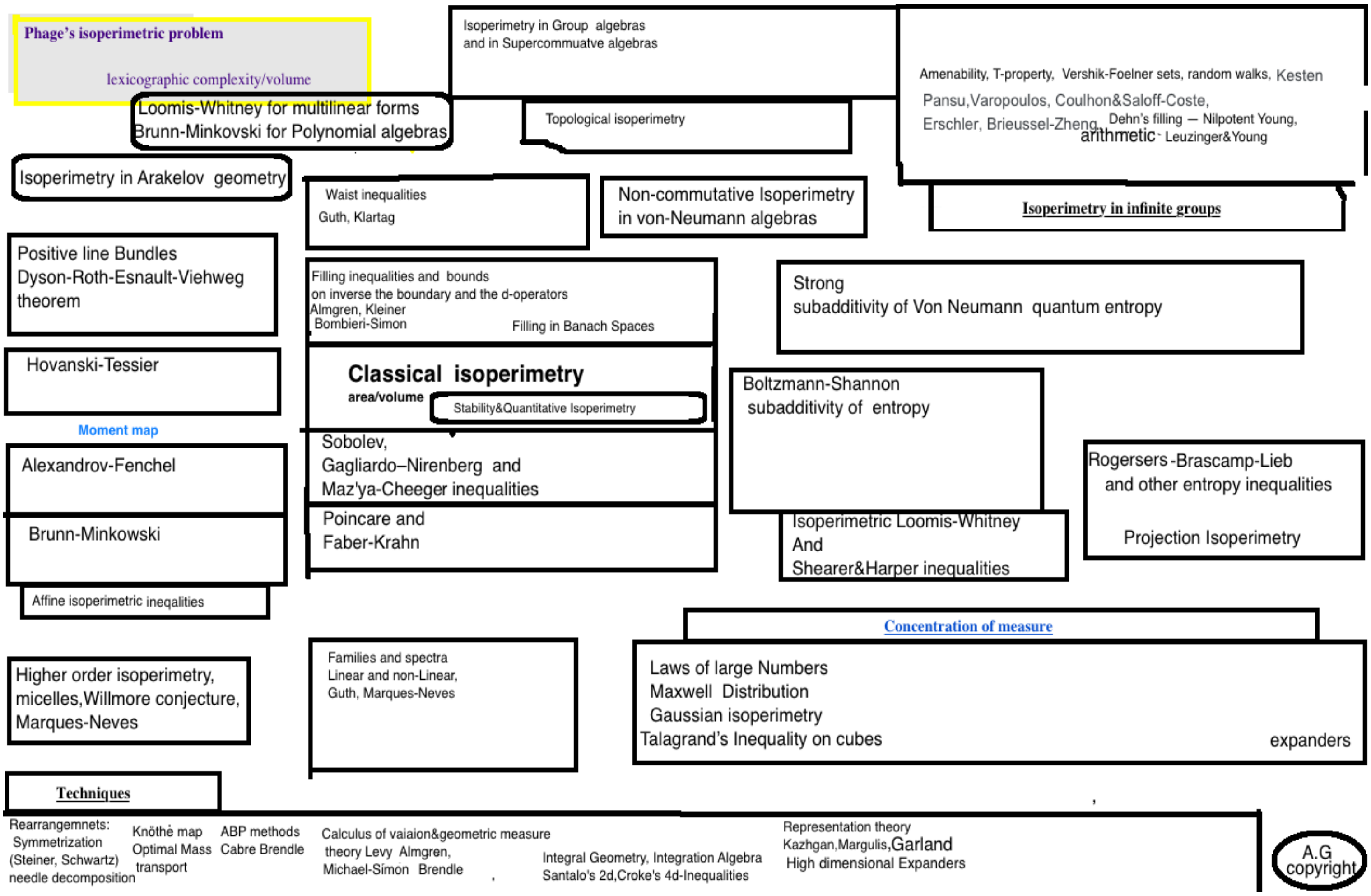
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Isoperimetric Inequalities in Geometry, Analysis, Algebra and Probability Theory, Fall 2023In The Process of rewriting and editing

Misha Gromov

December 6, 2023

Abstract

These notes are to accompany my lectures at the Courant Institute in the Fall 2023. Besides presenting basic theorems, we try to show several different proofs of most of them.

Contents

1 Magnificent Seven

1.1 Classical Eucleadean and non-Euclidean Isoperimetry

The volumes of all bounded domains $X \subset \mathbb{R}^n$ are bounded by "areas" their boundaries

$[Is]_X$ $$Vol_n(X) \leq C_n Vol_{n-1}(\partial X)^{n/n-1},$$

where the constant C_n is such that the unit balls $B^n = B^n(1) \subset \mathbb{R}^n$ satisfy the equality

$$Vol_n(\partial B^n) = C_n Vol_{n-1}(S^{n-1})^{n/n-1},$$

where $S^{n-1} = \partial B^n$ is the unit sphere. For instance $C_2 = \frac{1}{4\pi} = \frac{\pi}{(2\pi)^2}$ and $C_3 = \frac{1}{6\sqrt{\pi}} = \frac{(4/3)\pi}{(4\pi)^{3/2}}$.

Furthermore, the equality $Vol_n(\partial X) \leq C_n Vol_{n-1}(\partial X)^{n/n-1}$ implies that X is a round ball.

Thus

Among all domains with a given volume, balls and only balls have minimal surface area.

(This is obvious for $n = 2$ by calculus of variations: extremal $Y = \partial X$ are closed curves with *constant curvature*, hence, circles, where justification of regularity of extremal Y is easy for $dim(Y) = 1$.)

1.1.1 Sharp Isoperimetric Inequalities in Spheres, Balls, Hyperbolic Spaces and Gaussian SpacesMOVE?

Besides the above, the sharp isoperimetric inequalities are known in all simply connected *fully homogeneous* Riemannian manifolds X, where "fully homogeneous" signify that all isometries between subsets $X_1 \leftrightarrow X_2$ between subsets $X_1, X_2 \subset X$ extends to isometries $X \leftrightarrow X$ and where the *extremal hypersurfaces*, i.e.

hypersurfaces of given "areas" enclosing maximal volumes are metric spheres.

(The only fully homogeneous Riemannian, as well as *finite dimensional geodesic*, spaces besides the Euclidean ones are spheres, real projective spaces, and hyperbolic spaces, where the real projective spaces $S^n/\mp 1$ are non-simply connected.

