UNEDITED

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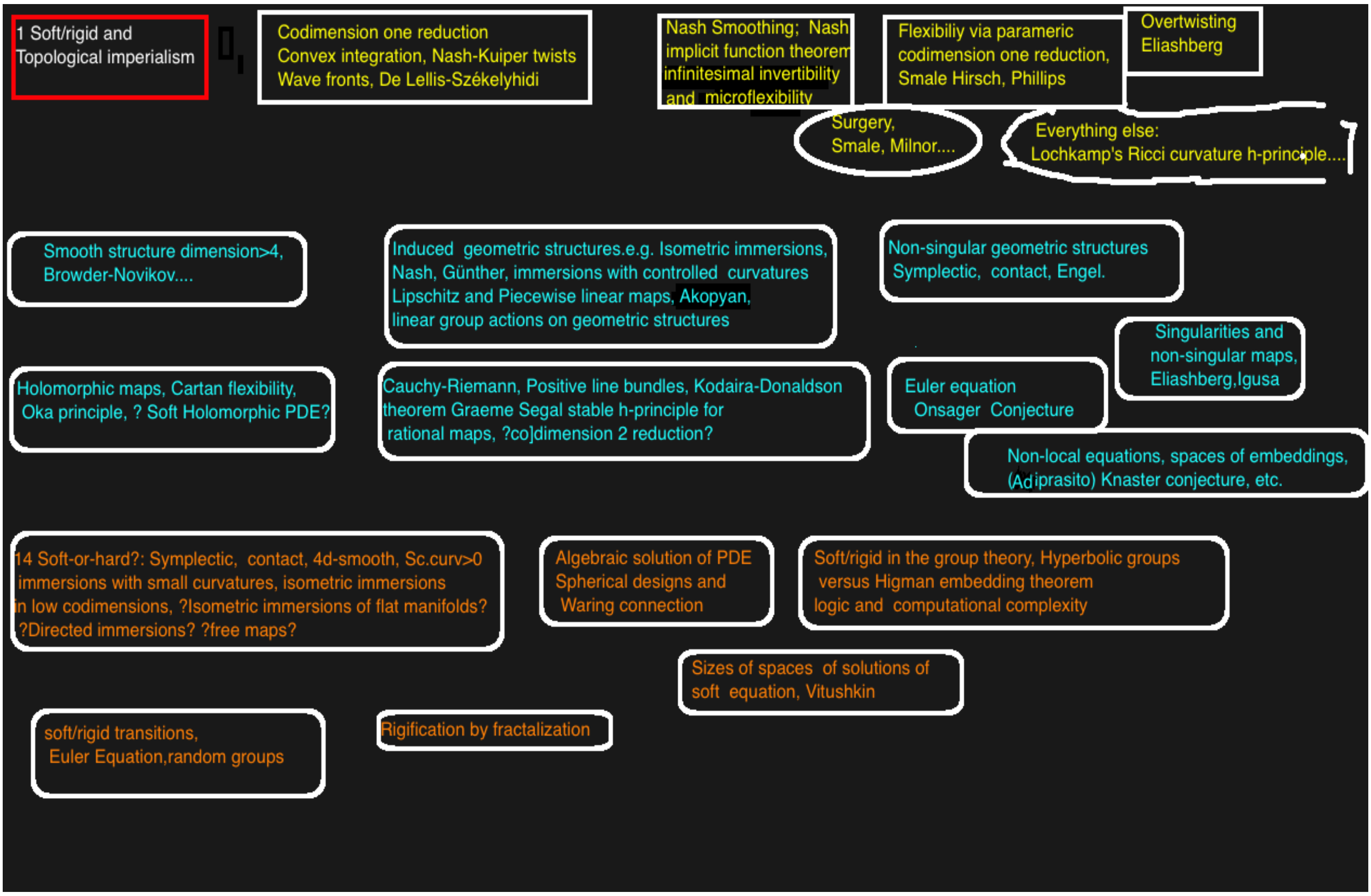
October 2, 2023

Contents



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18 ??? **47**

$\leadsto$

Solutions f of (systems of partial differential) equations E are expected to display varieties of global properties obtained by "integrating" the infinitesimal ones encoded by E.

But ever since the 1954 paper by Nash it was realized that there are "soft equations" E, which leave almost no trace on the global behaviour of their solutions: *almost all what remains "telescopically visible" in f of a presence of E is the homotopy properties of spaces of solutions resulting from pure algebra of equations in E, while "differential" fails to integrate to "global".*

Now we face the following.

Soft-versus-Rigid Problem. *Outline the softness domain $\mathcal{S}$ in the space $\mathcal{E}$ of all PDE and analyze equations on the borderline separating "rigid equations" from "soft ones".*

Experience shows that this borderline host most beautiful mathematics.

Random Historical Remarks

1909: Softness of solutions of certain diophantine, rather than differential, equations shows up in Hilbert's approach to the *Waring problem.*

1939: A "soft domain" in the complex analytic world was discovered by Oka.

1949: Onsager suggested Hölder $C^{\alpha<1/3}$-softness of the Euler equation as a reason for turbulence.

1954: Nash proved softness of isomeric C^1-immersions of Riemannian manifolds.

1958: Grauert proved the Oka h-principle for holomorphic maps from Stein manifolds to homogeneous spaces.

1959: Smale proved *flexibility of immersions in positive codimensions* and the **h**omotopy principle for immersions of spheres. Hirsch articulated and proved the general h-principle for immersions.

1967: Phillips proved the h-principle for submersions of open manifolds

1970: Eliashberg proved the h-principle for folded maps.

1974/76: Thurson proved the h-principle for foliations.

1993: Scheffer constructed *non-concervative* measurable weak solutions of the Euler equation.

1995: Lohkamp proved the h-principle for Riemannin metrics with $Ricci < 0$.

1996: Müller and Šverák Proved softness of Lipschitz solutions of certain non-linear elliptic equtions by *convex integration.*

1996: Donaldson proved an h-principle for almost holomorphic maps of symplectic manifols.

2009: De Lellis and L. Székelyhidi brought convex integration to the Euler Equation.

2014: Borman, Eliashberg and Murphy proved the h-principle for *overtwisted contact structures.*

2018: Isett constucted non-conservative $C^{\alpha<1/3}$-Hölder solutions of the Euler equation by convex integration. ....
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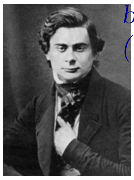
In fact it is *possible to show* that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition ... for any order n greater than 1/3; otherwise the energy is conserved (1949).[1]

My reflection, when I first made myself master of the central idea of [.....] was, 'How extremely stupid not to have thought of that!' I suppose that Columbus' companions said much the same when he made the egg stand on end (1888).[2]



Contents

1 C^1-Isometric Immersion Theorem

Preamble. Cauchy 1813, Minkowski 1903, Cohn-Fossen's 1927:

Rigidity Therem. *Closed convex surfaces*

$$X \subset \mathbb{R}^3,$$

[1] Lars Onsager

[2] Thomas Henry Huxley

e.g. unit spheres S^2, are C^2-rigid:

Isometric C^2-deformations of these are rigid motions,

where "isometric" means "preserving the lengths of all smooth curves in the sphere".

(A priori, modifications of the *Euclidean distances* between points in X are allowed, where one knows, for instance, that the half-sphere $S^2_+ \subset S^2$ admits many isometric C^∞-deformations, which do change all Euclidean distances in it. But no such deformation of the whole sphere is possible according the rigidity theorem.)

Folk Conjecture: $C^2 \ngtr C^1$.

For instance,

? the unit sphere $S^2 \subset \mathbb{R}^3$ can't be isometrically C^1-imbedded to the interior of the unit ball. ?

(C^2 is obvious by Gauss Theorema Egregium)

****

The ansver by Nash (Kuiper) 1954/55: Let

$$X = X(g_0) \text{ and } Y = (Y, h_0)$$

be Riemannian manifolds.

Euclidean Example. The papers by Nash and Kuiper were concerned with the case, where Y was a Euclidean space, while X could be *any* Riemannian manifold. But,

their arguments (almost) automatically *extend to non-Euclidean Y*,

while

the power of these arguments is fully displayed, where *both manifols are the ordinary Euclidean spaces*:

$$X = (\mathbb{R}^n, g_0 = g_{Eucl_n} = \sum_{i=1}^{n} dx_i^2)$$

and

$$Y = (\mathbb{R}^N, h_0 = g_{Eucl_N}) = \sum_{k=1}^{N} dy_k^2).$$

The Nash-Kuiper theorem claims the existence of of isometric C^1-maps and their deformations, where nothing of the kind is possible in the C^2-category.

Namely, let

$$f_0 : X \hookrightarrow Y$$

be a *g_0-isometric* C^1-embedding[3] where "isometric" signifies that the Riemannin metric induced on $X \overset{f_0}{\hookrightarrow} (Y, h_0)$ is equal to g_0:

[ISO] $$f_0^*(h_0) = g_0.$$

[3]This "C^1-imbedding" means that the image $f_0(X) \subset Y$ is a C_1-smooth submanifold and the map f_0 is a C^1-diffeomorphism on its image.

In general, opologists call a continuous map $f : X \to Y$ *embedding* if it is a homeomorphism from X to a subset in Y.

If X is compact, this is the same as being one-to -one (no double point), but, for instance, the obvious one-to-one immersion from the disjoint union of the (horizontal) closed segment $[-1/2, 1/2]$ and the (vertical) half-closed $(0, 2]$ onto the figure $\perp$ in the plane is not a topological embedding.

(This is the same "isometric" as above but now expressed in the infinitesimal terms.)

Euclidean Case. If $X = \mathbb{R}^n$ and $Y = \mathbb{R}^N$, then the [ISO] condition $f_0^*(g_{Eucl_N}) = g_{Eucl_n}$ says that the partial derivatives of the $\mathbb{R}^n$-valued vector-function $f_0(x_1, ..., x_n)$ have unit norms,

$$\left\| \frac{\partial f}{\partial x_i} \right\|_{\mathbb{R}^N} = 1, i = 1, ..., n,$$

and they are mutually orthogonal in $\mathbb{R}^N$,

$$\left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle_{\mathbb{R}^N} = 0, i \neq j.$$

Examples The simplest instance of an isometric embedding $\mathbb{R}^n \to \mathbb{R}^{N=n+k}$ is the standard one:

$$\mathbb{R}^n \ni (x_1, ..., x_n) \mapsto (x_1, ...x_n, \underbrace{0...0}_{k}) \in \mathbb{R}^N.$$

More interestingly, the map $\mathbb{R} \to \mathbb{R}^2$, where

$$y_1 = \sin(x) \text{ and } y_2 = \cos(x),$$

satisfies [ISO],(check it!) albeit it is not an embedding.

In fact this map sends the real line $\mathbb{R}$ onto the unit circle in the plane $\mathbb{R}^2$ and it defines an isomeric *embedding* of the quotient circle $\mathbb{R}/2\pi\mathbb{Z}$ to the plane.

Then one sees that the map $\mathbb{R}^n \to \mathbb{R}^{2n}$ given by n pairs of sin and cos,

$$(x_1, x_2, ..., x_n) \mapsto (\sin(x_1), \cos(x_1),$$

$$\sin(x_2), \cos(x_2), ..., \sin(x_n), \cos(x_1))$$

also satisfies [ISO].

Exercises. (a) Show that all C^1-smooth isometric maps $\mathbb{R}^n \to \mathbb{R}^n$, i.e. those preserving the lengths of curves, also preserve distances between all pairs of points.

(This fails to be true for non-C^1 maps, such as $x \mapsto |x|$ on the real line.)

(b) Construct an isometric C^∞-*embedding* of the real line to the unit disc in the plane and show that n-copies of such an embedding $\mathbb{R} \to B^2(1)$ define an isometric C^∞ -imbedding

$$\mathbb{R}^n \to B^{2n}(\sqrt{n}) \subset \mathbb{R}^{2n}.$$

Remark. One knows (Tompkins 1939) that if $N \leq 2n - 1$, then all isometric, i.e. satisfying [ISO], C^3-maps $\mathbb{R}^n \to \mathbb{R}^N$ have *unbounded* images.

But the Euclidean case of the Nash-Kuiper theorem as we shall see in **1**.A. below, delivers such C^1-maps for all $N \geq n + 1$.

Back to formulation of the general theorem, let

$$f_t : X \hookrightarrow Y, \ 0 \leq t \leq 1,$$

be an isotopy – a C^1-continuous family of C^1 embeddings.

For instance $f_t = (1-t)f_0$ for $f_0 : X \to Y = \mathbb{R}^N$.

Let g_t and h_t $0 \le t \le 1$, be continuous Riemannian metrics on X and on Y that are homotopies of g_0 and of h_0.

Example. The "no-homotopy" case, where $g_t = g_0$, $h_t = h_0$ for all t is already significant.

C^1-Isometric Approximation Theorem. *If the maps f_t for $t > 0$ are (g_t, h_t)- short,*

$$f_t^*(h_t) < g_t$$

and if

$$N = dim(Y) > n = dim(X),$$

then *there exists an isotopy of f_0 by (g_t, h_t)-isometric C^1-imbeddings arbitrarily C^0-close to f_t:*

$$f_{\delta,t} : X \to Y,\ f_{\delta,t}^*(h_t) = g_t,$$

$$dist_Y(f_{\delta,t}(x), f_t(x)) \le \delta(x,t).$$

for a given continuous function $\delta(x,t) > 0$. [4]

Amazing Corollaries

1.A. If $N \ge n+1$ then the map $f_t : \mathbb{R}^n \to \mathbb{R}^N$ for

$$(x_1, ..., x_n) \mapsto (tx_1, ..., tx_n, 0, ..., 0)$$

can be δ-approximated by isomeric C^1-imbeddings for all $t \in [0,1]$ and all $\delta = \delta(x) > 0$.

Thus, for example, $\mathbb{R}^n$ admits an isometric C^1-embedding to the unit ball in $\mathbb{R}^{n+1}$.

Exercise. Prove the C^∞-version of 1.A for $n = 1$.

1.B. *If a manifold X admits a topological C^1-embedding to $\mathbb{R}^N$ and if $dim(X) \le N-1$ then (X,g) also admits an* ISOMETRIC *C^1-embedding to $\mathbb{R}^N$ for all continuous Riemannian metrics g on X.* [5]

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WHAT?! : The equality

$$f^*(g_{Eucl}) = g$$

in local coordinaes on X and

$$g = \sum_{i \le j = 1, \ldots n} g_{ij}(x) dx_i dx_j$$

reads:

$$\sum_{k=1,\ldots,N} \frac{\partial f_k(x)}{\partial x_i} \cdot \frac{\partial f_k(x)}{\partial x_j} = g_{ij}(x).$$

These are

$s = \frac{n(n+1)}{2}$ equations

in N unknown functions.

NOT A CHANCE TO BE SOLVABLE FOR ALL *g if $N < s$,*

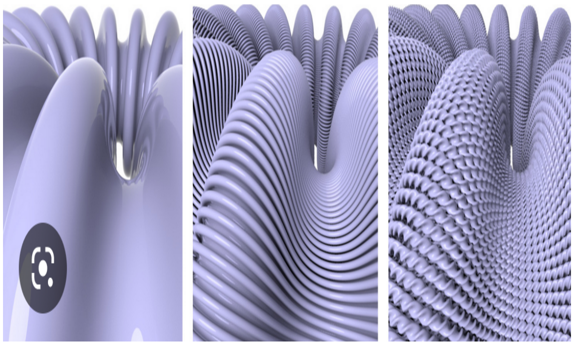
every PDE student knows this.

....

[4] John Forbes Nash Jr

[5] Derivation of his from C^1-isometric approximation theorem for compact manifolds X is achieved by an obvious homothetic scaling of a given embedding $X \to \mathbb{R}^N$ to a short one, while such a "scaling" for non-compact X requires an additional effort.

Science is the belief in the ignorance of experts.[6]



••••••••• OBVIOUS![7]
unprintable

1.1 Existence of (non-Isometric) Immersions

From the geo**metric** point of view the existence of a smooth, not necessary isometric, immersion or an embedding of an n-manifold X to the Euclidean space $\mathbb{R}^N$ may seem a trifle matter.

(Recall that a C^1-differentiable map $f : X \to Y$ is an *immersion* if the differential $df : T(X) \to T(Y)$ has everywhere *rank* n, i.e. if the linear maps $df(x) : T_x(X) \to T_{f(x)}(Y)$ are injective on all tangent spaces $T_x(X)$ of X or, equivalently, by the implicit function theorem, if f can be represented in some, depending on f, local coordinates $x_1, ..., x_n$ at all x in X and $y_1, ..., y_N$, $N = n + k$ at $f(x) \in Y$ by the standard linear embedding $(x_1, ..., x_n) \mapsto (x_1, ...x_n, \underbrace{0, ..., 0}_{N-n})$.)

In fact, in 1936 Whitney showed that *all C^∞ smooth n-manifolds X admit C^∞ -immersions to $\mathbb{R}^{2n}$.*

Here is the standard *proof*, simple and instructive

Step 1. Cover X by $n + 1$ mutually open subsets U_l, $l = 1, ..., n = 1$, where each U_i is the union of mutually disjoint very small subsets $B_{l,j} \subset X$, $j = 1, 2, ...$ (If X is compact here are finitely many of these $B_{l,j}$.)

