.*

Here is an example of how it goes.

WINOGRAD SCHEMA CHALLENGE

Amazingly (for some), Google, which can't be blamed for understanding the concept of size, tells you with 100% certainty what are the antecedents of *it* in the following two sentences.

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...the notion "probability of a sentence" is an entirely useless one, under any known interpretation of this term.

---

[34]A physicist's ambition is a theory based on few, say $k < 10$, basic rules with exponentially many, $k \rightsquigarrow k^n$, $n = 1, 2, 3, 4, 5, \ldots$. logically deducible empirical facts. In biology, one would be happy with $K \rightsquigarrow K^{n=2}$ for large (about million?) K with admissible 80% error rate. And in linguistic the realistic exponent, I guess, is $n = 3$.

[35]"Infinite", *"arbitrarily small"*, *"arbitrarily large"*, etc, are bona fide *mathematical* concepts. Their applications in physics (e.g. via differential equations) is (non-trivially) justifiable and (often inexplicably) successful.

But the idea of infinity applied to biology, psychology, linguistics and philosophy of AI, be it open or covert, is a common cause of speculative aberrations.

[36]10^9 words corresponds to a ten thousand average books, and 10^{14} to a hundred billion Google pages, where the number of all books in the world is of order 10^8 and the number of Google pages is about $3 \cdot 10^{13}$.

[37]This is, probably, how much the *inborn language learning programs/moduli*, most of them operating in parallel, occupy in the brains/subliminal minds of humans.