According to [Vil 2023], *the extremal hypersurfaces in* $S^n/\mp 1$*, are* $S^k \times S^l/\mp 1$ *for* $O(k+1) \times O(l+1)$*-equivariant (Clifford) embeddings.* $S^k \times S^l \hookrightarrow S^n$, $k + l = n - 1$.[1])

Also one knows (we prove this in section???) that

the extremal hypersurfaces S *in the (say open) balls* $B \subset X$ *in the above* (simply connected fully homogeneous Riemannian manifolds) X *are totally umbilical.*

(These, in the present case, are intersections of *fully homogeneous*, hence complete, hypersurfaces $Y \subset X$ with B.)

For instance, hypersurfaces, which divide Euclidean n-balls B in equal halves are intersection of B with hyperplanes.

(This brings to one's mind *Bourgain's Difficult Slicing Problem.* [2] Let $X \subset \mathbb{R}^n$ be convex body of unit volume. Does there exist a hyperplane $H \subset \mathbb{R}^n$, such that

$$vol_{n-1}(H \cap K) \geq \delta$$

for some universal $\delta > 0$, say for $\delta = 0.1$?)

1.1.??? Metric&Measure, μ-Isoperimetry&μ-Extremaliy. Let us extend the range of isoperimetric phenomena to Riemannin manifolds X, e.g,

[1] I have not studied yet the proof of this theorem.

[2] https://www.weizmann.ac.il/math/klartag/sites/math.klartag/files/uploads/bourgain_slicing_problem.pdf

$X = \mathbb{R}^n$ with smooth *non-Riemannian measures* $\mu(x) = \phi(x)dx$ on them, e.g. to measures μ on $\mathbb{R}^n$ with finite mass $M = \mu(\mathbb{R}^n)$.

Here a cooriented hypersurface $Y \subset \mathbb{R}^n$, which divide the space into halves with given masses, say M_- and M_+, $M_1 + M_+ = M$, is called *μ-extremal* if it minimizes the integral $\int_Y \phi(y)dy$. and thus solves the $-mu$-isoperimetry problem,

In general, the solution to such a problem seems fairly complicated but for *the Gaussian μ* it comes up with an unexpectedly neat solution.

1.1.??? Tsirelson-Sudakov-Borell Theorem. *If*

$$\mu(x) = e^{-\|x\|^2}dx,$$

then the μ-extremal hypersurfaces are affine hyperplanes.

This follows from the isoperimetric inequality for S^N for $N \to \infty$, since the Gauss measure on $\mathbb{R}^n$ is equal to the limit of the push-forwards of the normalized spherical measures on the spheres $S^{N+n-1}(R)$, $R = \sqrt{N+n}$, under the normal projections $\mathbb{R}^{N+n} \to \mathbb{R}^n$ (see section ???).

1.2 Sobolev and Gagliardo–Nirenberg

Smooth functions f with compact support in $\mathbb{R}^n$ satisfy

$$\left(\int |f(x)|^{n/n-1}dx\right)^{n-1/n} \le C_n \int \|df(x)\|dx$$

with the above constant C_n.

In fact, as we shall see presently, if all compact domain V in a Riemannian n-manifold X satisfy

$$vol(V) \le Cvol_{n-1}(\partial V)^{n/n-1} \Leftrightarrow$$

for some constant C, then the inequality

$$\left(\int |f(x)|^{n/n-1}dx\right)^{n-1/n} \le C_n \int \|df(x)\|dx$$

holds for the functions f with compact supports in X[3][Maz 1960)].

1.3 Minkowski Concavity

Given subsets $X, Y \subset L$ in a linear n-space $L(=\mathbb{R}^n)$, the Minkowski sum is

$$X + Y = \{x, y\} \subset L, x \in X, y \in Y,$$

that is the image of the product $X \times Y \subset L \times L$ under the addition map

$$L \times L \xrightarrow{+} L,\ (l_1, l_2) \mapsto l_1 + l_2$$

Minkowski $\frown^{1/n}$-Inequality. The volume of the Minkowski sum of arbitrary open subsets in $\mathbb{R}^n$ satisfies

$$[\frown^{1/n}]_M \qquad (vol(X+Y))^{1/n} \ge (vol(X))^{1/n} + (vol(Y))^{1/n},$$

[3]This, applied to powers of $|f|$ yields all Sobolev Inequalities $\sqrt[p]{\int |f|^p} \le cons \sqrt[q]{\int \|df\|^q}$ for $p, q \ge 1$ and $p \le nq/(n-q)$.

Equivalently,

$$\left(vol\left(\frac{X+Y}{2}\right)\right)^{1/n} \geq \frac{1}{2}\left(vol(X)^{1/n} + vol(Y)^{1/n}\right),$$

where $\frac{X+Y}{2}$ geometrically is the set of the centers of the segments $[x,y] \subset \mathbb{R}^n$ for $x \in X$ and $y \in Y$. [4]

If Y is an infinitesimal ball $Y = B^n(o(1))$ then $[\frown^{1/n}]_M$ implies the isoperimetric inequality for X.

1.4 Almgren's Sharp Filling Inequality

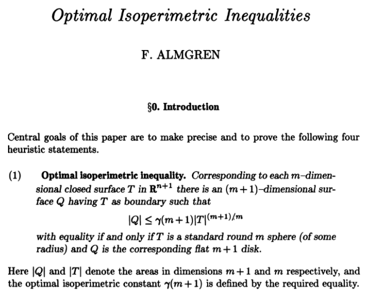
Let $Y \subset \mathbb{R}^N$ be a piecewise smooth $(n-1)$-cycle e.g. a smoothly embedded sphere $S^{n-1}(1) \overset{f}{\hookrightarrow} \mathbb{R}^N$. Then Y bounds a piece-wise smooth n-chain $X \subset \mathbb{R}^N$,

$$\partial X = Y,$$

which satisfy the above $[Is]_X$

$$Vol_n(\partial X) \leq C_n Vol_{n-1}(\partial X)^{n/n-1},$$

where the equality holds only for flat round spheres $S^n(R) \subset \mathbb{R}^n \subset \mathbb{R}^N$


Optimal Isoperimetric Inequalities

F. ALMGREN

§0. Introduction

Central goals of this paper are to make precise and to prove the following four heuristic statements.

(1) **Optimal isoperimetric inequality.** *Corresponding to each m–dimensional closed surface T in $\mathbf{R}^{n+1}$ there is an $(m+1)$–dimensional surface Q having T as boundary such that*

$$|Q| \leq \gamma(m+1)|T|^{(m+1)/m}$$

with equality if and only if T is a standard round m sphere (of some radius) and Q is the corresponding flat $m+1$ disk.

Here $|Q|$ and $|T|$ denote the areas in dimensions $m+1$ and m respectively, and the optimal isoperimetric constant $\gamma(m+1)$ is defined by the required equality.