[6] Richard Phillips Feynman

[7] Alfred Russel Wallace, Charles Robert Darwin

Step 2. Construct C^∞-smooth maps $f_{l,j} : X \to \mathbb{R}^n$, such that the restrictions of $f_{l,j}$ to $B_{l,j}$ are immersion, $B_{l,j} \to \mathbb{R}^n$, and such that the supports of f_{l,j_1} and $f_{l.j_2}$ are disjoint for all l and $j_1 \neq j_2$.

Step 3. Let

$$F_l = \sum_j fl, j$$

and observe that the map

$$F = (F_1, ...F_l, ...F_{n+1}) : X \to \underbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}_{n+1} =$$

$$= \mathbb{R}^{n(n+1)}$$

is an immersion.

Conclusion of the proof immediate by induction from the following.

Codimension one C^r-Projection Lemma. Let $F : X \to \mathbb{R}^N$ be a C^r-immersion, $r \geq 2$.

If $N \geq 2n + 1$, then there exists a 1-dimensional linear subspace $\Lambda \subset \mathbb{R}^N$, such that the composition of F with he quotient map,

$$X \xrightarrow{F} \mathbb{R}^N \to \mathbb{R}^N/\Lambda = \mathbb{R}^{N-1}$$

is an C^r-immersion $X \to \mathbb{R}^{N-1}$.

Proof. Let $U_F T(X) \subset T(X)$ be the set of tangent vectors $\tau \in T(X)$, which have unit lengt in $\mathbb{R}^N$,

$$\|dF(\tau)\| = 1$$

and observe that $U_F T(X)$ is a C^{r-1}-smooth $(2n - 1)$-dimensional submanifold in the (C^∞-smooth) tangent bundle $T(X)$. (One loses here one degree of differentiability, since $U_F T(X)$ is defined via the differential dF of F)

Transport the vectors $dF(\tau) \in T(\mathbb{R}^{N)}$ to the origin and thus obtain a C^{r-1}-map from $\tilde{U}_F(X)$ to the unit sphere, say

$$\tilde{T}_F : U_F(X) \to S^{N-1} \subset \mathbb{R}^N.$$

Since the map $\tilde{T}_F$ is (at least) C^1, the *dimension of the image of this map*

$$\tilde{T}_F(U_F(X)) \subset S^{N-1}$$

doesn't exceed $2n - 1 = dim(U_F T(X))$ (see below).

Thus, the map $\tilde{T}_F$ can't be onto for $N \geq 2n + 1$: there must exist a unit vector

$$\lambda \in S^{N-1} \setminus \tilde{T}_F(U_F(X));$$

then the line $\Lambda \subset \mathbb{R}^N$ spanned by this vector does the job.

About Dimension. The relevant dimension in the present context is the *Haussdorf dimension* that is defines for subsets $A \subset \mathbb{R}^M$ as the infimum of the numbers d, such that A can be covered by countably many balls of radii r_i, $i = 1, 2, ...,$, such that

$$\sum_i r_i^d < \infty.$$

In general, for subsets A in a smooth M-dimensional manifold Z, this is defined as the supremum of the Euclidean dim_{Haus} of the pullbacks of $A \subset Z$ under all smooth immersions $\mathbb{R}^M \to Z$.

It is an elementary exercise to check that

$dim_{Haus}(Z) = dim Z = dimHaus(\mathbb{R}^M)$ *for all smooth M-manifold Z* and that this dimension is non-increasing under C^1-maps $F : Z_1 \to Z_2$,

$$dim_{Haus}(f(A) \leq dim_{Haus}(A) \text{ for all } A \subset Z_1.$$

(This is a special case of Sard's theorem, which was refined by Yomdin as is briefly explained in section??? in PDR ???)

Exercises. (a) Show by adapting the above argument that *generic* C^2-maps $X^n \to Y^N$ are immersions for $N \geq 2n$, that is, such maps constitute an open *dense* set in the space $C^\infty(X, Y)$ of all C^∞ maps.[8]

(This also follows from the general Thom's transversality theorem.) (If X is non-compact and one insists on "open" one should use the fine topology in this space)

In 1944, Whitney proved that all n-manifolds of dimensions $n \geq 2$ can be smoothly immersed to $\mathbb{R}^{2n-2}$.

The proof is geometric and not very difficult but by no means obvious, while the follownig generalization, besides a use of an essenitally geometric Smale-Hirsch immersion theorem, heavily relies on algebraic topology.

Ralph Cohen's 1985-Solution of 1960 of William Massey's 1960-Conjecture. *All n-manifolds$X$ can be immersed to $\mathbb{R}^{2n-\alpha(n)}$, where $\alpha(n)$ is the number of 1's in the binary expansion of n.* (See`http://math.stanford.edu/~ralph/immersions-final.pdf`.)

1.2 Topological Obstructions to (non-Isometric) Immersions, Embeddings and other non-Singular Maps

The above theorem is optimal.

In fact, Massey proved in his 1960 paper that if $n = 2^{i_1} + ... + 2^{i_r}$, then the product of r real projective spaces,

$$X^n = \mathbb{R}P^{2^{i_1}} \times ... \times = \mathbb{R}P^{2^{i_r}}$$

can't be immersed to $\mathbb{R}^{2n-r-1}$.

(This follows from nonvanishing of the *normal Stiefel-Whitney class* $w^\perp_{2n-r}(\mathbb{X}^n)$ for this X^n.)

For instance (this goes back to Whitney) the projective space $\mathbb{R}P^n = S^n/\pm 1$, where $n = 2^i$, can't immersed to $\mathbb{R}^{2n-2}$.

We refer to the explanation of all this to `http://math.stanford.edu/~ralph/immersions-final.pdf` and limit ourself to the following illustration of an intervention of the algebraic topology in the immersion theory, which also helps us to understand Smale's h-principle.

.

[8]If X is non-compact and one insists on "open" one should use the so called*fine topology* in this space.

Start by recalling that, according to the Hirsch theorem, all smooth maps $f_0 : \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n+1$, can be *finely C^0-approximated by immersions $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^N$* that is, such that

$$\|f_0(x) - f_\varepsilon(x)\| \leq \varepsilon(x)$$

for a given continuous function $\varepsilon(x) > 0$, $x \in \mathbb{R}^n$.

But, as the following shows, some of these f can't be C^1-approximated by immersions,

Example. Let $X_0 \subset B^2(1) \subset \mathbb{R}^2$ be an annulus in the unit disc around the circle $S^1 = \partial B^2(1) \subset \mathbb{R}^2$ and let $f_0 : X_0 \to M_0 \subset \mathbb{R}^3$ be a double covering map of some Möbius strip in the space.

Then no smooth extension f of f_1 to a smooth map $f : B^2(1) \to \mathbb{R}^3$ admits a C^1-approximation by immersions.

Indeed, let us apply the differential df_1 of f_1 to the Euclidean coordinate 2-frame of tangent vectors on $\mathbb{R}^2$ restricted to $X_0 \subset \mathbb{R}^2$.

This defines a continuous map from X_0 to the of pairs of linearly independent (orthonormal for C^1-isomeric maps f) vectors in $\mathbb{R}^3$, call it

$$d_1 : X_) \to St_2(\mathbb{R}^3),$$

where $St_n(\mathbb{R}^N)$ denotes the space of n-tuples of linearly independent vectors in the Euclidean n-space.

Observe that $St_N(\mathbb{R}^N)$ is homeomorphic to the group $GL(N)$ of linear transformations of $\mathbb{R}^N$, since the natural action of $GL(N)$ on $St_N(\mathbb{R}^N)$ is free and transitive.

It is also clear that $St_{N_1}(\mathbb{R}^N)$ is homotopy equivalent to the *special linear group*) $SL(N) \subset GL(N)$ of orientation preserving linear transformations of $\mathbb{R}^N$, i.e. representable by $(N\times N)$-matrices with positive determinants $|x_{ij}|$, because, for given $(x_1, ..., x_{N-1} \in St_{N-1}(\mathbb{R}^N)$, the space of $x_N \in \mathbb{R}^N$, such that $|x_ij| > 0$, which is homeomorphic to the space of non-zero vectors in the halfspace $\mathbb{R}^N_+$, is (unlike $\mathbb{R}^N \setminus 0$) *contractible.*

Recall that the fundamental group of the *special linear group*) $SL(3)$ of linear transformations of $\mathbb{R}^3$ preserving orientation is

$$\pi_1(SL(3) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z},$$

where it is generated by the circle $S^1 = SO(2) \subset SO(3) \subset SL(3)$, and, observe, that the map d_1 applied to $S^1 \subset X$ represents (essentially) the same circle in $SL(3)$.

(To visualize this, represent all $s \in SO(3) \setminus \{id\}$ by counter-clock rotations around the axes of vectors $\overrightarrow{s}\mathbb{R}^3$ of length $\leq \pi$. Since the vectors $\overrightarrow{s}$ and $-\overrightarrow{s}$ for $\|\overrightarrow{s}\| = \pi$ represent the same spacial rotation s, this establishes a homeomorphism between $SO(3)$ and the projective 3-space: the ball $B^3(\pi)$ with the ±opposite points on the boundary identified.

Thus, $\pi(S)(3) = \pi_1(\mathbb{R}P^3) = \pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$.)

It follows that d_1 is *non-homotopic* to the constant map d_0 represented by the differential of standard embedding $X_0 \subset \mathbb{R}^2 \subset \mathbb{R}^3$; therefore the map d_1 doesn't extend to an immersion from $B^2 \supset X_0$, and the C^1-non-approximability property trivially follows.

Remark. This argument applies to those n and $N \geq n$ where the Stiefel manifold $St_n(\mathbb{R}^N)$, has a non-trivial homotopy group $\pi_l(St_n(\mathbb{R}^N)) \neq 0$ for some $l \leq n-1$, but fails otherwise.

(Note, that being an iterated fiber bundles of spheres of dimensions $N-1, ..., N-n$, the space $St_n(\mathbb{R}^N)$ has all homotopy group $\pi_l(St_n(\mathbb{R}^N)) = 0$ for $l \leq N-n-1$.)

For example, $\pi_2(SL(3)) = St_2(\mathbb{R}^3) = \pi_2(SO(3)) = 0$, since the double cover of $SO(3)$ is equal to $SU(2) = S^3$.

Therefore the differentials of all orientation preserving immersions from spherical annuli $X_0 \subset \mathbb{R}^3$ around $S^2(1) \subset \mathbb{R}^3$ define mutually homotopic maps from X_0 to $SL(3)$.

For instance if X_0 is the annulus between the spheres of radii 1 and 3 written in the polar coordinates as

$$X = \{s, 2+r\} \in \mathbb{R}^3\}_{s \in S^2(2), r \in [-1,1]}$$

then the map $f_1 : (s,r) \mapsto (-s,-r)$ is *orientation preserving*, and the corresponding map $X \to SL(3)$ is homotopic o the constant map which correspond to the original embedding, call it f_0 from X to $\mathbb{R}^3$.

Then by Smale's h-principle the map $f_1 : X_0 = S^2 \times [-1,1] \to \mathbb{R}^3$ is *regularly homotopic to* $f_0 : X_0 \to \mathbb{R}^3$:

f_1 can be joined with f_0 by a C^1-continuous family of immersions $f_t : X \to \mathbb{R}^3$, $0 \leq t \leq 1$, which, because of the switch $r \mapsto -r$ turns the axial sphere in X_0 inside out.

Exercise. Show that if $N \leq 2n-1$, then, among the C^2-maps $f : B^n \to \mathbb{R}^N$, such that $rankdf(0) = n-1$, the generic ones admit no C^1-approximation by smooth immersion.

2 A Few "Recent" References

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\begin{tikzcd}
\text{Thm } 1.1 & & \text{Prop } 3.8 \arrow[ll] \arrow[r] & \text{Thm } 3.12 \arrow[r] & \text{Thm } 1.2 \\
& \text{Prop } 3.9 \arrow[ul] & \text{Lemma } 4.3 \arrow[u] & \text{Prop } 3.11 \arrow[u] & \text{Lemma } 9.1 \arrow[ul] \\
\text{Prop } 3.1 \arrow[uu] & & \text{Prop } 5.4 \arrow[ul] \arrow[r] & \text{Prop } 5.3 \arrow[u] & \text{Prop } 3.10 \arrow[uul] \\
& & \S 8.1 \arrow[ull] \arrow[r] & \S 8.2 \arrow[ur] & \\
\text{Prop } 6.2 \arrow[uu] & \text{Prop } 6.12 \arrow[uul, Rightarrow] & \S 4.3.2 \arrow[u] \arrow[ur] & \text{Prop } 6.13 \arrow[u] \arrow[uur] & \text{Prop } 7.6 \arrow[uu] \\
\text{Lemma } 7.2 \arrow[u] & & \text{Thm } 7.1 \arrow[ull] \arrow[urr] & &
\end{tikzcd}
$$

The above diagram outlines the logical dependency of the major propositions in the paper. Notice that the left three columns together give the proof of Theorem 1.1, whereas the right three columns together prove Theorem 1.2. The double arrow between Propositions 6.12 and 3.1 indicates that 6.12 is used in the proof of 3.1 twice

Borman-Eliashberg-Murphy Existence and classification of overtwisted contact structures in all dimensions. Acta Math. 215 (2), 281-361, (2014)

GENERALITY, NATURALITY and
LEXICOGRAPHIC COMPLEXIY: **History** versus **Logic.**

Originally, the isometric immersion theorems were formulated and proven for embeddings $X \to \mathbb{R}^N$, while the intrinsic logic of the problem suggests $X \to Y$ for all Riemannin manifolds X and Y.

Here and everywhere, our preference is dictated by the relative simplicity of writing the corresponding statements in TeX.

For instance, "Y" is ten times more efficient TeX-wise than "$\mathbb{R}^N$", where "*complexity*" is measured by the number of symbols in LaTeX:

$$\frac{complexity\ "\mathbb{R}^N"}{complexity\ "Y"} = \frac{|\backslash\ \mathsf{m\ a\ t\ h\ b\ b\ R} \wedge \mathsf{n}|}{|Y|} = 10.$$

3 Riemannian Manifolds: Concepts, Terminology, Notation

Languages are true analytical methods.[9]

The limits of my language mean the limits of my world.[10]


A Riemannian metric/tensor g on a smooth manifold X can be regarded either as a positive definite *quadratic differential form* on X

$$g = g(x) = g_x = g_x(\tau_1, \tau_2) = \langle \tau_1(x), \tau_2(x) \rangle_{g_x}$$

or as a strictly positive *quadratic function* $g(\tau, \tau)$ on the tangent bundle $T(X)$,, i.e. such that g_x is positive definite for all $x \in X$:

$$g(\tau,\tau) > 0 \text{ for } T(X) \ni \tau \neq 0.$$

df^2-Example. The square

$$df^2 = (df)^2$$

of the differential $df : T(X) \to \mathbb{R}$ of a smooth function $f = f(x)$ on X, is an instance of a non-negative form.

If $n = dim(X) \geq 2$ this can't be strictly positive, since it vanishes on $ker(df)$, but sums of $N \geq n$ of these may be strictly positive.

Euclidean example.

$$g_{Eucl} = \sum_{k=1}^{n} dx_i^2 \text{ on } \mathbb{R}^N;$$

If $f = (f_1, ... f_k, ... f_N) : X \to \mathbb{R}^N$ is a C^1-map, then

$$f^*(g_{Eucl}) = \sum_{k=1}^{N} df_k^2,$$

where $g = f^*(g_{Eucl})$ is *strictly* positive if and only if f is an *immersion.*

More Notation

[9] Antoine Laurent Lavoisier

[10] Ludwig Josef Johann Wittgenstein

$B_{x_o}(\delta) \subset X = (X, g)$ is the δ ball in X,
$T_{x_o}(\delta) = T_{x_o}(X, \delta) \in T_{x_0}$ is the ball in the tangent space $T_{x_0}(X) = \mathbb{R}^n$,
$n = dim(X)$.
If g is C^2 and $\delta > 0$ is small, then the exponential map $e_o = \exp_{x_o}$

$$T_{x_0}(X) \supset T_{x_o}(\delta) \overset{e_o}{\to} B_{x_o}(\delta) \subset X,$$

is an *approximately isometric* C^1-diffeomorphism

$$|(e_o^*(g)/g_{Eucl}) - 1| \leq const_{X,x_o}\delta.$$

GW-Construction.[11] Let $\chi = \chi(t)$ be a smooth non-negative function with the support in $[0, \epsilon = \epsilon_\chi > 0]$ and, given a Riemannin manifold (X, g), let $\rho_{x_o}(x) = \chi(dist_g(x, x_o))$.

Integrate the squared differentials of the functions ρ_{x_o} over X and get

$$g_\chi(x) = \int_X (d\rho_{x_o}(x))^2 d_g x_o.$$

If $g_\chi = g_{Eucl}$ *for* $X = (\mathbb{R}^n, g_{Eucl})$,
then, for all (compact) X *and small* $\epsilon > 0$

$$|(g_\chi/g)| - 1 \leq const_{X,x}\epsilon_\phi.$$

Since integrals can be approximated by (Riemann) sums,
$[\star_\theta]$ *all* g *on (compact)* X *can be* C^0*-approximated by finite sums of* $\mathbb{R}$*-inducible forms, say* $\theta = d\phi^2$.
****
Nash proves $[\star_\theta]$ with his **twist formula:**

$$a(x)^2 df(x)^2 = d\varphi_\epsilon^2(x) + d\psi_\epsilon^2(x) - \varepsilon^2 da(x)^2,$$

where

$$\varphi_\epsilon(x) = \epsilon a(x)\sin\epsilon^{-1}f(x), \ \ \psi_\epsilon(x) = \epsilon a(x)\cos\epsilon^{-1}f(x).$$

Exercise. Let $f_\varepsilon : (X.g) \to \mathbb{R}^N$ be an isometric immersion with the image in the ε-sphere $S^{N-1}(\varepsilon) \subset \mathbb{R}^N$, i.e. $\|f(x)\| = 1$, $x \in X$. Check that the metric induced by the map $a(x)f_\varepsilon(x)$, for all functions $a(x)$, is

$$(af_\varepsilon)^*(g_{Eucl} = a^2 g + \varepsilon^2 (da)^2.$$

[11] David Hilbert, Hermann Klaus Hugo Weyl

****

Normal Exponential. Let $Y = (Y, h)$ be a Riemannian manifold, e.g. $Y = (\mathbb{R}^N, g_{Eucl})$ and let $f : X \hookrightarrow Y$ be a C^1-smooth immersion.

Let the normal bundle $T^\perp(X)$ of X in Y be *trivial*, e.g. X is homeomorphic to the (open or closed) ball $B^n \subset \mathbb{R}^n$ and let $\alpha : X \times \mathbb{R}^{N-n} \to \mathbb{T}^\perp(X)$, $n = dim(X)$, $N = dim(Y)$, be an isomorphism of vector bundles which implements a trivialization of $T^\perp(X)$.

Let $B^\perp(\delta) = B^\perp_X(\delta) \subset T^\perp(X)$ be the δ-balls subbundle and let

$$E_\delta = \exp^\perp_\delta : B^\perp(\delta) \to Y$$

be the normal exponential map.

Let $g = \varphi^*(h)$ be the induced Riemannin metic on X and $g^\oplus = g \oplus g_{Eucl_k}$ be the Riemannin sum metric on $X \times \mathbb{R}^k$.

$[\perp^*]$ *If the map α is C^2-smooth, then the map $E_\delta : B^\perp(\delta) \to Y$ is C^1 and approximately isometric for small δ:*

$$(\alpha \circ E_\delta)^*(h)/g^\oplus \to 1 \text{ for } \delta \to 0,$$

where $\alpha \circ E_\delta : X \times B^{N-n}(\delta) \to Y$.

+++++++++++++++++++++++++

From f to $g = f^*(h)$ and Back. C^1-small perturbations of embedding (and immersions) $f : X \to (Y, h)$ result in controllably C^0-small perturbations of the induced metric $g = f^*(h)$ and

the *converse is true for δ-small normal displacements f_1 of imbeddings* $f : X \to (Y, h)$ defined as follows.

A map $f_1 : X \to Y$ is called a *normal displacement of* $f : X \to (Y, h)$ if

for all x in X, the point $f_1(x)$ can be joined with $f(x) \in Y$ by a geodesic segment $\gamma = \gamma(x)$ normal to $X \overset{f}{\hookrightarrow} Y$ at x, where the unit tangent vectors to these γ at $x \in X$ (which are normal to X) are called the *directions* of the displacement and where f_1 is called i *δ-displacement* of f if $length(\gamma(x)) \leq \delta$, for all $x \in X$.

Observe that

unite normal fields on X define such displacements via the exponential map $B^\perp(\delta) \to Y$.

Now the above mentioned bound on the C^1-distance between f_1 and f reads:

$[\leftrightarrows]$ *If X is compact, if h is C^2 and $\delta > 0$ is small, then*

$$dist_{C^1}(f_1, f_0) \leq \lambda^* \cdot \log(f_1^*(h)/f_0^*(h)).$$

A quadratic differential form θ on X is $B^k(o(1)$*-inducible* if it can be induced by C^1-maps $\phi_\delta : X \to B^k(\delta)$ for all $\delta > 0$.

Since the real line $\mathbb{R}$ can be isometrically immersed to $B^2(\delta)$, $\delta > 0$.

$[o(1)^k]$ *$\mathbb{R}$-inducible forms are $B^k(o(1))$-inducible for $k \geq 2$.*

Smooth Immersions, Curvature and Gauss Theorema Egregium.

4 Proof of Nash Theorem

Summary off the above

$\mathbb{R}^N$-immersible $\iff \sum_1^N df_i^2$
df^2 is $B^2(o(1))$-immersible.
($\phi^2 df^2$ is also $B^2(o(1))$-immersible)
$\int_X df_\epsilon(x, x_o)^2 dx_o \implies \approx_\epsilon \sum_1^{N_\epsilon} df_i^2$
$(B_X^\perp(o(1)), g_Y) = (X, g_Y) \times B^{N-n}(o(1))$
$graph_\phi^*(g \oplus dt^2) = g + d\phi^2$

$|f_1 \underset{\perp}{-} f_0|_{C^1} < |g_1 - g_0|_{C^0}$.

Nash Stretch Lemma. Let $f : X^n \to (Y^N, h)$ be a smooth embeddings and θ be a $B^k(o(1))$-inducible form with support in a topological ball in X, where $k = N - n$, Then there exit δ-small normal displacements $f_\delta : X \to Y$ of f, such that

$$f_\delta^*(h) \to f^*(h) + \theta \text{ for } \delta \to 0.$$

Proof. Let $\phi_\delta : X \to B^k(\delta)$ induce θ, let

$$\psi_\delta : X \to X \times B^k(\delta)$$

be the graph of this map.

Since $\psi_\delta(g \oplus g_{Eucl_k}) = g + \theta$, for $g = f^*(h)$ the composed map and

$$f_\delta = (\alpha \circ E_\delta) \circ \psi_\delta$$

does the job due to $[\perp^*]$, which makes sense because the isomorphism $\alpha : X \times \mathbb{R}^k \to \mathbb{T}^\perp(X)$ is defined over the support of θ.

Nash C^1-Imbedding Theorem for $k \geq 2$. *Given a short embedding $f_0 : (X^n, g) \to (Y^N, h)$, where $k = N - n \geq 2$, there exists an isometric C^1-imbedding $f : (X, g) \to (Y, h)$.*

Proof. Since $\Theta_0 = g - f_0^*(h) > 0$, the form Θ_0 approximately decomposes into sum of $\mathbb{R}$-inducible forms θ, where, by the proof of $[\star_\theta]$, these θ can be chosen with arbitrarily small supports.

By $[o(1)^k]$ and stretch lemma, there exist an embedding $f_1 : X \to Y$ with an arbitrarily small positive difference $\Theta_1 - g = f_1^*)(h)$. Similarly one obtains imbeddings

$$f_2, ...f_i, ... : X \to Y,$$

where

$$\Theta_i = g - f_i^*(h) \to 0 \text{ for } i \to \infty,$$

and where, because of $[\ast\!\leftrightarrows]$, these imbeddings C^1-converge to the required isometric f.

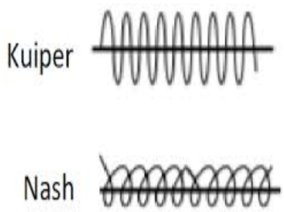
FRACTALS AND INFINITESIMALS

5 Codimension $k = dim(Y) - dim(X) = 1$

Let

$$\xi_\varepsilon : \mathbb{R} \to [-\varepsilon, \varepsilon]$$

be continuous piecewise linear map where the segments $[(i-1)\varepsilon, (i+1)\varepsilon]$, $i = ... - 1, 0, 1, ...$, are isometrically mapped onto $[-\varepsilon, \varepsilon]$ with and/or without reverse of orientation depending on parity of i.



Given a smooth function $\phi : X \to \mathbb{R}$, let

$$\phi_\varepsilon = \xi_\varepsilon \circ \phi : X \to [-\varepsilon, \varepsilon]$$

and

$$f_\varepsilon : X \to Y$$

be the normal ϕ_ε-displacement of a smooth imbedding $f : X \to Y$ in the direction of a unit normal vector field on $X \overset{f}{\hookrightarrow} Y$.

The map f_ε is smooth away from the $i\varepsilon$-levels $Z \subset X$ of the function ϕ, where it has corers along Z, while the induced form $f_\varepsilon^*(h)$ is continuous and it uniformly converges to $f^*(h) = d\phi^2$. for $\varepsilon \to 0$.

Smooth the corners an get smooth imbeddings , say $f_{\varepsilon,\epsilon} : X \to Y$, $\epsilon > 0$, such that

(i) $f_{\varepsilon,\epsilon}$ is equal to f_ε away from the ϵ neighbourhood of Z.

(ii) $f^*_{\varepsilon,\epsilon}(h) \to f^*_\varepsilon$ for $\epsilon \to 0$.

(iii) $dist_Y(f_{\varepsilon,\epsilon}, f_\varepsilon) \le \varepsilon$.

(iv) The distance between the differentials $df_{\varepsilon,\epsilon}$ and $df_{\varepsilon,\epsilon}$ is bounded by twice the jump of the differential at the corner.

Granted this, the above proof of the Nash C^1-imbedding theorem carries over to $k = N - n = 1$.

4???Pseudo-Riemannian Manifolds. The Nash-Kuiper stretching argument effortlessly generalizes to immersion of manifolds with indefinite "metrics".

Exercises. Let $Y = (Y, h)$ be a Pseudo-Riemannian manifold, with "metric" h of type (N_+, N_0), $N_+ + N_0 = N = dim(Y)$ and let $f_0 : X \to Y$ be a smooth imbedding.

4???*Show at if the induced metric $g_0 = f_0^*(g)$ is positive (definite), $g_0 > 0$, and if $n = dim(X) < N_+$, then, for all $g > g_0$, the map f_0 can be C^0-approximated by isometric embeddings $f : (X, g) \to)Y, h)$ isotopic to f_0.*

Hint. Use normal displacements directed by h-normal fields ν to Xm such that $h(\nu, \nu) > 0$.

4??? Let the induced "metric" $g_0 = f_0^*(h)$ have type (n_+, n_-), $n_+ + n_- = n = dim(X)$ and let g be of the same type as g_0.

Show that if

$N_+ > n_+$, $N_- > n_-$,

$n_+, n_- > 0$,

and if g is *homotopic to* g_0 in the space of (n_+, n_-)-"metric " on X (e.g X is contractible), then f_0 can be C^0-approximated by isometric embeddings $f : (X, g) \to (Y, h)$ isotopic to f_0.

Hint. Follow a homotopy of g by normal stretching f directed by normal vectors ν away from the isotropic directions, i.e. where $h(\nu, \nu) = 0$.

5??? Let X and Y be pseudo-Riemannian manifolds of types (n_+, n_-) and (N_+, N_-) and let $f_0 : X \to Y$ be a continuous map Let the tangent bundle $T(X)$ admit an isometric homomorphism to the induced bundle $f_0^*(T(Y)) \to X$.[12]

If $N_+ > n_+$, $N_- > n_-$, then f_0 can be C^0-approximated by isometric immersions $f : (X, g) \to (Y, h)$.

Hint. Start with the proof of the following proposition by using stretching in normal directions away from isotropic directions as earlier.

[12]This condition is satisfied for $N_\pm \ge n_\pm + n$ and also for contractible X, where $N_\pm \ge n_\pm$.

Homotopy Lemma. Let $f_0 : X \to Y$ be smooth embedding and let h_t, $0 \le t \le 1$, be a ıhomotopy of "metrics" on Y of a given type (N_+, N_-), $N_+ + N_- = N = dim(Y)$, where the induced $g_0 = f_0^*(h_0)$ is *non-degenerate* of type n_+, n_-, $n_+ = n_- = n = dim(X)$.

If either

(a) $n_\pm < N_\pm$,

or

(b) $n_- = 0$ and $n_+ < N_+$, then there exists an isotopy f_t, $0 \le t \le 1$, of f_0 such that the induced $g_t = f_t^*(h_t)$ are *non generate*, hence all of the same type (n_+, n_-).

The case (b) of the lemma yields the following.

Smale Hirsch' Homotopy Principle for Immersions. Let X and Y be smooth manifolds of dimension $n = dim(X)$ and $N = dim(Y)$.

Let $\Phi_0 : T(X) \to T(Y)$ be a *continuous fiberwise linear fiberwise injective map*[13] and let $f_0 : X \to Y$ be the continuous map which underlies Φ_0.[14]

If $n < N$, then f_0 can be *approximated by smooth immersions* $f : X \to Y$, such that the differentials $df : T(X) \to T(Y)$ can be joined with Φ_0 by *homotopies* of continuous fiberwise linear fiberwise injective maps $\Phi_t : T(X) \to T(Y)$.

To see how (b) helps, let V be the total space of the bundle $T^* = f_0^*(T(Y))$, where X is embedded to V by the zero section, say $X \overset{\psi_0}{\hookrightarrow} V$ and let the (co)normal bundle of $X \hookrightarrow V$ be identified with T^*.

Let f_0 be smooth and $F_0 : V \to Y$ be a smooth map, such that $F_0|X = f_0$ and such that the differential of F_0 on $T^* \subset T(V)|X$ is equal to the tautological map $T^* \to T(Y)$.

Employ Φ_t and construct a family h_t of "metrics" on V of type $(M+, M_-)$ for $M_+ = N$, $M_- = n$, such that

$\bullet_0$ the metric h_0 is positive on $T(V)$,

$\bullet_1$ the metric h_1 is negative on the kernel of the differential $dF_0 : T(V) \to T(Y)$,

Use the above (b) and approximate ψ_0 by smooth imbeddings $\psi : X \to V$ isotopic to ψ_0, such that $\psi * (h_1) > 0$.

Then observe that the composed maps $f = F_0 \circ \psi : X \to Y$ are immersions which approximate f_0.

Remark. The proof of (b) uses only a few lines in the Nash-Kuiper argument: the existence of stretches, which make the induced metric as large as you want and, since h is indefinite, one also needs to take care of keeping the displacement directions away from the isotropic ones.

Isometry on Subbundles

Exercise. Let (X, g) and (Y, h) be Riemannian manifolds and $\Theta \subset T(X)$ be a subbundle of rank $m \le n = dim(X)$.

Let $r < N = dim(Y)$ and generalize the Nash-Kuiper stretch argument to maps $f : (X, g) \to (Y, h)$, such that

$$f^*(h)\Theta = g|\Theta$$

.

[13]Such a Δ exists if $N \ge 2n$ and also if X is contractible of dimension $n \le N$.

[14]A fiberwise linear map $\Phi : T(X), T(Y)$ is a pair (f, η) where $f : X \to Y$ is a continuous map and $\eta : T(X) \to f^*(T(Y))$ is a vector bundle homomorphism.

Also extend Hirsch' h-principle to maps f where the differentialis injective on Θ.

Show, for instance that for a arbitrary independent continuous tangent vector fields $\theta_1, ..., \theta_m$ on X, $m < dim(Y)$,

there exists a C^1-map $f : X \to Y$, such that

$$||d(\theta_i)||_h = 1 \text{ and } \langle d(\theta_i), d(\theta_j)\rangle_h = 0$$

for all $i, j = 1, ..., m$, $j \neq i$.

6 Codimension Zero: dim(X) = dim(Y)

(A) Let X be a smoothly triangulated manifold with a continuous Riemannin metric g and $f_0 : (X, g) \to (Y, h)$ be a short C^1-map.

Then, f_0 can be approximated by continuous maps f, such that

• the maps f are C^1-smooth and *isometric on the interiors* of the simplices of dimension $m < n = dimX$ and f is *short on the interiors of n-simplices*;

• the induced Riemannin metrics $f^*(h)$ on X are *continuous.*

(B) Let (X, g_0) be a C^0-Riemannian manifold, let $g > g_0$ be another continuous metric and let $\epsilon_i(x) > 0$, $i = 1, 2, ...$, be continuous functions.

Then there exists smooth hypersurfaces $Z_i \subset X$, and continuous piecewise smooth maps

$$... \overset{f_{i+1}}{\to} X \overset{f_i}{\to} X \overset{f_{i-1}}{\to} ... \overset{f_1}{\to} X,$$

such that

• the maps f_i are smooth on Z_i and on the complements $X \setminus Z_i$; moreover, f_i are smooth *up to the boundaries* on the submanifolds with boundaries into which Z_i locally divide X;

• $dist_g(f_i(x), x) \leq \epsilon_i(x)$;

• the induced metrics

$$g_1 = f_1^*(g_0), ..., g_i = f_i^*(g_{i-1})...$$

are continuous

$$f_{i+1}^*(g_0) > f_i^*(g_0)$$

; and $g_i \to_{C^0}$ for $i \to \infty$.

It follows that the composed Lipschitz map $f = ...\circ f_i \circ ... \circ f_i : (X, g) \to (X, g_0)$ is isometric: it preserves the lengths of all rectifiable curves and $dim_{top}(f^{-1}(x)) = 0$, $x \in X$.

Can one make $dim_{Hau}(f^{-1}(x)) \leq \epsilon$, $x \in X$, for all $\epsilon > 0$?

7 Perspectives on Isometric Immersions and the h-principle

Mathematical phenomena are established by proofs and understood by generalization, or more respectfully, by finding underlying general principles/theories.

(1) In these lectures we emphasise the (quasi)analytic point view, which could elucidate general classes of *partial differential equations*, which behave similarly (or highly dissimilarly) to $f^*(h) = g$.