Moreover, if $Y = f(S^{n-1}(1))$, then the map $f : S^{n-1}(1) \hookrightarrow \mathbb{R}^N$ extends to a smooth map $F : B^n(1) \to \mathbb{R}^N$, such that

$$Vol_n(F : B^n(1)) \leq C_n (vol_{n-1} f(S^{n-1}))^{n/n-1},$$

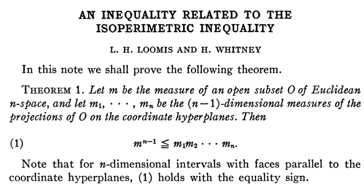
where this F can be chosen to be a smooth embedding for $2N > 3n$

1.5 Loomis-Whitney Inequality and Subadditivity of Entropy

Let $X \subset \mathbb{R}^n$ be a measurable subset and let $X_{\hat{i}} \subset \mathbb{R}^{n-1}_{\hat{i}} = \mathbb{R}^n, i = 1, \ldots, n,$ be the normal projections of X to the hyperplanes

$$\mathbb{R}^{n-1}_{\hat{i}} = \{(x_1, \ldots, x_n)\}_{x_i=0} \subset \mathbb{R}^n$$

$$[\mathsf{Loo-Whi}] \quad vol_n(X) \leq \bigtimes_{i=1}^{n} vol_{n-1}(X_i)^{1/n-1}.$$


AN INEQUALITY RELATED TO THE ISOPERIMETRIC INEQUALITY

L. H. LOOMIS AND H. WHITNEY

In this note we shall prove the following theorem.

THEOREM 1. *Let m be the measure of an open subset O of Euclidean n-space, and let $m_1, \cdots, m_n$ be the $(n-1)$-dimensional measures of the projections of O on the coordinate hyperplanes. Then*

(1) $$m^{n-1} \leqq m_1 m_2 \cdots m_n.$$

Note that for n-dimensional intervals with faces parallel to the coordinate hyperplanes, (1) holds with the equality sign.

This, almost obviously, implies a non-sharp isoperimetric inequality, namely

$$vol(X) \leq \frac{1}{(2n))^{n/n-1}} vol_{n-1}^{n/n-1}(\partial X)$$

with equality for cubes $X = [01]^n$.

In turn, [Loo-Whi] also almost obviously, follows from the *Shannon (Boltzmann?* inequality:

[4] This makes sense for subsets in Riemannian manifols, while the additive $[\frown^{1/n}]_M$ generalises to subsets in Lie Groups.

Strong Subadditivity of Entropy. Let $\mu = \mu_{123}$ be a probability measure on $\mathbb{R}^3$ let μ_1, μ_2, μ_3 be the push-forwards of μ to the coordinate lines (coordinate marginals of μ) and μ_{12}, μ_{13} and μ_{23} be the push-forwards of μ to the coordinate planes. (Thus, μ_1 is the marginal of μ_{12} as well as of μ_{13}, etc.)

3. Suppose there are two events, x and y, in question with m possibilities for the first and n for the second. Let $p(i,j)$ be the probability of the joint occurrence of i for the first and j for the second. The entropy of the joint event is

$$H(x,y) = -\sum_{i,j} p(i,j) \log p(i,j)$$

while

$$H(x) = -\sum_{i,j} p(i,j) \log \sum_{j} p(i,j)$$
$$H(y) = -\sum_{i,j} p(i,j) \log \sum_{i} p(i,j).$$

It is easily shown that

$$H(x,y) \le H(x) + H(y)$$

with equality only if the events are independent (i.e., $p(i,j) = p(i)p(j)$). The uncertainty of a joint event is less than or equal to the sum of the individual uncertainties.

Figure 1: From "*The mathematical theory of communication*" by Shannon

Then *Boltzmann's entropies* of these satisfy

$$ent(\mu_{123}) \le ent(\mu_{12}) + ent(\mu_{13}) - ent(\mu_1),$$

where the *entropy of a measure* $\mu = \mu(x) = \phi(x)dx$ for a positive function $\phi(x) \ge 0$, such that $\int_X \phi(x)dx = 1$ and such that $\log \phi$ is summable on the support $S = S(\phi)$, $\left(\int_S |\log \phi(x)| dx < \infty\right)$, are evaluated by the Boltzmann-Gibbs formula:

$$ent(\mu(x) = -\int \phi(x) \log \phi(x) dx^5$$

1.6 Poincaré Inequality on the unit n-Sphere $X = S^n(1)$.

If

$$\int_X f(x)dx = 0,$$

then

$$\int_X f^2(x)dx^2 \le \frac{1}{n^2} \int_X \|df\|^2 dx,$$

Equivalently, all smooth maps

$$F : X \to \mathbb{R}^N$$

satisfy

$$\frac{n^2}{2vol(X)} \int_{X\times X} \|F(x_1)-F(x_2)\|^2 dx_1 dx_2 \le \frac{1}{Vol(S^{n-1})} \int_{UT(X)} \|\partial_{\tau_x} F(x)\|^2 d\tau_x dx,$$

where $UT(X)$ is the unit tangent bundle of X and $\partial_{\tau_x} = dF(\tau_x)$, $\tau_x \in S_x^{n-1} = UT_x(X)$, is the derivative of F by the vector τ_x.

[5] This can be taken for the definition of entropy for all measure spaces (X, dx), e.g. for $X = \mathbb{R}^{6N}$ (Boltzmann's N-partical gas) and/or for Shannon's finite or countable sets $X = \{x_i\}$ with atoms x_i of equal weights, where $ent(\mu) = -\sum_{i\in I} \log_2 \mu(x_i)$, compare `https://www.crmarsh.com/pdf/Charles_Marsh_Continuous_Entropy.pdf`.

1.7 Selberg Theorem, Selberg $\frac{1}{4}$-Conjecture and Ramanujan Graphs.

Preparation to Selberg. There is a strictly decreasing sequence of subgroups of finite index in the free group on two generators,

$$F_2 = \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_l \supset \ldots,$$

with the following property.

Let X be a compact connected Riemannian manifold, such that the fundamental group $\pi_1(X)$ admits a homomorphism onto F_2, e.g. X is a smooth bounded planar domain with at least two holes, such as a small neighbourhood of the figure ∞.