(2) A topologist would be mainly intersted in *homotopy and homology* of "natural" (e.g. $Diff(X)$-invariant) *mapping spaces and "natural" sheaves solutions of differential relations – equations and inclusions over* X in the spirit of the Smale-Hirsch h-principle.

Definition of Relations $\mathcal{R}$ *of the First Order for Maps* $X \to Y$. Let $\mathcal{H} = \mathcal{H}(X,Y) \to X \times Y$ be the vector bundle with

$$\mathcal{H}_{x,y} = hom(T_x(X) \to T_y(Y)),$$

$\mathcal{H}$*-morphisms* are continuous fiberwise linear maps $T(X) \to T(Y)$ or equivalently sections $X \to \mathcal{H}$, where $\mathcal{H}$ is fibered over X via the projection $X \times Y \to X$.

Given a subset $\mathcal{R} \subset \mathcal{H}$ – a *differential relation* in our terms – an $\mathcal{H}$-morphism $X \to \mathcal{H}$ is *a* $\mathcal{R}$*-morphism* if its image is contained in $\mathcal{R}$, ,

A *solution of* $\mathcal{R}$ is a C^1-map $X \to Y$, the differential of which $df : T(X) \to T(Y)$, regarded as a section of $H \to X$, is an $\mathcal{R}$-morphism.

Isometric Example. If X and Y are Riemannian manifolds then the isometry relations consists of the *isometric* homomorphisms $T_x(X) \to T_y(Y)$.

Definition of the h-Principle. A Relation $\mathcal{R} \subset \mathcal{H}(X,Y)$ and/or its solutions *satisfy the h-principle* if all continuous sections $X \to \mathcal{R} \subset \mathcal{H}$, are homotopic to differentials of solutions of $\mathcal{R}$, by continuous homotopies of $\mathcal{R}$-morphisms.

Exercises. (a) Show that short immersions $X^n \to \mathbb{R}^N$, $N > n$, satisfy the h-principle.

Hint. This follows from Hirsch theorem by homothetic scaling immersions of *compact* manifols X, while shortening of immersions of non-compact X needs special auxiliary immersions $f_0 : X \to \mathbb{R}$, such that

$$f_0(x) \to 0 \text{ for } x \to \infty,$$

where the existence of suitable f_0 follows from the Hirsch theorem.

(b) Derive the h-principle for isometric C^1-immersions $X^n \to \mathbb{R}^N$ for $M > n$ from (a) by the Nash-Kuiper argument.

(c) Give examples of *open* Riemannian manifolds X^n smoothly embedded to $\mathbb{R}^N$, $N > n$, which admit no short embeddings to $\mathbb{R}^N$.

Hint. Look at the Möbius band

$$X = \mathbb{R}P^2 \setminus p_0.$$

(d) Show that proper embeddings $X^n \hookrightarrow \mathbb{R}^N$, $N > n$, can be transformed to short ones, where the latter can be made proper as well as short for complete X.

Parametric h-Principle. Let $C^0(\mathcal{R})$ be the space of continuous sections $X \to \mathcal{R}$ and $Sol^1(R)$ be the space of C^1-solutions of $\mathcal{R}$.

Then $\mathcal{R}$ and its solutions are said to abide by the parametric h-principle if the differential $d : Sol^1(R) \to C^0(\mathcal{R},\ f \mapsto df$, induces an *isomorphism between the homotopy groups* of these two spaces.

Exercise. Show that isometric C^1-immersions $X^n \to \mathbb{R}^N$, $n < N$, abide the parametric h-principle.

(3) From the metric/convexity perspective, isometric immersions $(X,g) \to \mathbb{R}^N$ are *extremal points* in the space of distance decreasing maps; accordingly, one asks what are similar points for distance decreasing maps between more general metric spaces.

(4) If you think of g as an instance of a *contravariant tensor*, you turn to manifolds equipped with such tensors of a given type $\mathcal{T}$ (e.g. symmetric and/or antisymmetric differential forms of a given degree) and the corresponding category $\mathcal{C}_{\mathcal{T}}$ of "$\mathcal{T}$-isometric" maps.

A prominent example is that of *symplectic immersions* between symplectic manifolds,

$$f : (X, \omega) \to (Y, \eta),$$

which, for $dim(X) < dim(Y)$, satisfy the h-principle with a properly incorporated cohomology condition $f^*[\eta] = [\omega]$, $[\eta] \in H^2(Y; \mathbb{R})$, $[\omega] \in H^2(X; \mathbb{R})$.

(5) From the classical differential geometric point of view the isometry condition for $f : X \to Y$ prescribes the *first fundamental form* I_1 on X defined by the Y-scalar products between the *first* derivatives of f.

This suggests the study of maps $f : X \to Y$ with given forms $II_2, II_3 ...$, where the most attractive one is II_2, which characterizes the curvature of $f(X) \subset Y$.

(✳) Your may dreams, of extending your "soft ideas" to the "rigid worlds" of complex manifolds and even further to algebraic and diophantine geometry. [15]

If you succeed your may delight in the great unity of mathematics or be humbled by realizing how repetitive our mathematical ideas are.

8 Euler/Onsager

V. Scheffer (1974,1993) [16] Müller-Šverák(2003), [17] De Lellis-Székelyhidi(2007).[18]

V. Scheffer (1974,1993) [19] Müller-Šverák(2003), [20] De Lellis-Székelyhidi(2007).[21]

Euler Equation on $(v = v(x,t), p = p(x,t))$, where v is a time dependent *vector field* on a Riemannin manifold X (e.g. on the flat 3-torus), $v : X \times \mathbb{R} :$

[15] The h-principle is vaguely reminiscent of the *Hasse local-to-global principle* and the Nash proof of the C^∞- isometric immersion theorem can be compared to Gilbert's solution of the Wring problem; then one wonders if the ideas behind the *Hardy-Littlewood circle method* can be useful in PDE.

[16] Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities

An inviscid flow with compact support in space-time.

[17] Convex integration for Lipschitz mappings and counterexamples to regularity.

[18] The Euler equations as a differential inclusion.

[19] Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities

An inviscid flow with compact support in space-time.

[20] Convex integration for Lipschitz mappings and counterexamples to regularity.

[21] The Euler equations as a differential inclusion.

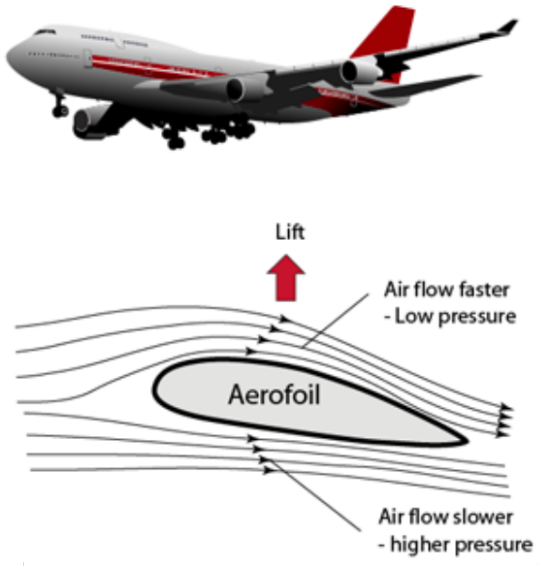


Figure 1: plane

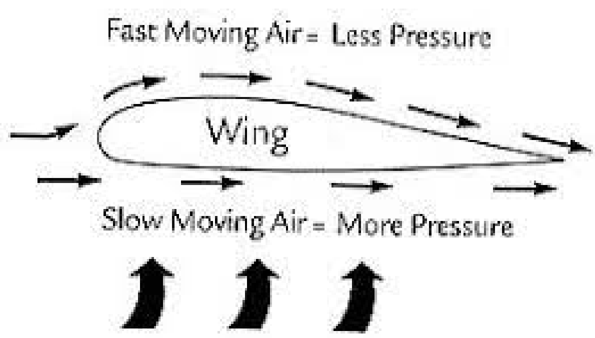


Figure 2: B... principle

$T(X)$, and $p : X \times t \to \mathbb{R}$ is a *function* on X:

$$\partial_t(v) + \nabla_v(v) + \mathsf{grad}_{\mathsf{X}}(p) = 0,$$

$$\mathsf{div}(v) = 0.$$

where $\nabla_v(v)$ is the covariant derivative ∇_v of v:
$\nabla_v(v)_i = \sum_j v_j \partial_j v_i =$
$= \sum_j \partial_j(v_i v_j) - v_i \sum_j \partial_j v_j,$
or
$\nabla_v(v) = $ "div" $(v \otimes v) - \mathsf{div}(v)v$
for "div" $\{v_i \otimes v_j\}_i = \sum_j \partial_j(v_i v_j)$

Energy Conservation.

$$\partial_t \int_X \langle v, v\rangle = \int_X -div\langle v, ||v||^2 + p\rangle = 0$$

Degression 1: Example of an Onsager Relation.[22] Heat flows from the warmer to the colder parts of a liquid system and matter flows from high-pressure to low-pressure regions.

But temperature differences can also cause matter flow (convection) and pressure differences can cause heat flow.

The heat flow per unit of pressure difference and the density (matter) flow per unit of temperature difference are equal.

This equality follows from *microscopic reversibility.*

[22] In 1931, L. Onsager after Kelvin and Helmholtz.

Scheffer-Shnirelman "Paradox". There exists a weak bounded measurable solution of Euler in dimension 2, with a compact support in $X \times \mathbb{R}$.

Digression 2. Trees Hight Paradox.

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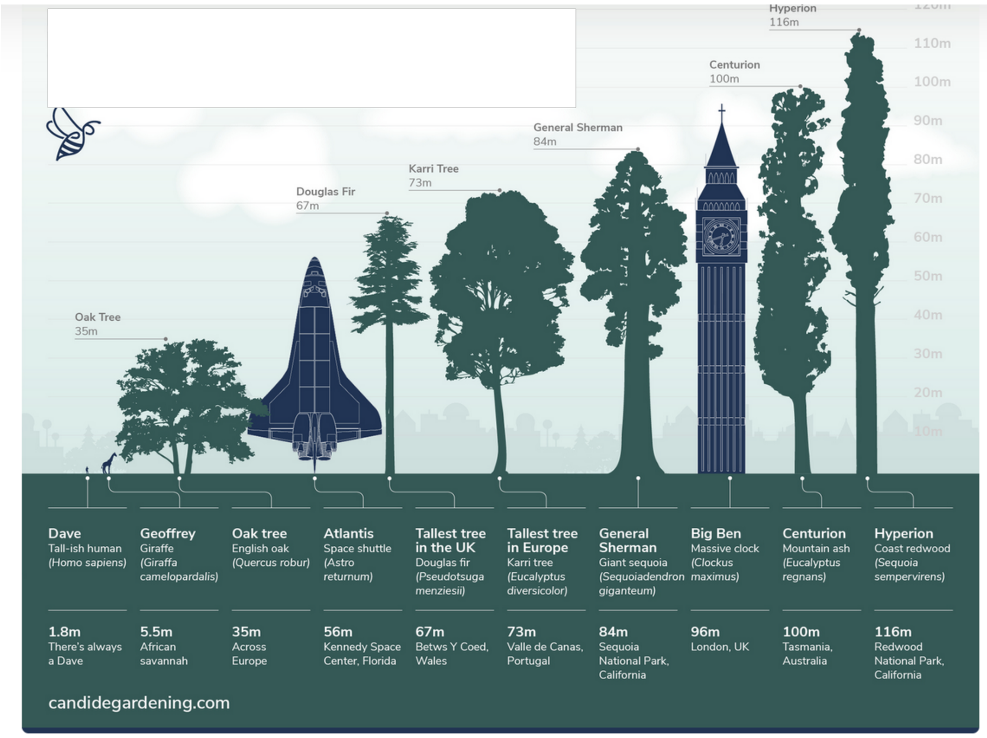

Onsager's H older 1/3 Conjecture (1949)

Positive Direction. If $\alpha > 1/3$, then every weak C^{α}-solution $v(x,t)$ to Euler conserves energy: $E(t) = \int v^2(x,t)dx$ is constant in time.

(Final 2-page Proof [23]

[23] G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Phys. D, 78(3- 4):222–240, 1994.

Negative Direction. For every $\alpha < 1/3$, there exist (periodic) weak C^{α}-solutions, such that the conservation of energy fails.

Isett-.....De Lellis-Székelyhidi ($(\frac{1}{10}$ - $\frac{1}{5})$ - $\frac{1}{3}$) Theorem. For all $\alpha < 1/3$, there is a nonzero weak C^{α}-Hölder solution $v(x,t)$ on the 3-torus $X = T^3$ with $C^{2\alpha}$-pressure p, where v is 0 outside a finite time interval.

References

B. Kirchheim, S. Muller, Sverák(2003) Studying Nonlinear pde by Geometry in Matrix Space

CÉDRIC VILLANI Paradoxe de Scheffer-Shnirelman revu sous l'angle de l'intégration convexe [d'après C. De Lellis et L. Székelyhidi] Astérisque, tome 332 (2010), Séminaire Bourbaki, exp. no 1001, p. 101-134

Philip Isett, A Proof of Onsager's Conjecture arXiv:1608.08301 [math.AP]

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9 Convex Integration

Introduction to the H-Principle - Eliashberg, Y., Mishachev.

I. Dimension One. Let X and Y be smooth manifolds and ∂ be a non-vanishing vector field on X.

Let

$$A = \bigcup_{(x,y)\in X\times Y} A_{x,y} \subset X \times T(Y).$$

$$A_{x,y} \subset x \times T_y(Y)$$

and

$$B = conv.hull_T(A) = \bigcup_{(x,y)\in X} conv.hull(A_{x,y})$$

$\mathbf{C^1}$-Case. Let

$\bullet_{regA}$ the obvious projection from A to $X \times Y$, call it $P_{|A} : A \to X \times Y$ is a *topological submersion*[24], (e.g. a locally trivial fibration);

$\bullet_{regB}$ the projection $P_{|B} : B : X \times Y$, call it is a a locally trivial fibration);

$\bullet_{connect}$ the fibers $A_{x,y}$ are path connected;

$\bullet_{lift}$ the manifold $X \times Y$ admits a *continuous lift* to A, that is a continuous map $L : X \times Y \to A$, such that the composed map $P \circ Q : X \times Y \to X \times Y$ is the identity map.

Then

the space of $A\partial$-*directed* C^1-maps $f : X \to Y$, i.e. such that

$$\partial f(x) = df(\partial_x) \in A_{x,f(x)},\ x \in X,$$

is C^0-*dense* in the space of $B\partial$-directed C^1-maps.

Or, in an analyst's terms, *∂-subsolutions of A can be approximated by regular solutions*".

Example Let X be the circle S^1, let $Y = \mathbb{R}^k$ and $A = A_o \times S^1 \times T(\mathbb{R}^k)$, for $A_o \subset \mathbb{R}^k = R^k_{x,y} = T_y(\mathbb{R}^k)$.

P. CONSTANTIN, W . E & E. S. TITI, Onsager's conjecture on the energy conservation for solutions of Euler's equation, Comm. Math. Phys. 165 (1994),p. 207-209

[24]Each point in$A$ admits a neighbourhood which fibers over its image in $X \times Y$.

If A_o is path connected and $conv.hull(A_o)$ contains a neighbourhood of $0 \in \mathbb{R}^k$, then $A\frac{d}{ds}$-directed maps $f: S^1 \to \mathbb{R}^k$, i.e. with $\frac{df}{ds} \in A$, do exist.

Lipschitz Case. Let A be a closed subset, such that

$\bullet_{strA}$ $X \times Y$ and A admit stratifications such that for each stratum $S \subset A$ there is a stratum $\underline{S} \subset X \times Y$ such that

$\bullet_{AB}$ the projection $P_{|A} : A \to X \times Y$ sends $S \to \underline{S}$, where this map is a topological submersion and the corresponding B-map over $\underline{S}$,

$$B \cap P_{|A}(\underline{S}) \to \underline{S}$$

is a fibration'

$\bullet_{XY}$ the projections of $\underline{S}$ to X and to Y are topological submersions.

Then the space of almost everywhere *$A\partial$-directed* Lipschitz maps $f: X \to Y$, is *C^0-dense* in the space of a.e. $B\partial$-directed Lipschitz maps. [25]

Remark. The analytically most essential case, of the above ??? and ??? where $X = [01]$, is proven by A. F. Filippov: Classical solutions of differential equations with multi-valued right-hand side, SIAM J. Control 5 (1967), p. 609-621.)

Our multidimensional formulation is needed for applications to partial differential equations and inclusions.

Convex Decomposition. Let U and V be compact smooth manifolds, let $f: U \to \mathbb{R}^m$ be a C^l-map, and $\Phi: V \to \mathbb{R}^m$ be a C^r-map, such that the image $f(U) \subset \mathbb{R}^m$ is contained in the interior of the convex hull of the image of Φ,

$$f(U) \subset inter.conv.hull(\Phi(V)).$$

If U is connected and $r \geq 1$, then there exit finitely many C^k-maps

$$\psi_i : U \to V,\ k = \min(l, r),$$

such that f is equal to a convex combination of the composed maps

$$f_i = \Phi \circ \psi_i : U \to \mathbb{R}^m,$$

$$f = \sum_i p_i f_i,\ p_i \geq 0,\ \textstyle\sum_i p_i = 1.$$

Remark. This is not true for $r = 0$, not even for generic Lipschitz maps $\Phi: [0,1] \to \mathbb{R}^{m=1}$.