Then there exist compact Riemannian manifolds X_l and l-sheeted coverings $X_l \to X$,[6] *where $l \to \infty$* and such that all smooth functions $f(x_l)$ on X_l with $\int_{X_l} f(x_l)dx_l = 0$ satisfy

$$\int_{X_l} \|df(x_l)\|^2 dx_l \geq \mathsf{const} \cdot \int_{X_l} f^2(x_l)dx_l^2,$$

where the constant $\mathsf{const} = \mathsf{const}(X) > 0$ *doesn't depend on l.*

Equivalently maps $F : X_l \to \mathbb{R}^N$ satisfy

$$\frac{\frac{1}{vol(X_l)^2}\int_{X_l\times X_l} \|F(x_l) - F(x_l')\|^2 dx_l, dx_l'}{\frac{1}{vol(X_l)}\int_{X_l} \|dF(x_l)\|^2 dx_l} \leq \frac{2}{\mathsf{const}}{}^7$$

In truth, the above are "coarse corollaries" of a particular instance of a precise form of such an inequality for a specific family of complete non-compact Riemann surfaces X_l with constant curvatures -1 and with finite volumes proved by Selberg in 1965.

These Selberg's X_l are the quotients,

$$X_l = H^2/\Gamma_l,$$

where H^2 is the hyperbolic plane and Γ_l are subgroups in the group of (2×2)-matrices (a_{ij}) with integer entries and determinants one,

$$\Gamma_l \subset SL(2,\mathbb{Z}),$$

where $SL(2,\mathbb{Z})$ naturally acts on H^2 by isometries and where where Γ_l consists of matrices congruent to upper triangular ones $\mod l$, i.e. where the entry $a_{12} = 0 \mod l$.[8]

---

[6] These are locally isometric maps with l pullbacks of all $x \in X$. In particular $vol(X_l) = d \cdot vol(X)$.

[7] One's experience in the classical PDE – (*Hersch S^2-eigenvalue theorem, Rayleigh–Faber–Krahn Inequality...*) points to the opposite:

$$\frac{\int_{X_l\times X_l} \|F(x_l) - F(x_l')\|^2 dx_l, dx_l}{\int_{X_l} \|dF(x_l)\|^2 dx_l} \to 0$$

for l-sheeted coverings of compact manifolds: the ground frequency of an oscillating membrane X must tend to zero for $size(X) \to \infty$. This is true in the "real world", and, probbaly, true in mathematics under some reasonable assumptions on X, but... this is not so in general.

[8] Strictly speaking, these X_l are not quite coverings of $X_1 = H^2/SL_{\mathbb{Z}}(2)$, since torsion elements in $SL(2,\mathbb{Z})$ do not act freely on the hyperbolic plane H^2, but this needs only a minor adjustment of our terminology.

Selberg proved that the above constant in his case, call it $\underline{\lambda} = const(X_{Selb})$ is bounded from below by 3/16 and *conjectured that* $\underline{\lambda} \geq 1/4$; The best current bound is $\underline{\lambda} \geq \frac{1}{4}\left(\frac{7}{64}\right)^2$[9]

Remark. Mathematics behind $\underline{\lambda}$ is fundamentally different from what is seen in the other six famous "isoperimetric theorems".

The later essentially depend on similarities between the geometries of the spaces these theorems apply to with the geometries of the corresponding *Euclidean models*, where everything boils down to the inequality $|f(1) - f(0)\| \leq \int_0^1 |f(t)|dt$ with occasional use of the $O(n)$-symmetries.

But the geometry of the spaces X_l for $l \to \infty$ is maximally non-Euclidean,[10] which, in fact, follow from the inequality

$$\liminf_{l\to\infty} \underline{\lambda}(X_l) > 0.$$

2 Methods of Proofs

2.1 Coarea Inequality, Volumes of Cones, Divergence and Green's Formula

2.1.A. Coarea Equality. Let X be a Riemannian manifold, e.g. the Euclidean N-space $X = \mathbb{R}^N$ let $X_0 \subset X$ be a subset, e.g. the origin $\{0\} \subset \mathbb{R}^N$ and let

$$d_0(x) = dist(x, X_0)$$

be the distance function to X_0 (e.g. $d_0(x) = \|x\|$ for $X_0 = \{0\} \in \mathbb{R}^N$).

Let $V \subset X \setminus X_0$ be a measurable, e.g an open subset in the complement of X_0. Then the $(N-1)$-volumes, $N = dim(X)$, of the intersections $V_r = V \cap d_0^{-1}(r) \subset V$ of V with the r-levels of the function d_0 satisfy:

$$\int_0^\infty vol_{N-1}(V_r)dr = vol_N(V).$$[11]

2.1.B. Coarea Inequality. Let $V \subset X$ be a smooth n-dimensional submanifold. Then the intersections $V_r = V \cap d_0^{-1}(r) \subset V$, which are smooth $(n-1)$-submanifolds for almost all r by the *Sard theorem*, satisfy:

$$\int_0^\infty vol_{n-1}(V_r)dr \leq vol_n(V).$$[12]

2.1.C. Cone Inequality. Let $\mathsf{Cone}_{x_0}(Y) \subset \mathbb{R}^N$ be the cone over a submanifold $Y = Y^{n-1} \subset \mathbb{R}^N$ from a point $x_0 \in \mathbb{R}^n$.

Then the n-volume of this cone is

$$\mathsf{vol_n}(\mathsf{Cone}_{x_0}(Y)) = \frac{1}{n}\int_Y \|x_0 - y\| \sin\angle(x_0 - y, T_y(Y))dy \leq \frac{1}{n}\int_Y \|x_0 - y\|dy$$

[9]Kim & Sarnak 2003).https://www.ams.org/notices/199511/sarnak.pdf

[10]These X_l admit no approximately isometric embeddings to the Hilbert space. In fact, 1-Lipschitz maps $f: X_l \to \mathbb{R}^\infty$ satisfy $vol(X_l)^{-1}\int_{X\times X} dist(f(x), f(y))/dist(x,y)dxdy \to 0$ for $l \to \infty$.

[11]This, for $d_0(x) = \|x\|$ in the 3-space, may be attributed to Cavaleri (1635) and in general to Fubini (1907) and Fedrer (1959), see https://www3.nd.edu/~lnicolae/Coarea.pdf .

[12]This is seen clearly for intersections V_r of a curve or surface $V \subset \mathbb{R}^3$ with (a family of) parallel planes $H_r \subset \mathbb{R}^3$. In fact, this inequality applies to Lipschitz maps $d: X \to R$ between general metric spaces and measures dv in X, see https://en.wikipedia.org/wiki/Eilenberg%27s_inequality.

$$\le \frac{1}{n} vol_{n-1}(Y) \sup_{y \in Y} \|x_0 - y\| \le \frac{1}{n} diam(Y) \cdot dist(x_0, Y).$$

Therefore, the volume of the cone from some point x_0 is bounded by the volume and the diameter of Y as follows

$$[Cone] \qquad \mathsf{vol_n}(\mathsf{Cone}_o(Y)) \le \frac{N}{n\sqrt{2(N+1)}} diam(Y) vol_{n-1}(Y),$$

since

there exists a ball $B_o(r) \subset \mathbb{R}^n$, $o \in \mathbb{R}^N$, of radius $\frac{N \cdot diam(Y)}{\sqrt{2(N+1)}}$, which contains Y by Jung's theorem.

If $N = n$ and $Y \subset \mathbb{R}^n$ is a closed naturally (say, inward) cooriented hypersurface, then the angle between the vector $x - y \in \mathbb{R}^n$ and he tangent space $T_y(Y)$ comes with a $\mp$-sign and the volume of the domain $X \subset \mathbb{R}^n$ bounded by Y is equal to the absolute value of the "signed volume" of the cone, i.e.

$$[Cone]_{\mp}. \qquad |vol(X)| = \frac{1}{n} \left| \int_Y \|x_0 - y\| \sin \angle (x_0 - y, T_y(Y)) dy \right|$$

This yields a non-sharp isoperimetric inequality for $n = 2$ by Jung's theorem,

$$area(X) < \frac{1}{4\sqrt{3}} length(Y)^2.$$

(Yung's heorem for closed curves gives you $area(X) \le \frac{1}{8} length(Y)^2$.[13])

But since the diameters of (connected) hypersurfaces for $n \ge 3$ are not controlled by their $(n-1)$-volumes for $n \ge 3$, this *only indirectly* leads to non-trivial bounds on $vol(X)$ by $vol_{n-1}(\partial X)$ and actual proofs of isoperimetric inequalities often amounts to particular specifications of this "indirectly".

2.1.G Gauss-Green Formula. If τ is a vector field in $\mathbb{R}^n$ with divergence n, then

$$[\mathsf{div} = \mathsf{n}] \qquad vol(X) = \frac{1}{n} \int_Y \langle \tau(y), \nu(y) \rangle dy \le \frac{1}{n} \int_Y \|\tau\| dy,$$

where ν is the unit normal vector field on the boundary $Y = \partial X$ and which reduces to the above $[Cone]_{\mp}$ applied to $\mathsf{grad}\|\mathsf{x_0} - \mathsf{x}\|^2$, that is the

gradient of the squared distance function $x \mapsto \|x_0 - x\|^2$.

2.2 Volumes of Radial Projections and Isoperimetry in Balls

Let Y be a smooth. (or piecewise smooth) $(n-1)$-dimensional submanifold in the unit ball

$$Y = Y^{n-1} \subset B^N(1) \subset \mathbb{R}^N$$

Since $N > n - 1$, the integral $\int \frac{1}{\|x\|^{n-1}} dx$ *converges* at zero in $\mathbb{R}^N$ and the mean of the $dist^{n-1}$-function in the ball $B^N(r) \subset \mathbb{R}^N$ satisfies,

$$\frac{1}{vol_N(B^N(1))} \int_{B^N(1)} \|x - x_0\|^{-(n-1)} dx \le const_N{}^{14}$$

[13] The sharp inequality for $n = 2$ is $area(X) \le \frac{1}{4\pi} length(Y)^2$.

[14] Since the measures of balls concentrate near their boundaries, this $const_N \to 1$ for $N \to \infty$.

for all $x_0 \in \mathbb{R}^N$.

It follows, that for all submanifolds Y in $\mathbb{R}^N$, (not necessarily contained in particular balls) and all r-balls $B^N(r) \subset \mathbb{R}^N$, there exist $x \in B^N(r)$, such that

$$\int_Y \|y - x\|^{-(n-1)} dy \le \frac{const_N vol_{n-1}(Y)}{r^N}.$$

Therefore, for all domains $V \subset \mathbb{R}^N$, which contain an r-ball $B^N(r)$,

$$B^N(r) \subset V \subset \mathbb{R}^N,$$

the *cylinder of the radial projection*, cal it $\psi = \psi_x$, $x \in B^N(r)$, of a submanifold $Y = Y^{n-1} \subset B^N(r)$ from some point $x \in B^n(r)$ to the boundary ∂V of V satisfies:

$$vol_n(\mathsf{cyl}_\psi) \le vol_{n-1}(Y) \frac{const_N}{n} \left(\frac{diam(V)}{r}\right)^n,$$

2.1.D. Corollary: Smaller Half Inequality. Let $V \subset \mathbb{R}^N$ be a connected domain and let a hypersurface $Y = Y^{N-1} \subset V \subset \mathbb{R}^N$ divide V in two subdomains. $V_1, V_2 \subset V$ with common boundary $V_1 \cap V_2 = Y$ in V. Then

$$\min(vol_N(V_1, vol_N(V_2)) \le vol_{n-1}(Y) \frac{const_N}{n} \left(\frac{diam(V)}{inrad(V)}\right)^n.$$

2.1.D. *Exercises.* (a) Let $V \subset \mathbb{R}^N$ be a *convex* domain and $Y = Y^{n-1} \subset V$. Show that there exists a point $x \in V$???

(b) Let V satisfy ???

2.2.1 "Involutive" Proof of Isoperimetry in Convex Sets and Cheeger Constant in Manifolds with $Ricci \ge -const$

Cheeger Constant for Convex Sets. Let $Y, W_1, W_2 \subset \mathbb{R}^n$ be closed subsets. of finite volumes , such that all straight segments $[w_{1,2}] \subset \mathbb{R}^n$, $w_i \in W_i$, i=1,2 intersect Y.

Then – this 99% obvious – one of the subsets, say W_1 contains a point w_1 and the second subset contains a subset $W_2' \subset W_2$, such that

- $vol_n(W_2') \ge \frac{1}{2} vol W_2$;
- each segment $[w_1, w_2']$, $w_2' \in W_2'$, intersect Y at a point $y \in Y$, such that

$$dist(w_1, y) \ge dist(y, w_2')$$

.

Conclusion. Let Y a smooth hypersurface divide a convex domain $W \subset \mathbb{R}^n$ into two parts W_1 and W_2. Then

$$\min(vol(W_1), vol(W_2)) \le 2^n vol_{n-1}(Y) \times diam(W).$$

2.1 E. Cones, Mapping Cylinders and Volumes /Measures of non-Injective Maps ???

2.1.E. Geodesic Cones: Riemannian and non-Riemannian ???

2.3 Parallel Displacement of Volume and Isoperimetry

15

Given a a bounded domain $V \subset \mathbb{R}^n$ with a smooth boundary, (or any Borel subset for this matter) it is *geometrically obvious* and is justified below that V *can't be almost invariant* under parallel translations by vectors $x \in \mathbb{R}^n$ with norm $\|x\| \le d$ for $d >> vol(V)^{1/n}$.