The convex decomposition serves in the inductive steps in following.

Codim 1 Reduction and $C^\perp$-Approximation . The convex integration of certain differential relations – equations and inclusions for vector valued functions $f(x_1, ..., x_n)$ can be implemented by treating f as functions in a single variable, say in x_n with values in the space of functions in the remaining $n-1$ variables.

Such reduction is present in the proofs of the h-principle in the variety of cases, starting with its implicit use in the Nash-Kuiper C^1-isometric immersion argument and explicit in the Smale-Hirsch proof of the topological immersion theorem.

[25]This is true under weaker conditions, but $\bullet_{strA}$, $\bullet_{AB}$,$\bullet_{XY}$ are not so bad since they are satisfied in many geometric cases, e.g, where A is a semialgebraic set.

$C^{\perp}$**-Example.**

Also a version of this is present in constructions of isometric C^{∞}-immersions and, with in the modern Oka theory (see section??), where $\mathbb{C}$ takes place of $\mathbb{R}$.

But when this reduction becomes impossible, (maybe only invisible?) the proofs of the h-principle become more difficult e.g. for construction of foliations and metrics with *Ricci* < 0.

Two simple(?) questions.

(1) Directed Immersions. Let $U \subset S^2$ be a *connected open* subset in the unit sphere, such that $U \cup -U = S^2$.

Does the 2-torus $\mathbb{T}^2$ *admit an immersion to* $f : \mathbb{T}^2 \to \mathbb{R}^3$, *such that the tangential Gauss map* $G_f : \mathbb{T}^2 \to S^2$ *lands in* U*?*

More generally, let $Gr_n(\mathbb{R}^N)$ be the Grassmann manifolds of (oriented) n-subspaces in $\mathbb{R}^N$.

Under what conditions on $U \subset Gr_n(\mathbb{R}^N)$ *do immersion of (oriented)* n*-manifolds* $f : X \to \mathbb{R}^N$ *with* $G_f(X) \subset U$ *satisfy the h-principle?*

For instance let U be an open subset which contains the spherical image $G_f(X) \subset Gr_n(\mathbb{R}^N)$ of some immersion $f_0 : X_0 \to \mathbb{R}^N$, of a closed n-manifold X e.g. of the n-torus $\mathbb{T}^n$.

Do then all parallelizable n-manifolds X admit immersions $f : X \to \mathbb{R}^N$ with $G_f(X) \subset U$?

Differential Inclusions: Polyhedral and Lipschitz Solutions.[26]

If the convex hull of a subset $G \subset SN - 1$ coneains a neighbourhood of zero $\mathbf{0} \in \mathbb{R}^{\mathbf{N}}$, then there exists a compact convex polyhedron $P = P_G \subset \mathbb{R}^N$ withe the faces normal to some $u \in U$, in writing, $G(\partial P) \subset U$.

It follows that every smooth immersion of an oriented n-manifold, $n = N - 1$,

$$f : X \to \mathbb{R}^N$$

can be ε approximated by piecewise linear immersions

$$f\varepsilon X \to \mathbb{R}^N$$

with $G_{f_\varepsilon} \subset U$

To see this, pretend that f is an imbedding, cover $f(X) \subset \mathbb{R}^N$ by translated and ε-scaled copies of $P = P_G$,

$$\bigcup_i \varepsilon P_i + y_i \supset f(X).$$

Then let $f_\varepsilon(X) \subset \mathbb{R}^N$ be a connected component of the boundary of the union $\bigcup_i \varepsilon P_i + y_i \supset f(X)$.

If X *is non-orientable*, this applies to the complement of a hypersurface $X^(n-1) \subset X = X^n$ and delivers a Lipschitz immersion $X \to \mathbb{R}^N$, which is piecewise linear on the the complement $X \setminus X^(n-1)$ and which has all faces from $X \setminus X^(n-1)$ normal to vectors $u \in U$.

This Lipschitz f_ε can be upgraded to a piecewise linear map if G contains "sufficiently many" symmetric pairs $(u, -u)$, but...

If $N > n + 1$, and $f : X \to \mathbb{R}^n$ is an immersion with a trivial normal bundle, then (again pretend f is an embedding) the submanifold $f(X) \subset \mathbb{R}^N$ is

[26]Compare with ???[Cellina Inclusions 2005]

a transversal intersection of (co)-oriented hypersurfaces and transversal intersections of the above ∂P-piecewise linear approximations to these approximate $f(X)$, where this approximation may *change the topology* of X.

However this doesn't happen if the dihedral angles of P are $> \pi - \frac{[}{\pi}2(N-n)$ but I am not certain what is the true condition on P needed for this purpose.

Also it is unclear, what are $U \subset Gr_n(\mathbb{R}^N$, such that all immersions of all $X^n \to \mathbb{R}^N$ admit piecewise linear approximations by maps with all n-faces parallel to these in U.

Exercise. Generalize the above to (approximations by) piecewise smooth immersions between arbitrary manifolds, $f : X \to Y$, such that the differentials of these f at smooth points send X to a given subset U in the Y-tangent bundle over $X \times Y$,

$$U \subset X \times T(Y) \to X \times Y,$$

where this U is the union of finitely many smooth hypersurfaces $U_i \subset X \times T(Y)$ transversal to the $T(Y)$ fibers.

Problems

Relate directed *p.l.* immersions with triangulations of n-manifolds, where the links of all vertices belong to a given set of triangulations of S^{n-1}

Study directed piecewise smooth immersions with singular loci of codimension 2.

(2) Free Maps. Does $\mathbb{T}^2$ admit a *free immersion* to $\mathbb{R}^5$?

(A map $f : \mathbb{T}^2 \to \mathbb{R}^5$ is free if the five partial derivatives,

$$\partial_1 f(x), \partial_2 f(x), \partial_{1,1} f(x),$$

$$\partial_{1,2} f(x), \partial_{2,2} f(x) \in \mathbb{R}^5$$

are linearly independent at all

$x \in \mathbb{T}^2$.)

More generally what are n, r and N, $n > 1$, $r > 1$, such that the maps $f : X \to \mathbb{R}^N$, for which the rth osculation spaces coincide with R^N,

$$osc_r(f(x)) - N\mathbb{R}^N, x \in X,$$

satisfy the h-principle?

Straight and Localized Elimination of Singularities. [G-E], 2.1.5 Embedding Haefliger

Haefliger, A., Plongements différentiables dans le domaine stable, Commentarii Math. Helv. 1962/1963, 37, 155–167

Gromov, M. and Eliashberg, 1.(1971), Construction of nonsingular isoperimetric films, Trudy Steklov Inst. 116, pp. 18-33.

Approximation in Sobolev Spaces.[GE]Gromov, M. and Eliashberg, 1.(1971), Construction of nonsingular isoperimetric films, Trudy Steklov Inst. 116, pp. 18-33.

10 Seymour-Zaslavsky-Hilbert Rationality Theorem

The the above ??? and ??? as well as their proofs are similar to that of the Seymour-Zaslavsky theorem stated below and, where as we shall see later on, the Hilbert's (spherical design) case of this theorem applies to the h-principle for isometric immersions with controlled curvatures.

A point z in the convex hull of $X \subset \mathbb{R}^n$ is called *X-rational* if it is equal to a convex combination of points from X with *rational weights*,

$$[p_j] \qquad z = \sum_{j=1}^{N} p_j x_j,\ x_j \in X,$$

where $p_i \geq 0$ are rational numbers , such that $\sum_j p_i = 1$.

Equivalently, X-rational points $z \in conv(X)$ are *centers of mass* of finite multi-sets[27] from X,

$$[1/M] \qquad z = \frac{1}{M}\sum_{k=1}^{M} x_k,$$

where $[p_j] \implies [1/M]$ for M equal the common denominator of the numbers p_j.

I. SZ Theorem.[28] If a compact subset $X \subset \mathbb{R}^M$ contains $2M$ point $\underline{x}_i, \underline{y}_i \in X$, $i = 1, ..., n$, such that the n vectors $\underline{x}_i - \underline{y}_i \in \mathbb{R}^M$ are *linearly independent* and such that $\underline{x}_i$ and $\underline{y}_i$ lie in *the same connected component of* X for all $i = 1, ..., M$, then all points x in the *interior* of the convex hull of X, are X-rational.

Moreover, these z are representable by centers of mass of finite subsets (rather than multi-sets) in X.

Hilbert Theorem.[29] For all k and d there exit $N = N(k,d)$ rational ponts $s_i \in S^k = S^k(1) \subset \mathbb{R}^{k+1}$, such that all polynomial functions $P(s)$ of degrees $\leq d$ satisfy

$$\frac{1}{N}\sum_{i=1}^{N} P(s_i) = \int_{S^k} P(s)ds$$

where ds stands for the $O(k+1)$-invariant probability measure on S^k.

Exercise. Show that the existence of not-necessarily rational points s_i with $\frac{1}{N}\sum_{i=1}^{N} P(s_i) = \int_{S^k}$ follows from the SZ-theorem applied to the sphere S^k imbedded to some $\mathbb{R}^M$ by a polynomial map $Q : S^k \to \mathbb{R}^M$.

11 Applications and Generalizations of Convex Integration

Convex integration serves many open relations $\mathcal{R}$, where, for instance, it yields the following.

Short Immersions Theorem. Let $f_0 : X \to Y$ be a smooth *short* (e.g constant) map between Riemannin manifolds.

If f_0 homotopic to an immersion and $N > n$, then

f_0 can be C^0-approximated by short immersions.[30]

[27] A multiset is an mage of a map $I \to X$, written as $\{\underline{x}_i\} \subset X$, $i \in I$, $\underline{x}_i \in X$.

[28] Seymour, P. D. and Zaslavsky, T., *Averaging set. A generalization of mean values and spherical designs*, Adv. Math. 52 (1984), 213-246.

[29] This reduces the Waring problem in degree 2d to that for d, *Hilbert, D., "Beweis fiir die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem)" Math. Ann. 67 (1909), 281-300.*

[30] PDR

Independent Forms Theorem If a manifold $X = X^n$ admits M linearly independent differential forms of degree d and if

$$2 \leq d \leq n - 1,$$

then X admits M linearly independent *exact forms* N of the same degree D.[31]

Odd d Decomposition Theorem. Let g be a continuous symmetric differential form on $X = X^n$ of odd degree d.

If $N \geq 2s(n+1, d-1) + 2n$, where

$$s(n+1, d-1) = \frac{(n+d-1)!}{(n)!(d-1)!},$$

is the dimension of the space of homogeneous polynomials of degree $d-1$ on $\mathbb{R}^{n+1}$, then there exists C^1-function $f_1, ..., f_N$ on X, such that

$$g = \sum_{i=1}^{N} df_i^d.$$[32]

Remark. If $N < s(n,d) = \frac{(n+d-1)!}{(n-1)!d!}$,,which may happen for

$$\frac{(n+d-1)!}{(n-1)!d!} > 2\frac{(n+d-1)!}{n!(d-1)!} + 2n,$$

e.g. for $d \geq 2$ and $n > 4d$, the PDE system $g = \sum_{i=1}^{N} df_i^d$ is *overdetermined and has no C^∞-solution for generic C^∞-smooth g.*

Question. Can one significantly improve the bound $N \geq 2s(n+1, d-1) + 2n$ in our C^1-case.?

Even d Decomposition Problem. Here the h-principle has not been proved so far and only limited results are available.

Example.[33] Let $d = 2r$ and $g = \underline{g}^r$, where $\underline{g}$ is a positive definite differential quadratic form on $X = X^n$,

(a) If

$$N = s(n,d) + s(n, d-1-n) - n,$$

then locally, in a neighbourhood $U = U(v) \subset X$ of each point $x \in X$,

$$g = \sum_{i=1}^{N} df_i^d$$

for some C^1-functions f_i on U.

(b) If

$$N \geq (n+1)s(n,d) - n$$

then g decomposes into the sum

$$g = \sum_{i=1}^{N} df_i^d$$

[31] Eliashberg, Y., Mishachev, N.

[32] This follows from the h-principle for hyperegiuar g-isometric maps, see 2.4.(3') in PDR.

[33] See 2.4(4) in PDR and section 10.1.8,10.19 in [Sbornik 1972] for corresponding C^∞ results.

with C^1-functions f_i on all of X.

Remark Both (a) and even more so (b) are underdetermined, which raises the following

Question. Does the space of continuous forms of even degree $d \geq 4$ on X^n contain a non-empty open subset of forms g which decompose as

$$g = \sum_{i=1}^{N} df_i^d$$

for $N < s(n,d)$ with C^1=functions $f_i = f_i(x)$?

The basic first order realations for C^1-maps $X \to Y$, to which the convex integration doesn't apply are

$$\mathcal{R} = \{\mathcal{R}_{x,y}\} \subset \mathcal{H} = \{\mathcal{H}_{x,y}\},$$

$$\mathcal{H}_{x,y} = hom(T_x(X) \to T_i(Y),$$

where $\mathcal{R}_{x,y}$ are linear or affine subspaces in $H_{x,y}$. However, such $\mathcal{R}$ may lie in the range of the h-principle.

1-d***Dimensional Example.*** Let $\Theta \subset T(X)$ and $\Xi \subset T(Y)$ be smooth subbundles of ranks $n_o \leq n = dim(X)$ and $N_0 \leq N = dim(Y)$ and let $\mathcal{R}_{x,y} = R_{x,y}(\Theta{\to}\Xi)$ consist of homomorphisms $T_x(X) \to T_y(Y)$, which send $\Theta_x \to \Xi_y$ for all $x \in X$ and $y \in Y$.

[$\star$]. Let Ξ by *fully non-integrable*, i.e the consecutive commutators of tangent vector fields from Θ span all of the tangent bundle $T(Y)$.

If $n_o = rank(\Theta) = 1$, then the relation $\mathcal{R} = \{\mathcal{R}_{x,y}\}$ satisfies the h-principle.

(It is common knowledge in the Carnot-Caratheodory community but I don't know to whom it must be attributed to.)

There are instances, where the h-principle has been proved for $n_o > 1$, e.g. for *immersions between contact manifolds* or where $N = dimN$ and $N_o = rank(Xi)$ are large depending on n_o and the corank $k = rank)T(Y)/\Xi$ of X. The true lower bounds on N_o and $N = dim(Y)$ needed for the h-principle remains problematic but, without bothering to think hard, $N_o \geq 2(k+n_0)^2$ will do.

$\mathcal{R}^\perp$($\Theta{\to}\Xi$)-Relations. The simplest such a relation on $f : X \to Y$ is where the differential $df : T(X) \to T(Y)$ is *injective* on all linear (sub) spaces $\Theta_x \subset T(X)$ and sends them *transversally* to $\Xi_{f(x)} \subset T_{f(x)}(Y)$.

Question. Do all these $\mathcal{R}^\perp$ satisfy the h-principle?

(If $n_o < N - N_o$, this follows by convex integration e.g. by the Nash-Kuiper stretch as in the proof of the Smale-Hirsch theorem; also the case $N - N_o = n_o = 1$ is easy.)

More General/Difficult Question. Let $\mathcal{R}^\natural \subset \mathcal{R}(\Theta{\to}\Xi)$ be defined by imposing bounds on the dimensions of $df(\Theta_x) \subset T_{f(x)}(Y)$ and on of the intersections $df(\Theta_x) \cap \Xi_{f(x)}$,

$$r_1 \leq dim(df(\Theta_x)) \leq r_2,$$

$$r_1^\perp \leq dim(df(\Theta_x) \cap \Xi_{f(x)}) \leq r_2^\perp$$

When does such a relation

$$\mathcal{R}^\natural = \mathcal{R}^\natural(r_1, r_2, r_1^\perp, r_2^\perp)$$

satisfy the h-principle?

Non-smooth Ξ. Let $\Xi \subset T(Y)$ be a generic continuous (hence, nowhere differentiable) subbundle.

When do relations $\mathcal{R}(\Theta{\to}\Xi)$ and $\mathcal{R}^{\natural}(r_1, r_2, r_1^{\perp}, r_2^{\perp})$ satisfy the h-principle.

Convex Integration in $\mathcal{R}(\Theta{\to}\Xi)$

Every class of first order relations for maps $f_o : X_o^{n_o} \to Y_o^{N_o}$, $\{\mathcal{R}^o_{x_o,y_o}\} \subset \{\mathcal{H}_{x_o,y_o} = hom(T_{x_o}(X_o) \to T_{y_o}(Y_o)\}$, has a counterpart with Θ instead of $T(X_o)$ and Ξ instead of $T(Y_o)$ for $rank(\Theta) = n_o$ and $rank(\Xi) = N_o$.

For instance, one may endow Θ and Ξ with symmetric forms g and h of degrees d and let

$$\mathcal{R}(g,h) \subset \mathcal{R}(\Theta,\Xi) \subset \mathcal{H}$$

consist of (g,h)-respecting homomorphisms $(\Theta_x, g_x) \to (\Xi_y, h_y)$ at all $(x,y) \in X \times Y$.

Thus, solutions of $\mathcal{R}(g,h)$ – call them (g,h)-isometric maps $f_X \to Y$ – satisfy:

$$df(\Theta \subset \Xi \text{ and } (df)^*(h) = g.$$

A relevant h-principle is proven [Da2000] for a class of *"suitably regular"*[34] isometric immersions for contact structures $(\Theta_x, g_x) \to (\Xi_y, h_y)$ with quadratic forms on them by adapting the Nash stretching argument.

Also, a similarly adapted Nash stretching delivers the h-principle for "suitably regular" (g,h)-isometric immersions where Ξ is fully non-integrable of sufficiently large dimension depending on $n_0 = rank(\Theta)$, $d = deg(h) = def(g)$ and $corank(\Xi) = rank(T(Y)/\Xi)$.

In fact, this argument adapts to other geometric situations,[35] including those where the general convex integration theorems don't apply, e.g. to *symplectic isometric embeddings* [DL2002].

In a similar spirit, one can prove an h-principle for *connection inducing maps*(see 2.2.6 in [PDR]) augmented with an isometry condition.[36]

Exercise. Formulate and prove a 1-dimensional h-principle simultaneously generalizing ??? and ???.

12 H-Principle beyond Convex Integration: Foliations, Ricci Curvature, Holomorphic and almost Holomorphic Maps

??A. Oka(1939) and Stein(1951).

A complex n-manifold X is Stein if it possesses "the same kind of abundance" of holomorphic functions $\to \mathbb{C}$ as the Euclidean space $\mathbb{C}^n$ does.

In concrete terms, X is Stein if and only if it is bi-holomorphic to a complex analytic submanifold in C^N.

[34]This is typical : proofs (possibly the validities), of h-principles often apply not to relations $\mathcal{R}$ themselves but to subrelations $\mathcal{R} \setminus \Sigma$ for some $\Sigma \subset \mathcal{R}$ of positive codimensions in Σ.

[35]See [The2019] for generalizations and applications of the Nash-Kuiper stretch argument.

[36]Given a vector bundle $V \to X$ with a Euclidean connection ∇_X one proves an h-principle for maps X to the Grassmann manifold, $f : X \to Y = Gr_m(\mathbb{R}^M)$, which induce (V, ∇_X) from the canonical bundle (W, ∇_Y) over this Y; if X is endowed with a Riemannin metric, one can also prove for large M the h principle with the isometry condition for some metric on Y. But the optimal bound on M remains problematic.

For instance, all complex algebraic submanifolds in $\mathbb{C}^N$ are Stein.

A complex n-manifold Y is Oka if it possesses "the same kind of abundance" of holomorphic lines $\mathbb{C} \to Y$ as $\mathbb{C}^n$ does.

In precise terms, a connected Y is Oka (elliptic) manifold if the following mutually equivalent conditions are satisfied for all N and all relatively compact convex open subsets $U \subset \mathbb{C}^N$.

RAP(Forstnerič) All holomorphic maps $U \to Y$ can be (Runge) uniformly approximated on compact subsets in U by maps, which holomorphically extend to $\mathbb{C}^N \supset U$.

$\mathbb{C}$-CONNectivity (Kusakabe) Given two holomorphic maps $f_0, f_1 : U \to Y$, there exits a holomorphic map $F : U \times \mathbb{C} \to Y$, such that $F(u,0) = f_0(u)$ and $F(u,1) = f_1(u)$.

$\mathbf{ELL_1}$(Kusakabe) For all holomorphic maps from Stein manifolds $f : X \to Y$, there exist holomorphic maps $F : X \times \mathbb{C}^M \to Y$, such that $F(x,0) = f(x)$ and the differentials $dF(x,0) : T_{x,0}(X \times \mathbb{C}^M : \mathbb{C}^M = x \times \mathbb{C}^M \to Tf(x(Y)$ have ranks $N = dimY$ at all $x \in X$.

Examples. Complex *homogeneous* spaces, (obviously) satisfy ELL_1, while RAP and CONN were proven here by Grauert (1958).

Smooth toric algebraic varieties are Oka, Larusson(2011)

Complements to *compact holomorphically convex* subsets, [37] in complex semisimple Lie groups of dimensions $N \geq 3$, e.g. in $\mathbb{C}^N$ or in $GL(N,\mathbb{C})$, are Oka (Kusakabe 2020).

H-Principle (Forstnerič) Holomorphic maps from Stein manifolds to Oka manifolds satisfy the parametric h -principle, which for holomorphic maps reads as follows:

every continuous map $X \to Y$ is homotopic to a holomorphic one.

None of the above, including the equivalence

$$\textbf{RUN} \Longleftrightarrow \textbf{CONN}$$

is trivial.

TWO REFERENCES.

F. Forstnerič *Recent developments on Oka manifolds*(2023), arXiv:2006.07888 [math.CV]

Finnur Larusson, Eight lectures on Oka manifolds,(2014) arXiv:1405.7212v2

Conjecture. Let Y be Oka and Stein and let it also satisfy the *density property*:

complete holomorphic vector fields on Y are dense in the space of all holomorphic fields for uniform convergence on compact subsets, e.g. Y is a semisimple Lie group.[38]

[37] A subset K in a complex space Y is *holomorphically convex* if, for all $y_o \in Y \setminus K$, there exists a holomorphic function f_o on Y which separates y_o from K, that is

$$|f_o(y_o)| > \sup_{y \in K} |f(y)|.$$

[38] Dror Varolin: *The density property for complex manifolds and geometric structures*. J. Geom. Anal., 11(1):135–160, 200,

A general notion of shears, and applications, Michigan Math. J. 46 (1999), no. 3, 533–553.

Riccardo Ugolini, Joerg Winkelmann, *The Density Property for Vector Bundles* arXiv [2209.05763].

Then holomorphic maps from Stein manifolds X to Y the ranks of which is every where bounded from below by a given $m \leq n = dim(X)$,

$$rank(df(x)) \geq m, x \in X,$$

satisfy the h-principle.

Remark. if $N > m$, then the *elimination of singularities*[39] used for maps $X \to \mathbb{C}^N$, may work with the density property for *nonvanishing* vector fields, if $N > m$, but the case $N = m$ seems very difficult.

Problem. Let g be a holomorphic quadratic differential form on a Stein manifold X, e.g $g = 0$, and let h be such a nonsingular form on Y, e.g. $h = \sum_{i=1}^{N} dy_i^2$ on $Y = \mathbb{C}^N$.

Under what conditions on (Y, h) (Oka, density,...) do free [40] isometric holomorphic maps $f : X \to \mathbb{C}^N$ satisfy the parametric h-principle, at least for $N >> 2n + 2n(n+1)/2$?

Remark. This is motivated by possible reduction of the (quadratic) differential equations such as $f^*(h) = g$ for $h = \sum_{i=1}^{N} dy_i^2$ on $Y = \mathbb{C}^N$ to algebraic ones, Compare with 10.1.3 in [G. Smoothing 1972] and 5.4.A in [G. Oka 1989]. but the problem hasn't been resolved even in the case of $Y = \mathbb{C}^N$.

Also it remains unclear what happens to similar equations of degrees > 2.

More References.

[Oka] K. Oka: *Sur les fonctions des plusieurs variables. III: Deuxi'eme probl'eme de Cousin.* J. Sc. Hiroshima Univ. 9, 7–19 (1939)

Franc Forstnerič *The homotopy principle in complex analysis: a survey* arXiv:math/0301067v2 [math.CV] 3 Mar 2003

Franc Forstnerič, *Oka manifolds* arXiv:0906.2421v2

F Forstneric *What is an Oka manifold?* https://users.fmf.uni-lj.si › Forstneric-Krems-2011 PDF

Book © 2017 by F Forstneric *Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis*

Franc Forstnerič *Oka manifolds: From Oka to Stein and back* Annales de la faculté des sciences de Toulouse Mathématiques (2013)

Volume: 22, Issue: 4, page 747-809 ISSN: 0240-2963

arXiv:2301.01268v1 [math.CV] 3 Jan 2023 Proper holomorphic maps in Euclidean spaces avoiding unbounded convex sets Barbara Drinovec Drnovˇsek and Franc Forstneriˇc

Yuta Kusakabe 2020 Oka properties of complements of holomorphically convex sets arXiv:2005.08247 [math.CV]

Yuta Kusakabe (2020) An implicit function theorem for sprays and applications to Oka theory, International Journal of MathematicsVol. 31, No. 09, 205007.

Elliptic characterization and localization of Oka manifolds Yuta Kusakabe (or arXiv:1808.06290v1 [math.CV] for this version)

[39] See 2.1.5 in [PDR] and ??? below.

[40] A holomorphic map $f : X \to Y$ is free if the second holomorphic osculating spaces of $X^n \overset{f}{\hookrightarrow} \mathbb{C}^N$ have dimension $n + n(n+1)/2$.

Frank Kutzschebauch, Finnur Larusson, Gerald W. Schwarz Gromov's Oka principle for equivariant maps, arXiv[1912.07129]

Smooth toric varieties are Oka Finnur Larusson

arXiv:1107.3604v3 [math.AG] for this version)

Approximation and interpolation of regular maps from affine varieties to algebraic manifolds Finnur Larusson, Tuyen Trung Truong

Cite as: arXiv:1706.00519 [math.AG] (or arXiv:1706.00519v3 [math.AG] for this version)

The Density Property for Vector Bundles Riccardo Ugolini, Joerg Winkelmann

Dror Varolin. The density property for complex manifolds and geometric structures. J. Geom. Anal., 11(1):135–160, 200 *Foliations.*

The Density Property for Complex Manifolds and Geometric Structures II Internat. J. Math. 11 (2000), no. 6, 837–847.

A general notion of shears, and applications Michigan Math. J. 46 (1999), no. 3, 533–553.

journal of symplectic geometry Volume 18, Number 3, 733–767, 2020 H-principle for complex contact structures on Stein manifolds Franc Forstneric ˇ

Regular Algebraic Ell_1-property makes sense for algebraic manifolds Y over all fields K of characteristic zero and, probbaly, can be meaningfully extended to characteristic>0 as well. See [Larusson-Truong 2017] [41] for $K = \mathbb{C}$.

Questions. What can be said in the spirit of Larusson-Truong on (possibly,stabilized) spaces of regular maps from affine manifolds X to elliptic Y, (e.g. to algebraic groups and homogeneous spaces) over more general fields?

Is there an algebraic counterpart of the Hirsch h-principle for immersions $X \to Y$?

If $Y = K^N$, $N > dimX$, this follows by applying "straight" elimination of singularities section ???), but the holomorphic version of "localization" is probematic.

Is there a general theorem for regular maps $f : (X, g) \to (Y, h)$ where g and h are

Is there an h-principle kind of theory for regular maps $f : (X, g) \to (Y, h)$ generalizing Hilbert's theorem on representation of rational polynomials g of degree d by sums of d-th powers of rational linear forms?

Is there such a theory for maps from projective varieties X to elliptic ones, along the lines Gream Segal's 1979-theorem for rational functions?[42]

Here one may start with developing "straight" elimination of singularities for N-tuples of sections of *sufficiently ample vector bundles* over projective varieties.

Is there an h-principle. where the role of "continuous" is taken by morphisms between "etale homotopy types" of algebraic manifols?[43]What I mean

[41] Finnur Larusson, Tuyen Trung Truong,Approximation and interpolation of regular maps from affine varieties to algebraic manifolds, arXiv:1706.00519v3.

[42] The topology of spaces of rational functions Acta Math. 143: 39-72 (1979). Jacob Mostovoy, Spaces of rational maps and the Stone-Weierstrass Theorem (2003)arXiv:math/0307103v2 [math.AT] for this version)

Alexis Aumonier An h-principle for complements of discriminants Alexis Aumonier(2022) arXiv:2112.00326v2.

[43] (

here, probbaly naively, is the classifying space of the category of non-ramified coverings of Zariski open subsets in X.

Donaldson's Almost Holomorphic Maps Of Symplectic Manifolds

Similarly to abundance of holomorphic maps of high degrees from projective manifolds X to $\mathbb{C}P^N$[44] it almost holomorphic maps from symplectic manifolds with adapted complex structures display a similar behavior.[45].

This is reminiscent of a similar abundance of approximately isometric C^1-immersions $X^n\mathbb{C}^N$, where the Donaldson-Kodaira-Bergman argument is analogous to the GW-construction in section 1.

Apparently, the quasi-complex (as well as complex) flexibility depends on production of (real) codimension two bubbles, that takes place of the codimension one *corrugations.*

But a comprehensive unified treatment of these two classes of overdetermined PDE is not available yet.

References

S. Bergman, "The kernel function and conformal mapping" , Amer. Math. Soc. (1950)

Jean-Paul Mohsen Limit holomorphic sections and Donaldson's construction of symplectic submanifolds, arXiv:1610.06111v4 [math.SG] 5 Jun 2021

Vicente Muñoz, Fran Presas, Ignacio Sols, Almost holomorphic embeddings in Grassmannians with applications to singular simplectic submanifolds
arXiv:math/0002212 [math.DG]

Foliations.

⋆★ For all k and n, there exists a (Haefiger universal) foliated (non-empty!) manifold $Y = (Y(n,k), \mathcal{F}_{n,k})$ with *k-codimensional leaves*, such that smooth maps f of n-manifolds X to Y, such that these f are *transversal* to the leaves of the foliation $\mathcal{F}$ on Y, satisfy the h-principle.

This theorem for open manifolds X is due to Haefliger (1970) and for closed ones to Thurston,(1974, 1976), where the proof for codimension $k = 1$ is more delicate.[46]

References.

W.P. Thurston, The theory of foliations of codimension greater than one, Comm. Math. Helvet. 49 (1974), 214–231.

[Th2] , Existence of codimension-one foliations, Ann. of Math. 104 (1976), 249–268.

N. M. Mishachev and Y.M. Eliashberg, Surgery of singularities of foliations, Funct. Anal. Pril. 11 (1977), 43–53.

[44]K. Kodaira, (1954), "On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)", Annals of Mathematics, Second Series, 60 (1): 28–48.

I am not certain what a most general h-principle known in this context.

[45]S. K. Donaldson (1996) Symplectic submanifolds and almost-complex geometry. S. K. Donaldson ... J. Differential Geom. 44(4): 666-705 (1996)

[46]In 1970, Yasha Eliashberg came up with an idea of how to use his surgery of singularities for proving the above h-principle for foliations of codimensions ≥ 2.

There was nothing wrong with the idea but Yasha made a mistake of consulting then the best "expert" on foliations in Leningrad. The "expert" convinced him that the idea can't work, being in contradiction with the Reeb *global* stability theorem erroneusy applied by this "expert" to codimension > 1; Yasha dropped his project. To conclude, the "expert" was myself.

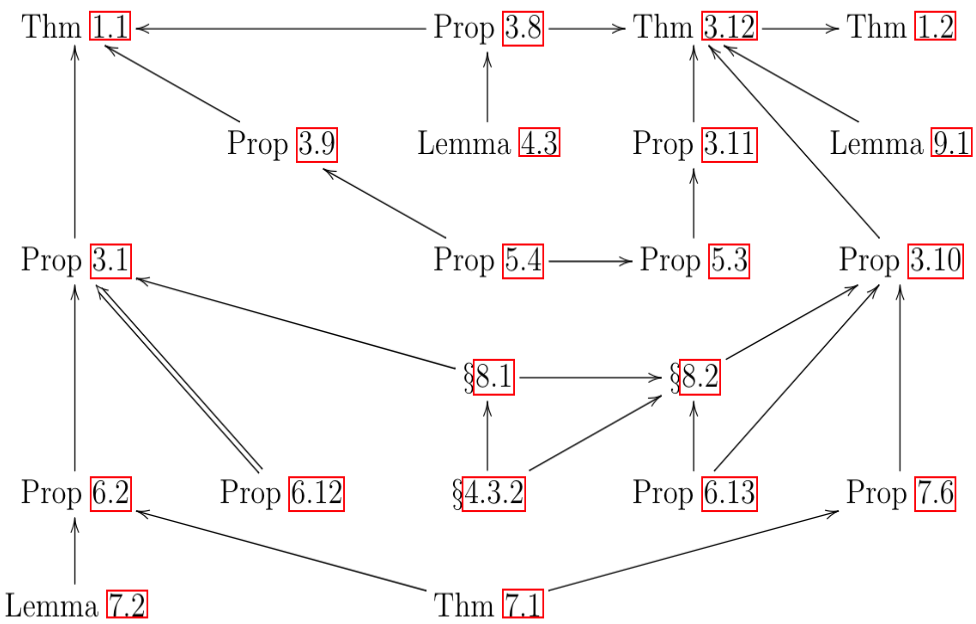
1998 Wrinkling of smooth mappings III. Foliations of codimension greater than one. Y. Eliashberg, N. M. Mishachev Topological Methods in Nonlinear Analysis Journal of the Juliusz Schauder Center Volume 11, 1998, 321–350

Contact Structures *Overtwisted contact structures on all manifolds X satisfy a parametric h-principle, Thus all X admit contact structures in all homotopy classes of almost contact structures.*

In particular, all orientable $(2n-1)$-manifolds immersible to $\mathbb{R}^{2n}$ admit contact structures.

Reference.