For instance if $d \ge R$, where $R = R_V$ is the radius of the ball $B^n(R = R_V) \subset \mathbb{R}^n$ such that $vol(B^n(R)) = 2vol(V)$, that is

$$R = \frac{(2vol(V))^{1/n}}{vol(B^n(1))^{1/n}},$$

then

★1/2 at least *half of the volume* of V is *transported out of* V by an x with $\|x\| \le R_V$.

This means that

$$vol(V \cap V + x) \le \frac{1}{2} vol(V),$$

where $V + x = \{v + x\}$ is the x-translate of V.

Then clearly, since $\|x\| \le R$,

$$vol(V_x) \le R \cdot vol_{n-1}(\partial V) = \frac{(2vol(V))^{1/n} \cdot vol_{n-1}(\partial V)}{vol(B^n(1))^{1/n}}$$

and

$$vol(V) \le 2vol(V_x) \le \frac{2(2vol(V))^{1/n} \cdot vol_{n-1}(\partial V)}{vol(B^n(1))^{1/n}},$$

which can be rewritten as an

isoperimetric inequality with a non-sharp constant,

$[Iso_2]$. $$vol(V) \le \frac{2^{n+1/n-1}}{vol(B^n(1))^{1/n-1}} vol_{n-1}(\partial V)^{n/n-1}$$

Proof of ★1/2. Let $D \subset V \times B^n(2R) \subset \mathbb{R}^{2n}$ be the subset of the pairs (v, x), such that $v + x \in V$ and let us evaluate the $2n$-volume of D in two ways.

$$vol_{2n}(D) = \int_V vol_n(V \cap (B_v(2R))dv \le vol_n(V)^2,$$

where $B^n_v(2R) \subset \mathbb{R}^n$ is the $2R$-ball with center v. Thus,

$$vol_{2n}(D) = \int_{B^n(2R)} vol_n(V \cap (V + x)dx \le vol(V)^2 \le \frac{1}{2} vol(V) \times vol(B^n(R)),$$

since $vol(B^n(R) = 2vol(V)$.

Therefore, there exits an $x \in B^n(R)$, such that

$$vol_n(V \cap (V + x) \le \frac{1}{2} vol(V)$$

[15] Compare with ???

by the mean value theorem. QED. ???Reference to Minkowski (for small R) and to Saloff-Coste???

???similar proof for subdomains in a convex sets.

???

Euclidean Geometry and Descendants : Semi-Algebraic Integral Formulas, Calculus of variations, Rearrangements

Non-Euclidean Symmetry: Amenability, T-property and ...

Probability ???

Linear Algebra and Algebraic Geometry

2.4 Sharp Santalo's Argument for $n = 2$ and non-Sharp for $n \geq 3$

Let $X \subset \mathbb{R}^2$ be a bounded planer domain with smooth boundary $Y = \partial X$ and proceed with the proof of the *isoperimetric inequality*

$$area(X) \leq \frac{length(Y)^2}{4\pi}$$

as follows.

???A. Let $\phi(x,y)$ be the norm of the differential of the radial projection from Y to the unit circle $S^1_x(1) \subset \mathbb{R}^2$.

Let $V \subset X \times Y$ be the set of pairs (x,y), such that the segment $[x,y] \subset \mathbb{R}^2$ is contained in X and let

$$V_x = V \cap \{x\} \times Y \subset V \text{ and } V_y = V \cap X \times \{y\} \subset V.$$

Let X' be the disk $B^2 = B^2(R(a))$ with

$$area(X' = X'(a)) = a = area(X).$$

Then, clearly,

[lenght] $$\int_{V_x} \phi(x,y)dy = 2\pi \text{ for all } x \in X$$

and

[area] $$\int_{V_y} \phi(x,y)dx \leq \int_{X'} \phi(x',y')dx' = c(v) \text{ for all } y \in Y,$$

since the levels of the x-function $\phi_y(x) = \phi(x,y)$ are r-circles tangent to $Y = \partial X$ at y and ϕ is monotone *decreasing* in r.

It follows that on the one hand

$$\int_V \phi(x,y)dxdy = \int_X dx \int_{V_x} \phi(x,y)dy \geq 2\pi \cdot area(X)\cdot$$

and on the other hand

$$\int_V \phi(x,y)dxdy = \int_Y dy \int_{V_y} \phi(x,y)dx \leq length(Y)c(a)$$

Thus

$$length(Y)c(a) \geq 2\pi \cdot area(X)$$

and since this becomes the equality for X',

$$c(v) = \frac{2\pi \cdot area(X')}{length(Y')} (= \sqrt{\pi a}),$$

this implies that

$$\frac{area(X)}{length(X)^2} \le \frac{area(X')}{length(X')^2} (= 1/4\pi).$$

QED.

Commentary. Santalo's argument is, *logically*, the most elementary among known proofs of the sharp 2d-isoperimetric inequality; besides, this proof gives an exemplary form of the deviation of the ratio $area(X)/length(\partial X^2)$ from that for the ball $X' = B^2 \subset \mathbb{R}^2$.

??? [area], [length]-*Divergence and Greens Formula.* There is a (unique) vector field $\nu = \nu_y = \nu_y(x)$, for all $y \in Y = \partial X$, *normal to the levels* of the function $\phi_y(x) = \phi(x,y)$, such that

$$\mathsf{div}(\nu_y)(x) = \mathsf{div}_X(\nu_y) = \phi_y(x) \text{ and } \|\nu_y(x)\| = 1/2 \text{ for all } y \text{ and } x$$

and the Green's formula delivers an alternative proof of the key inequality [area] in the *length* form

$$\int_{V_y} \phi(x,y)dx \le length(Y)/2.$$

If sect.curv ≠ 0. The above generalizes to surfaces with non-zero, e.g constant<0, curvatures, as follows

Non-Sharp Santalo for all dimensions n. Let $X \subset \mathbb{R}^n$ be a compact domain with a smooth boundary Y and let

$$\psi(x,y) = dist(x,y)^{-(n-1)}$$

(instead of the above $\phi(x,y)$). Then, clearly,

$$\int_Y \psi(x,y)dy \ge a_n = vol_{n-1}(S^{n-1}(1)),\ x \in X,$$

and

$$\int_X \psi(x,y)dx \le b_n r = \int_{B^n(r)} \|x\|^{-(n-1)}dx,\ y \in Y = \partial X.$$

where $B^n = B^n_0(r(v)) \subset \mathbb{R}^n$ is the ball with volume $v = vol_n(X)$.