Matthew Strom Borman, Yakov Eliashberg, Emmy Murphy (2014) *Existence and classification of overtwisted contact structures in all dimensions. (86 pages)* arXiv:1404.6157 [math.SG]



The above diagram outlines the logical dependency of the major propositions in the paper. Notice that the left three columns together give the proof of Theorem 1.1, whereas the right three columns together prove Theorem 1.2. The double arrow between Propositions 6.12 and 3.1 indicates that 6.12 is used in the proof of 3.1 twice

Roger Casals, Jose Luis Pérez, Álvaro del Pino, Francisco Presas, Existence h-principle for Engel structures,(2015)
arXiv:1507.05342 [math.SG]

Geometry & Topology 24 (2020) 2471–2546 The Engel–Lutz twist and overtwisted Engel structures ÁLVARO DEL PINO THOMAS VOGEL

Question. Which integrability/non-integrability conditions $\mathcal{C}$ ion C^∞-subbundles

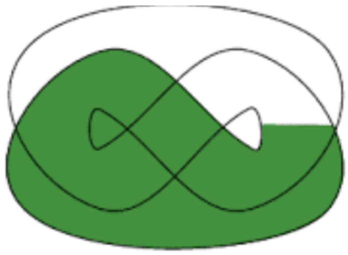


Figure 3: Milnor's Discs

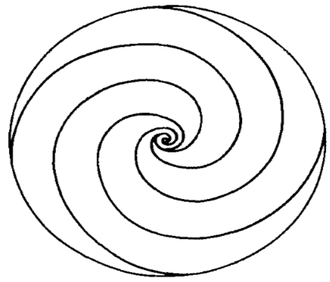


Figure 4: Overtwisted

$\Theta \subset T(X)$ satisfy the h-principle?

Here $\mathcal{C}$ must be expressed by equalities and non-equalities imposed on r-jets of germs of Θ, which are invariant under diffeomorphisms of X.

For instance, $\mathcal{C}$ may say that Θ is everywhere locally generated by m tangent vector fields, such that the ranks ρ_l of the subbundles in $T(X)$ generated by commutators of orders $\leq l$ of these fields, $l = 1, ..., r$, are contained in given intervals,

$$m \leq m_{l,1} \leq \rho_l \leq m_{l,2} \leq n = dim(X).$$

Alternatively, if Θ is expressed represented as intersection of the kernels $n-m$ linear differential forms λ_i on X,

$$\Theta \bigcap_i ker(\lambda_i),$$

then $\mathcal{C}$ may be described in terms by the (isomorphism class of) the differential algebra generated by λ_i.

(This suggests a similar problem for forms of degriees $d \geq 1$.)

T Shin · 2021, Directed immersions for complex structures

https://comptes-rendus.academie-sciences.fr › item PDF

Maximally non-integrable almost complex structures: an h-principle and cohomological properties R. Coelho, G. Placini & J. Stelzig

Research Article Open Access Luis Fernandez, Tobias Shin, and Scott O. Wilson* Almost complex manifolds with small Nijenhuis tensor https://doi.org/10.1515/coma-2020-0122 Received September 10, 2021; accepted October 2, 2021

Lohkamp-Ricci h-Principle. *If $dim(X) \geq 3$ then the space of Riemannin metricsgon X with $Ricci(g) < \rho$ on on X are contractible for all ρ.*

Furthermore, these metric are C^0-dense in the space of all metrics[47]

Curvature h-Principles Joachim Lohkamp

Annals of Mathematics , Nov., 1995, Second Series, Vol. 142, No. 3 (Nov., 1995), pp. 457-498

Metrics of Negative Ricci Curvature Author(s): Joachim Lohkamp

[47]These g are not C^1 dense, which, along with the inequality $dim(X) \geq 3$, indicate that a direct codimension one reduction doesn't work here.

Yet the h-principle may be grounded in "C^0-local concavity" of the space of metrics g with $Ricci(g) < 0$, as opposed to "C^0-local convexity" of $Ricci(g) \geq 0$, which results in rigidity of the latter class of metrics.

Annals of Mathematics , Nov., 1994, Second Series, Vol. 140, No. 3 (Nov., 1994), pp. 655-683

Question. For which n and N do C^∞-immersions $f : X^n \to \mathbb{R}^N$ with $Ricci(f^*(g_{Eucl})) < 0$ and/or with $scal.curv(f^*(g_{Eucl})) < 0$ satisfy the h-principle?[48]

13 De Lellis - Székelyhidi Rendition of Convex Integration

[49]

Version V → W. Let $V, W \to X$ be smooth vector bundles, let $\mathcal{A} \subset V$ be a subset and $\mathcal{D}_V : C^\infty(V) \to C^\infty(W)$ be a linear differential operator with smooth coefficients. Let

$$B = conv_V(A) = \bigcup_{x \in X} conv.hull(V_x) \subset V$$

be the fiberwise convex hull of V.

Call a continuous section (lift) $f : X \to A$, for which $\mathcal{D}_V(f) = 0$, where this equality understood in the sense of distribution a weak $A\mathcal{D}_V$-solution and a section $f : X \to B$, for which $\mathcal{D}_V(f) = 0$, a *subsolution.*

The ***Conv****ex* ***Int****egration*-property of $(A\mathcal{D}_V)$ is the *density of the space of solutions in the space of subsolutons with a weak* [50] *topology.*[51]

Isometric $U \to W$ *Example.* Let $V = hom(T(X) \to \mathbb{R}^N)$, i.e. $V_x = hom(T_x(X) \to \mathbb{R}^N)$, let $A = A_g \subset V$ consist of isometric homomorphisms for a given metric g on V and $\mathcal{D}_V(f) = df$, where d is the exterior differential applied to the N components of f, which are 1-forms,

$$f = (f_1, ..., fi, ...f_N \text{ and } df = (df_1, ..., df_i, ...df_N).$$

Here, solutions are representations of g by sums of the squares of closed (rather than exact) 1-forms and subsolutions are weakly approximable by solution for $N \geq dim(X)$ by the Nash-Kuiper C^1-theorem.

Remark. The above applies to isometric immersions $X \to Y$ for flat Y, such as $\mathbb{R}^N$ and $\mathbb{T}^N$, while the case of non-flat Y needs a (slight) generalization of the above setting.

Euler $V \to W$ *Example.* Write the Euler equation as an algebraic equation, which define A,

$$u = v \otimes v$$

and the differential one $\mathcal{D}_V(u, v, p) = 0$ for

$$\mathcal{D}_V(u, v, p) =$$

[48] Compare with Luis A. Florit, Bernhard Hanke,*Scalar positive immersions*, arXiv:1910.06290 [math.DG]

[49] C. D E LELLIS, L. SZÉKELYHIDI - The Euler equations as a differential inclusion arXiv:math/0702079.

[50] This needs to be specified. For instance, $f_i \to 0$ weakly if f_i are bounded and $\int f_i(x) d\mu x \to 0$ for all measures supported on smooth segments in X with continuous densities on these segments.

[51] Analysts, unlike topologists, do not care for the approximating D_V-solutions $f : X \to A$ to be homotopic to given continuous $\phi_0 : X \to A$.

$$= \partial_t(v) + \text{"div"} u + \mathsf{grad}_{\mathsf{X}}(p), \mathsf{div} v.$$

Here the approximation of (suitably defined) subsolutions by solutions (without mentioning homotopies) is a de Lellis- Székelyhidi's result.

Version $\mathbf{U} \to \mathbf{V}$. Here instead of $W \to X$ and an operator $\mathcal{D}_V$ on section $X \to V$ we are given a vector bundle U and a differential operator $\mathcal{U} : C^\infty(U) \to C^\infty(V)$.

Now solutions of $A \subset V$ are defined as sections $u : X \to U$, such that

$$\mathcal{D}_U(u) \subset A$$

and where subsolutions are u with

$$\mathcal{D}_U(u) \subset B = conv.hull_V(A)$$

If one insists on C^r-regularity of solutions for r being the order of $\mathcal{D}_U$ one can formulate the h-principle without ever mentioning weak topologies.

Isometric $U \to V$ Example. Here $U = X \times \mathbb{R}^N$ and $\mathcal{D}_U(f)$ is the differential df, where f is an N-tuple of functions on X. Thus we return to isometric immersions $(X, g) \to \mathbb{R}^N$ regarded as representations of g by sums of squares of *exact* 1-forms.

Euler $U \to V$ Example. The $V \to W$ Euler equation can be rewritten in terms of $(n+1)$-tuples of exterior $(n-1)$-forms, $n = dim(X)$, on X with divergence replaced by the exterior differential

$$d : \bigwedge^{n-1}(X) \to \bigwedge^{n}(X).$$

Then one passes to the $U \to V$ Euler with the bundle U equal the Whitney sum of N copies of $\wedge^{n-2} T(X)$ and the differential

$$d : \bigwedge^{n-2}(X) \to \bigwedge^{n-1}(X).$$

This suggests the following.

Question. Which systems of polynomial equations imposed on (tuples of) exterior forms on smooth manifolds are solvable by *exact or closed* C^1-forms?

****************************** ******************************* [52]

14 Two Convex Integrations Theorems for the Euler Equation by De Lellis-Székelyhidi

Euler Equation on X (e.g. $X = \mathbb{T}^n$),

$$\partial_t(v) + \nabla_v(v) + \mathsf{grad}_X(p) = 0,$$

$$\mathsf{div}(v) = 0.$$

[52] D. Spring, Convex integration theory(1998), Y. Eliashberg, N. Mishachev, Introduction to the H-Principle(2003), Mélanie Theillière *Convex integration theory without integration. (2019)*

$\nabla_v(v)_i = \sum_j v_j \partial_j v_i =$
$= \sum_j \partial_j(v_i v_j) - v_i \sum_j \partial_j v_j,$
or
$\nabla_v(v) = \text{"div"}(v \otimes v) - \text{div}(v)v$
for $\text{"div"}\{v_i \otimes v_j\}_i = \sum_j \partial_j(v_i v_j)$

Energy Conservation.

$$\partial_t \int_X \langle v, v\rangle = \int_X -div\langle v, \|v\|^2 + p\rangle = 0$$

====================

$U = \begin{pmatrix} v \otimes v + p\mathbf{1_n} \\ v \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} + q\mathbf{1_n} \quad \begin{matrix} v \\ v \quad 0 \end{matrix}$

$\text{div}_{x,t}(U = (u_{i,j}, v_i, q)) = 0$

$u = v \otimes v - \frac{1}{n}\|v\|^2 \mathbf{1}_n$

$q = p + \frac{1}{n}\|v\|^2$

============================

Linear space: $\mathcal{U} = \mathbb{R}^m = \{u, v, q\} = \{U_{i,j}\},\ i, j = 1, ..., n+1,\ U_{ij} = U_{j,i},$ $U_{n+1,n+1} = 0,\ m = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)^2}{2},$

Linear Operator:

$\mathcal{D} = \mathcal{D}_{Eul} = \text{div} : \{X \times \mathbb{R} \overset{C^\infty}{\to} U\} \to \{X \times \mathbb{R} \overset{C^\infty}{\to} \mathbb{R}^{n+1}\}.$

$\mathcal{D}$-Neutral directions $\overrightarrow{r} \in \mathbb{R}^{m+1}$ of codimension k in X. A nonzero vector $\overrightarrow{r}$ is such a direction if all points in X admit *accommodating* local coordinates $x_1, ..., x_k, ...x_n$, such that the maps from these neighbourhoods to $\mathbb{R}^{n+1}$ of the form

$$f(x) = \rho(x_1, ..., x_k)\overrightarrow{r}$$

satisfy

$$\mathcal{D}(f) = 0$$

for all smooth function ρin k variable

This definition, which makes sense for all linear differential operators on manifolds, is usually concerns $k = 1$, where it goes under the heading of "wave cone" with references to the work by Tartar, Di and Perna and Murat on "compensated compactness" that is opposite to ConvInt.[53]

Examples: Exterior Differential (a) let $\mathcal{D}$ be an exterior differential on N-tuples of differential 1-forms on a manifold $X = X^n$

$$\mathcal{D} : (\phi_1, ..., \phi_N) \mapsto (d\phi_1, ..., d\phi_N)$$

Then such an N-tuple $\overrightarrow{r} = (\phi_1, ..., \phi_N)$ is a $\mathcal{D}$-neutral direction of codimension 1, if all ϕ_i are multiples of the same form, say $\phi_i = p_i(x)dx_1$, with accommodating coordinates $x_1, ..., x_n$.

[53]Tartar, L. The compensated compactness method applied to systems of conservation laws. In Systems of nonlinear partial differential equations (Oxford, 1982), vol. 111 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Reidel, Dordrecht, 1983, pp. 263–285. and Ronald J. Di Perna Compensated Compactness and General Systems of Conservation Laws

Transactions of the American Mathematical Society, Vol. 292, No. 2 (Dec., 1985), pp. 383-420 (38 pages).F . Murat - Compacité par compensation, Ann. Scuola Norm. Sup. Pisa CI. Sci. 5 (1978), p. 489-507, partie II : Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, p. 245-256.

(These correspond to *principal directions* in convex integration, where the latter apply to sections f of smooth bundles over X with N-dimensional fibers, where the differentials of such sections are locally represented by N-tuples of closed 1-forms.)

(b) Let $\mathcal{D}$ be the exterior differential on N-tuples of exterior d-forms. Then such a tuple $\overrightarrow{r} = (\phi_1, ..., \phi_N)$ is a $\mathcal{D}$-neutral direction of *codimension* k, if all ϕ_i are divisible by $dx_1, ..., dx_k$ for some coordinate system $(x_1, ..., x_k, x_{l+1} ..., x_N)$.

This, for $d = n-1$, applies to tuples of *divergence free* vector fields, where such fields naturally correspond to *closed* $(n-1)$-forms on X.

Non-Linear: $\mathcal{E} \subset \mathcal{U} = \{u = u_{ij}, v = v_i, q\}$ consist of those (u, v, q), which satisfy

$u = v \otimes v - \frac{1}{n}||v||^2 \mathbf{1}_n$

$q = p + \frac{1}{n}||v||^2$, where

$dim(\mathcal{E}) = n + 1$,

$codim(\mathcal{E}) = \frac{n(n+1)}{2}$

and

$conv.hull(\mathcal{E}) = \mathcal{U}$.

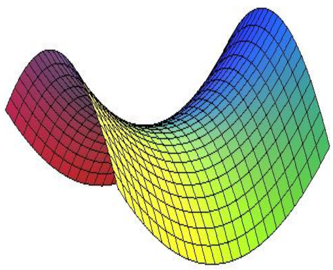


Figure 5: Schematic $\mathcal{E}$

Solutions of the Euler equations in these terms are maps $f : X \to \mathcal{E} \subset \mathcal{U}$ such that $\mathcal{D}_{Eul}(f) = 0$.

R-Directed (Convex) Hulls. Given a subset $R \subset \mathbb{R}^k$, the R-directed hull of a subset $E \subset \mathbb{R}^k$ is the minimal subset $hull_R(E) \subset \mathbb{R}^k$, wich contains E and such that all straight segments parallel to vectors $r \in R$ with the ends in $hull_R$ are contained in $hull_R$.

Examples. (a) If $R = \mathbb{R}^k$, then this is the ordinary convex hull, $hull_{\mathbb{R}^k}(E) = \mathsf{conv.hull}(E)$.

(b) If R consist of a single non-zero vector and $E \subset \mathbb{R}^k$ is a closed convex hypersurface, then also $hull_R(E) = \mathsf{conv.hull}(E)$.

(c) let $\mathcal{E}$ and $\mathbb{R}$ be a smooth submanifolds in *general position.*

If $2dim(E) + dim(R) + 2 \leq k$, then $hull_R(E) = E$.

Exercise. Evaluate the dimension of $hull_R(E)$ for generic smooth submanifolds E and R in $\mathbb{R}^k$ of given dimensions.

Convexity Lemma. Let $\mathcal{E}_e \subset \mathcal{E} \subset \mathcal{U} = \{u, v, q\}$ be the subset of those u, v, q where $\frac{1}{2}||v||^2 = e$ for a given $e > 0$.

Let $R_{\mathcal{D}} \subset \mathbb{R}^{n+1}$ be the set of the $\mathcal{D}$-neutral directions for the above $\mathcal{D} : U \mapsto div(U)$. Then

$$hull_{R_{\mathcal{D}}}(\mathcal{E}_e) = \mathsf{conv.hall}(\mathcal{E}_e).$$

Moreover, the convex hull of $\mathcal{R}$ is equal to the union of segments which are parallel to vectors $r \in R_{\mathcal{D}}$ and which have their ends in $\mathcal{E}$.

The proof is straightforward see [DL-S]??? but, combined with the following property of the linear operator $\mathcal{D}_{Eul} : \{X \times \mathbb{R} \overset{C^\infty}{\to} U = \mathbb{R}^m\} \to \{X \times \mathbb{R} \overset{C^\infty}{\to} \mathbb{R}^{n+1}\}$, $n = dim X$, $m = \frac{(n+1)^2}{2}$, it yields Scheffer's paradox.

Localization Lemma. Let $\mathcal{U}_0 \subset \mathcal{U} = \mathbb{R}^m = \{u = u_{ij}, v = v_i, q\}_{i,j=1,\dots,n}$ be an *open convex centrally symmetric* subset and let

$$\overrightarrow{r_0} = (u_0, v_0, q_0) \in \mathcal{U}_0$$

be a $\mathcal{D}_{Eul}$-neutral vector of codimension one.

Then, for all $\varepsilon > 0$ exists a C^∞-map

$$F_\varepsilon : X \times \mathbb{R} \to \mathcal{U}_0$$

with *support in the ε-ball* $B = B_{\mathbf{0}}(\varepsilon) \subset X \times \mathbb{R} = \mathbb{R}^{n+1}$ such that

$$\mathcal{D}_{Eul}(f) = 0,$$

and such that the v-component of $F(x,t) \in \mathcal{U}_0$, denoted $v_\varepsilon(x,t)$, satisfies

$$\int_B \|v_\varepsilon(x,t)\| dxdt \geq const_n \|v_0\|, \ const_n > 0.$$

The proof is achieved with a representation of the kernel of $\mathcal{D}_{Eul}$ by the image of some differential operator.[54]

(Thus would be immediate if $\mathcal{U}$ consisted of *all* $(n+1) \times (n+1)$-matrices, since closed $(n-1)$-forms locally are d-images of $(n-2)$-forms.

But the conditions $U_{i,j} = U_{j,i}$ and $U_{n+1,n+1} = 0$ require a specific (linear algebraic) construction of a suitable operator Δ (with constant coefficients as well $\mathcal{D}_{Eul}$).

The above convexity and localization lemmas allow a consecutive Nash-style corrections of subsolutions of the Euler equation, which weakly converge to "wild" weak solutions and deliver measurable weak solutions v of Euler with given energies,

$$\frac{1}{2}\|v\|^2(x,t) = e(x,t) > 0.^{55}$$

.

Continuous and Hölder Continuous. If $X = \mathbb{T}^3$, then any smooth subsolution of the Euler equations can be weakly approximated by Hölder-continuous weak solutions with given energies.

***Eu**ler-**Re**ynolds system.*

$$\partial_t(v) + \mathsf{div}(v \otimes v) + \mathsf{grad}_X(p) = \mathsf{div}(\breve{R}),$$

$$\mathsf{div}(v) = 0,$$

where $\breve{R} = \breve{R}(x,t)$ is (like $v \otimes v$) a symmetric trace free $(n \times n)$-matrix function.

Smoothing: $(v,p) \mapsto (\bar{v}, \bar{p})$. $v = \bar{v} + w$

[54] See ???DL-S,[Vil]

[55] see ???

$$\partial_t(\bar{v}) + \mathsf{div}(\bar{v} \otimes \bar{v}) + \mathsf{grad}(p) = \mathsf{div}(\check{R}),$$

$\check{R} = \bar{v} \otimes \bar{v} - \overline{v \otimes v} = -\overline{w \otimes w}(??)$

"Wild" continuous solutions to the Euler equation are obtained from solutions of the EuRe system by iteration process consecutively diminishing the $\check{R}$-terms, were the main building blocks are *Beltrami flows* – particular stationary periodic solutions to the 3D Euler equation.

Stationary Flows:

$$v \times \mathsf{curl}(v) = \mathsf{grad}(\beta)$$

$\beta = p + ||v||^2/2$

Beltrami flows are where $\beta = 0$ and where the above equation becomes linear:

$$\mathsf{curl}(v) = \lambda v$$

For constant λ these are eigen vectors of the operator $v \mapsto \mathsf{curl}(v)$.

Universality of the Euler flows.

[56]

15 Hölder Immersions

7.A. Problem. Let $X = X^n$ be a smooth Riemannian manifold, which admits an immersion $f_0 : X \to \mathbb{R}^N$.

For which $0 < \alpha \leq 1$, do short maps $X \to \mathbb{R}^N$ admit C^0-approximation by isometric $C^{1+\alpha}$ immersions?

7.B. Borisov Conjecture (1965).[57] (a) If $\alpha \leq \frac{1}{2}$, such an approximation $X \to \mathbb{R}^N$ exists for all X, f and $N > n$.

(b) If $\alpha > \frac{1}{2}$, and $n \geq 2$, then $C^{1+\alpha}$-immersions $f : X \to \mathbb{R}^{n+1}$ are smooth for most smooth Riemannian manifolds X.

For instance, if X^n, $n \geq 2$, admits a smooth isometric immersion to $\mathbb{R}^{n+1}$, where the Gauss map $X \to S^n$ has $rank \geq 2$, or at least $rank = n$, e.g $X = S^n$, then, probably, all isometric $C^{1+\alpha}$-immersions $f : X \to \mathbb{R}^{n+1}$ are C^∞. (This is unclear even for C^2-immersions f)

7.C. De Lellis-Székelyhidi-Borisov Hölder Immersion Theorem.[58] Short immersions between compact smooth Riemannin manifolds,

$$f_0 : X^n \to Y^N, \; N = n + 1,$$

can be uniformly approximated by isometric Hölder $C^{1+\alpha}$-immersions in the following cases.

(i_n) $\alpha < \frac{1}{1+n(n+1)^2}$;

[56] Robert Cardona Aguilar. *The geometry and topology of steady Euler flows, integrability and singular geometric structures* https://upcommons.upc.edu/bitstream/handle/2117/349573/TRCA1de1.pdf?sequence=1

Steady Euler flows and Beltrami fields in high dimensions Robert Cardona arXiv:2003.08112 [math.DS]

[57] Yu. Borisov, $C^{1+\alpha}$-isometric immersions of Riemannian spaces,

[58] S. Conti, C. De Lellis. L. Székelyhidi Jr. *h-Principle and Rigidity for $C^{1+\alpha}$-Isometric Embeddings*, Also see *High dimensionality and h-principle in PDE* by De Lellis and László Székelyhidi.

(ii_n) X is homeomorphic to the n-ball and $\alpha < \frac{1}{1+n(n+1)}$;

(iii_2) X is homeomorphic to the 2-ball and $\alpha < \frac{1}{2s_2-1} = \frac{1}{5}$.

Remarks.(a) Let $X = X^n$ admit a smooth immersion to $\mathbb{R}^n$, e.g. it is obtained by removing a point or a ball from a closed connected hypersurface in $\mathbb{R}^{n+1}$. [59] Then there exit $s_n = \frac{n(n+1)}{2}$ smooth functions ϕ_i on X, $i = 1, ..., s_n$, such that the linear combinations

$$g = g(x) = \sum_{i=1}^{s_n} a_i(x) d\phi_i^2(x),$$

where $a_i(x) > 0$, $i = 1, ...s_n$, are C^2-functions, make an *open cone* in the space of continuous Riemannian metrics on X.

It follows (unless I am mistaken) that if $\alpha < \frac{1}{1+n(n+1)^2}$, then the proof of ($ii_n$) in ??? delivers isometric $C^{1+\alpha}$-immersions $(X, g) \to \mathbb{R}^{n+1}$ for all C^2-smooth metrics g on X.

In fact, it seems that

(b) Let X be a stably parallizabe n-manifold,[60] then it admits a *folded map* $\Phi : X \to \mathbb{R}^n$ by *Poenaru's folding theorem.* Therefore,

there exits $s_n + 1$ smooth functions ϕ_i on X, $i = 0, ..., s_n$, where s_0 vanishes on the folding locus $\Sigma_\Phi \subset X$ and such that the linear combinations

$$g(x) = a_o \phi_0 + \sum_{i=1}^{s_n} a_i(x) d\phi_i^2(x),$$

where $a_i(x) > 0$, $i = 1, ...s_n$, are continuous functions and $a_0 >$ is a constant, make an *open cone* in the space of continuous Riemannian metrics on X,

Since immersions of orientable manifolds X to $\mathbb{R}^{n+1}$ carry unit normal fields, the argument in ??? (unless I misunderstood it) shows that

if $\alpha < \frac{1}{2+n(n+1)}$. then (X, g) admits an isometric $C^{1+\alpha}$-immersions $(X, g) \to \mathbb{R}^{n+1}$ for all C^2-smooth metrics g on X.

In fact, it seems that the argument in ??? yields the following relative version of (1_n).

Let $\phi_i : \mathbb{R}^n$, $i = 1, ..., s_n = \frac{n(n+1)}{2}$, be C^∞-functions with linearly independent $d\phi_i^2(x)$, $x \in \mathbb{R}^n$, and let

$$g = (1 - ||x||^2)^2 \sum_{i=1}^{s_n} a_i(x) d\phi_i^2(x)$$

for C^2-smooth functions $a_i(x) > 0$.

Let $Y = (Y, h)$ be a C^∞-smooth N-dimensional Riemannian manifold and let $f_0 : \mathbb{R}^n \to Y$, be a $C^{1+\beta}$-immersion, which is C^∞ on the open unit ball $B \subset \mathbb{R}^n$.

If $\alpha < \min\left(\beta, \frac{1}{1+n(n+1)}\right)$ then f_0 can be C^0-approximated by $C^{1+\alpha}$ -immersions $f : \mathbb{R}^n \to Y$, such that

$$f^*(h) = f_0^*(h) + g.$$

Granted this, the induction by skeleta argument (as in 2410 and 2411 in [PDR]) upgrade the above inequality (i^n) to the (ii^n)-level:

[59] By Hirsch' immersion theorem all open parallelizable X immerse to $\mathbb{R}^n$.

[60] That is an orientable n-manifold immersible to $\mathbb{R}^{n+1}$ by Hirsch theorem.

if $\alpha < \frac{1}{2+n(n+1)}$, *then short immersions* $X^n \to Y^N$, $N > n$ *can be approximated by isometric* $C^{1+\varepsilon}$*-immersions.*

7.D. Borisov Hölder $2/3$**-regularity Regularity Theorem.**[61] *If*

$$\alpha > \frac{3}{2}$$

then $C^{1+\alpha}$*-surfaces where the induced metrics are smooth and have positive curvatures, are smooth.*[62]

W*enger-Young maps Carnot Spaces*

Besides isometric immersions and dynamics of liquids "soft and wild" Hölder solutions of PDE appear among maps between Carnot spaces as is demonstrated by Stefan Wenger, Robert Young in *Constructing Hölder maps to Carnot groups*, arXiv:1810.02700 (2018)

16 Soft C^∞

C^∞-immersion of a smooth manifold X to a smooth Riemannian $Y = (Y, h)$,

$$f : X \to Y$$

is called $\mathcal{II}_h$ or just $\mathcal{II}$, if the Riemannian metric inducing operator

$$\mathscr{I} = \mathscr{I}_h : \mathcal{F} = C^\infty(X, Y) \to \mathcal{G}_+(X)$$

for

$$f \overset{\mathscr{I}}{\mapsto} g = f^*(h)$$

is ***i****nfinitesimally* ***i****nvertible.*

This means that .the differential/linearization of $\mathscr{I}$,

$$\mathscr{L}_f : T_f(\mathcal{F}) \to T_{\mathscr{I}(f)}(\mathcal{G}),$$

of $\mathscr{I}$ is *right invertible* by a *differential operator*

$$\mathscr{M}_f : T_{\mathscr{I}(f)}(\mathcal{G}) \to \mathscr{T}_f(\mathcal{F}),\ \mathscr{L}_f \circ \mathscr{M}_f = Id : T_{\mathscr{I}(f)}(\mathcal{G}) \to T_{\mathscr{I}(f)}(\mathcal{G}).$$

This, if $Y = \mathbb{R}^N$, (and in local coordinates for all Y, in general) can be written as an operator on maps $\overrightarrow{f} : X \to Y$,

$$\mathscr{L}_f(\overrightarrow{f})) = \mathscr{I}(f + \epsilon\overrightarrow{f}) - \mathscr{I}(f) + o(\epsilon),\ \epsilon \to 0,$$

and where $\mathscr{M}_f(\overrightarrow{g})$ is a differential operator in $(f, \overrightarrow{g})$, which is linear in $\overrightarrow{g}$ and which satisfies

$$\mathscr{L}_f(\mathscr{M}_f(\overrightarrow{g})) = \overrightarrow{g}.$$

"Free" Example. Free immersions f, i.e. where (second) osculating spaces $osc_2(f(x)) \in T_{f(x_(Y)}$ have dimensions

$$\frac{dim(X)(dim(X)-1)}{2} + dim(X)$$

[61] Yu Borisov, The parallel translation on a smooth surface.

[62] Compare with Sören Behr, Heiner Olbermann; *Extrinsic curvature of codimension one isometric immersions with Hölder continuous derivatives* arXiv:1601.05959 [math.DG]

at all points $x \in X$, are $\mathcal{II}$ by the Janet-Burstin-Nash Lemma.

Consequently,

generic f are $\mathcal{II}$ for $dim(Y) \geq \frac{dim(X)(dim(X)-1)}{2} + 2dim(X)$.[63]

Generalized Nash Implicit Function Theorem for $\mathcal{II}$ operators.

Isomeric C^2-Immersions with Prescribed Curvature.

Bisymmetric 4-forms Φ on $\mathbb{R}^n$ are symmetric bilinear forms on the symmetric square $(\mathbb{R}^n)^{ⓢ}$,

$$\Phi \in \mathcal{S}_2 = ((\mathbb{R}^n)^{ⓢ})^{ⓢ}.$$

$$dim(\mathcal{S}_2) = \frac{n(n+1)}{4}\left(1 + \frac{n(n+1)}{2}\right) \approx n^4/8.$$

$\Phi(x_1, x_2, x_3, x_4)$ are symmetric for $x_1 \leftrightarrow x_2$, $x_3, \leftrightarrow x_4$, $(x_1, x_2) \leftrightarrow (x_3, x_4)$.

Bisymmetric forms split into fully symmetric and anti symmetric ones for

$$\mathcal{S}_2 = \mathcal{S}_2^+ \oplus \mathcal{S}_2^-,$$

where , $\Phi = \Phi^+ = \Phi^-$, $\Phi^-(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_3, x_4) - \Phi(x_1, x_4, x_3, x_2)$ and where

$$dim\mathcal{S}_2^+ = \frac{n(n+1)(n+2)(n+3)}{24}) \approx n^4/24,$$

,

$$dim\mathcal{S}_2^- = \frac{n^2(n^2-1)}{12} \approx n^4/12.$$

$\Phi^-(x_1, x_2, x_3, x_4) = \Phi(x_1, x_2, x_3, x_4) - \Phi(x_1, x_4, x_3, x_2), \Phi = \Phi^+ + \Phi^-$

Isometric C^2-immersions $f : X \to Y$ come with the "second fundamental" forms on X

$\Phi_f(\partial_i, \partial_i, \partial_j, \partial_k, \partial_l) = \langle \nabla_{ij} f, \nabla_{kl} f \rangle_Y$, where (anti symmetric) Φ^- is equal to the curvature tensor of X by the Gauss formula.

A. C^2-Curvature Immersion Theorem. *Given a free isometric C^∞-immersion $f_0 : X \to Y$ and a form $\Phi^+ > \Phi^+_{f_0}$ on X, there exists a C^1-approximation by free isometric C^2-immersions $f : X \to Y$, such that $\Phi^+_f = \Phi^+$, provided* $N = dim(Y) \geq (n+2)(n+5)/2$, where the corresponding PDE system of the second order contains more than $n^4/24$ equations in N-variables.

Question. Is the condition $N = dim(Y) \geq (n+2)(n+5)/2$ necessary for $n \geq 2$?

B. Euclidean Example/Corollary. *The standard embedding $f_0 : \mathbb{R}^n \to \mathbb{R}^{(n+2)(n+5)/2}$ can be C^1-approximated by isometric C^2-embeddings f with a given strictly positive curvature $\Phi^+_f > 0$.* ($\Phi^-_f = 0$ for isometric f, since $Riem.curv(\mathbb{R}^n) = 0$.)

(B reduces to A, with a C^∞-approximation of f_0 by *free* isometric embeddings.)

17 Immersions with Bounded Curvature

1.G. Small Curvature Approximation Theorem . Let $X^n = (X^n, g)$ and $Y^N = (Y^N, h)$ be smooth Riemannian manifold and $f_0 : X \to Y = (Y, h)$ be a

[63]See [Gr1986], [Gr2017] and references therein.

smooth *strictly short map*, i.e, the quadratic differential form $g - f^*(h)$ is positive definite.

If X^n is compact and

$$N \geq \frac{(2n-1)(2n-2)}{2} + 3n \sim 2n^2,$$

($3n = 2n - 1 + 1 + n$) then there exist δ_i-*approximation* of f_0 for $\delta_i \leq \frac{1}{i}$, $i = 1, 2, \ldots$, by isometric C^∞-immersions $f_i : X^n \to Y$ with

$$curv(f_i((X)) \leq i \cdot C_n + o(i).$$

If X imbeds to $\mathbb{R}^{n+1}$, the same is true for $N \geq \frac{n(n+1)}{2} + 2n + 3 \sim n^2/2$

If $N \geq 10n^2$ then $C_n < \sqrt{3}$

18 ???

Comparison with the group theory, such as conjectural "phase transition" in Burnside problem from finite: exponents 2,3,4, 6, problematic for 5 and conjecturally "quasihyperbolic" starting from 7.

.... we do not have the mathematical power today to analyze them except for very small Reynolds numbers—that is, in the completely viscous case. That we have written an equation does not remove from the flow of fluids its charm or mystery or its surprise.

Perhaps the fundamental equation that describes the swirling nebulae and the condensing, revolving, and exploding stars and galaxies is just a simple equation for the hydrodynamic behavior of nearly pure hydrogen gas.

The next great era of awakening of human intellect may well produce a method of understanding the qualitative content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders.

Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality—or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

VOLUME 78, NUMBER 6 P H Y S I C A L R E V I E W L E T T E R S 10 FEBRUARY 1997 Chemical Kinetics is Turing Universal Marcelo O. Magnasco

Convex integration and phenomenologies in turbulence Tristan Buckmaster * Vlad Vicol

http://www.science.unitn.it/cirm/Bardos-course2.pdf

arXiv:2303.02036v1 [math.AG] 3 Mar 2023

Gromov Ellipticity and subellipticity Shulim Kaliman, Mikhail Zaidenberg