Evaluate the integral of ψ over $X \times Y$ (instead of $V \subset X \times Y$). as earlier,

$$a_n \cdot vol_n(X) \le \int_{X \times Y} \psi(x,y)dxdy \le b_n \cdot r \cdot vol_{n-1}(Y).$$

Write $vol_n(X) = vol_n(B^n(r)) = \beta_n r^n$ and conclude to the inequality

$$vol_n(X)/r = vol_n(X)^{n-1/n} \le C_n vol_{n-1}(Y),\ \text{for } C_n = a_n^{-1} b_n \beta_n^{1/n}.$$

??? On Divergence and Green's Formula

2.5 Santalo Formula and Croke Inequality

A point x in a Riemannin manifold X, e.g. $x \in X = \mathbb{R}^n$, is *ray-surrounded* by a subset $Y \subset \mathbb{R}^n$ if all geodesics in X issuing from x intersect Y.

Let $\overleftrightarrow{[Y]} \subset X$ be the set of all x surrounded by Y.

Observe that $\overleftrightarrow{[\partial V]} \supset V$ for all bounded domains $V \subset \mathbb{R}^n$.

Theorem.[16] Let X be a complete simply connected Riemannian n-manifold with

$$sect.curv(X) \leq 0,$$

e.g. $X = \mathbb{R}^n$, and let $Y = Y^{n-1} \subset X$ be a smooth submanifold. Then

$$(vol_n \overleftrightarrow{[Y]})^{n-1} \leq const_n (vol_{n-1}(Y))^n$$

for where

$$const_n = \frac{vol_{n-2}(S^{n-2})^{n-2} \cdot \left(\int_0^{\pi/2} \cos(t)^{n/n-2} \sin(t)^{n-2} dt \right)^{n-2}}{vol_{n-1}(S^{n-1})^{n-1}}$$

Proof.???

Sharp 4d-Isoperimetric Corollary. Bounded domains V in complete simply connected Riemannian 4-manifolds X with $sect.curv(X) \leq 0$ satisfy:

$$vol_4(V) \leq (const_4(vol_{n-1}(\partial V))^{4/3}$$

for

$$const_4 = \frac{vol(B^4)^3}{vol(S^3)^4} = \frac{vol_2(S^2)^2 \cdot \left(\int_0^{\pi/2} \cos(t)^2 \sin(t)^2 dt \right)^2}{vol_{n-1}(S^3)^3}.$$

****

Bound on $vol_{m_1+m_2+1}$ of the set

$$\overleftrightarrow{[Y_1, Y_2]} \subset UT(\mathbb{R}^N) = \mathbb{R}^N \times S^{N-1}(1)$$

of the unit tangent vectors to the segments $[y_1, y_2] \subset \mathbb{R}^N$ joining points $y_1 \in Y_1$, and $y_2 \in Y_2$, $Y_1 = Y_1^{m_1}, Y_2 = Y_2^{m_2} \mathbb{R}^N$ by

$$const_N vol_{m_1}(Y_1) \cdot vol_{m_2}(Y_2).$$

Measure of flags of affine $(k, k-1)$-subspaces $A^k, A_k - 1)$ in $\mathbb{R}^{n+k}$, such that both half-spaces $A^k_\mp \subset A^k$ intersect $Y = Y^{n-1} \subset \mathbb{R}^{n+k}$,

(Alternatively, where A^{k-1} is linked with Y.)

2.6 Steiner Symmetrizations and Isoperimetry in Balls with Constant Curvature

[16] `https://link.springer.com/content/pdf/10.1007/BF02566344.pdf`.

??? A. Mirror Symmetry by Parallel Rearrangement .[17] Let $V \subset \mathbb{R}^n$ be a bounded Borel (e.g closed or open) subset and let $H \subset \mathbb{R}^n$ be a hyperplane. Then there exists a unique subset

$$V_H = \mathsf{symm}_H(V) \subset \mathbb{R}^n,$$

such that

$\bullet^{\perp}$ the intersections of V_H with all lines $L \subset \mathbb{R}^n$ normal to H are closed segments, the Lebesgue measures (lenghts for segments) of which are equal to those of the intersections $L \cap V$.

$\bullet_{sym}$ V_H is symmetric under the reflection in H, that is all segments $L \cap V_H$ intersect H at their middle points.

Ist durch die vorgetragene Schlußweise STEINERS, wenn wir uns die darin verwendeten Begriffe wie „geschlossene ebene Kurve“, „Bogenlänge“ und „Flächeninhalt“ genau umgrenzt denken — worauf wir bald zurückkommen — wirklich der Nachweis für die isoperimetrische Eigenschaft des Kreises erbracht? Wiederholen wir, es wurde gezeigt: Ist K eine geschlossene ebene Kurve, aber kein Kreis, so kann man durch das Viergelenkverfahren dazu immer eine neue geschlossene ebene Kurve K^* konstruieren, die *gleichen* Umfang und *größeren* Flächeninhalt besitzt. K kann also keine Lösung des isoperimetrischen Problems sein.

Wenn es also unter allen geschlossenen ebenen Kurven gegebenen Umfangs eine gibt, deren Flächeninhalt $\geqq$ dem Flächeninhalt jeder anderen ist, so kann sie nur ein Kreis sein.

Die Voraussetzung aber, daß eine solche Lösung unserer Aufgabe wirklich existiert, wird man zunächst als selbstverständlich erfüllt ansehen. Bei tieferem Eindringen jedoch zeigt sich, daß gerade in diesem Punkte eine Hauptschwierigkeit verborgen ist.

???B. Central Symmetrization. If we symmetrize V successively in the n mutually orthogonal linear hyperlanes, then the resulting subset, call it $V_{\square^n}$ will be centrally symmetric with respect to the origin $\mathbf{0} \in \mathbb{R}^n$ and this central symmetry will be preserved by all further H-symmetrizations "$\square^n$*-symmetrizations.*"

??? C. Trivial Observation and Useful Corollary. Let $V \subset \mathbb{R}^n$ be a bounded centrally symmetric. domain Then H-symmetrizations increase the volumes of the intersections of symmetrized sets $V_H = \mathsf{symm}_H(V)$ with balls,

$$vol(V_1 \cap B^n_{\mathbf{0}}(r) \geq vol(V \cap B^n_{\mathbf{0}}(r),\ r \geq 0,$$

for all (linear) hyperplanes $H \subset \mathbb{R}^n$.

Moreover, if ε-much of addtional measure of V could have been put to $B^n_{\mathbf{0}}(r)$, that is if

$$\min(vol(V \smallsetminus B^n_{\mathbf{0}}(r)), vol(B^n_{\mathbf{0}}(r), \smallsetminus V) \geq \varepsilon > 0,$$

then there exists an H such that the above inequality is *controllably strict*;

$$vol(V_1 \cap B^n_{\mathbf{0}}(r) \geq vol(V \cap B^n_{\mathbf{0}}(r),\ r \geq \delta > 0,$$

where this strictly positive δ depends (only) on $\varepsilon > 0$ as well as on R and $vol(V)$.

Corollary. If $Vol(V) = vol_n(B^n_{\mathbf{0}}(R))$, then there exists a sequence of hyperplanes H_i, such that the sequence V_i of H_i-successive symmetrizations of V,

$$V_i = \mathsf{symm}_H(V_{i-1})$$

volume-wise converges to $B^n_{\mathbf{0}}(R)$, that is

$$vol(B^n_{\mathbf{0}}(R)) \smallsetminus V_i \to 0 \text{ for } i \to \infty.$$

Now comes the basic step in Steiner's (attributed to Steiner?) symmetrization proof of the isoperimetric inequality.

??? B. Area Decrease of the Boundary. The boundaries of symmetrized of polyhedral domains[18] satisfy

$$vol_{n-1}(\partial(V_H) \leq vol_{n-1}(\partial V)).$$

In fact this inequality trivially reduces to the special case, where V is a trapezoid in the plane.

[17] `https://www.math.utah.edu/~treiberg/Steiner/SteinerSlides.pdf`

[18] ???

??? A. Symmetric Trapezoid Lemma. Among all trapezoid $T \subset \mathbb{R}^2$ with a given hight and the lengths of the two parallel bases, an *isosceles T has the minimal sum of the lengths of its two side legs* because the differential of the distance function $\phi_0 : x \mapsto dist(x, x_0)$ depends only on the direction (but not the length length) of the segment $[x, x_0]$, namely

$$d\phi_0(\tau) = \frac{\langle \tau, x - x_0 \rangle}{\|x - x_0\|},$$

the horizontal derivative of the sum of the m lengths of the legs of T vanishes if and only if T is isosceles.

In fact, the derivative of the sum of distances $dist(x, x_1) + dist(x, x_2$ in an arbitrary Riemannian manifold X, where x runs along a geodesic $L \subset X$ vanishes iff the angles α_1 and α_2 of minimizing segments $[x, x_1]$ and $[x, x_2]$ with $L \ni x$ at x are π*-complementary* i.e. $\alpha_1 + \alpha_2 = \pi$.

However simple, this is sole of Steiner's symmetryzation, which, together with the cone inequality 2.1.C??? yield the following.

??? B. Isoperimetric Conclusion. *All bounded polyhedral domains in* $\mathbb{R}^n$ *satisfy*

$$vol(V)^{n-1}/vol(\partial V)^n < vol(B^n(1))^{n-1}/vol(S^{n-1}(1))^n.$$

Complexity Remark The statement and the above proof of this inequality for $n = 3$ is limited to the high school level of geometry and if you comfortable with calculus and rudimentary differential geometry, this trivially generalizes to domains V with almost everywhere smooth boundaries in all complete simply connected spaces with constant sectional curvatures (i.e. inspheres and hyperbolic spaces).

Strangely however, this (19th century) argument was considered incomplete and difficult by the early 20th century mathematicians, e.g. Blaske (above is from his book "*Kreis und Kugel*", 1916) and Loomis and Whitney 1949[19]

> With any reasonable definition of the $(n-1)$-dimensional measure s of the boundary of O, $s \geqq 2m_i$ for each i, so that (1) gives
>
> (2) $$m^{n-1} \leqq s^n/2^n;$$
>
> this is the isoperimetric inequality, without the best constant. Since the proof of the isoperimetric inequality with the best constant is difficult,[1] and since its applications do not necessarily require the best constant, our elementary proof of the theorem may be of some interest.

Correction Term in Symmetrization and Isoperimetric Stability of Balls.

2.7 Formal Schwarz Symmetrization

The Geometric Schwarz Symmetrization transforms domains $V \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ to $V_\circ \subset \mathbb{R}^n$, such that

[19]Here is a quote from their 1949 paper`https://www.ams.org/journals/bull/1949-55-10/S0002-9904-1949-09320-5/S0002-9904-1949-09320-5.pdf` where they refer to E.Schmidt's 99 pages 1939-paper `https://link.springer.com/article/10.1007/BF01210681`Compare with `https://maa.org/sites/default/files/pdf/upload_library/22/Ford/blasjo526.pdf`.

$V_\circ$ are invariant under the orthogonal group $O(n-1)$ of rotation around the $\mathbf{0}\times\mathbb{R}$ axes

and

the intersections of $V_\circ$ with the hyperplanes parallel to $\mathbb{R}^{n-1}\subset\mathbb{R}^n$, say $H_t=\mathbb{R}^{n-1}\times\{t\}$ are balls, say $B^{n-1}(R_t)=H_t\cap V_\circ$ with the $(n-1)$-volumes equal to these of the intersections $V\cap H_t$.

$$vol_{n-1}(B^{n-1}(R_t))=vol_{n-1}(V\cap H_t).$$

The isoperimetric inequality in H_t, along with the trapezoid lemma ???A and an obvious integral (same Schwartz?) inequality show that

$$vol_{n-1}(\partial V_\circ)\le vol_{n-1}(\partial V).$$

This, technically speaking, seems as a trivial generalization of the area decrease property for the Steiner symmetrization, but it has a much wider range of application when combined with the 1-dimensional calculus of variation.

In fact, a single Schwarz symmetrization reduces the general isoperimetric problem for $V\subset\mathbb{R}^n$ to that for $O(n-1)$-invariant domains, that are obtained by rotating a domain $V_\bullet$ in the plane $P=\mathbb{R}^2\supset\mathbb{R}^2\supset\mathbb{R}^n$ which contains the $\mathbf{0}\times\mathbb{R}$ axes.

Thus, the (first) variation extremality condition for the isoperimetric problem, that is the is the constancy of the mean curvature of the hypersurface ∂V_{extr}, translates to a certain second order *ordinary* differential equation for the boundary of $V_\bullet\subset\mathbb{R}^2$.

Then (by the same argument Newton shown that the elliptic orbits are *the only* solutions of the second law with the inverse quadratic attraction):

among all $O(n-1)$-invariant closed Euclidean hypersurfaces S with given $(n-1)$-volumes, spheres maximize the volumes of domains $V\subset\mathbb{R}^n$ bounded by S.

This implies, by Schwratz, similar extremality of spheres among all domains S in the Euclidean space $\mathbb{R}^n$.

Isoperimetry in Euclidean and Non-Euclidean Balls. The Schwartz symmetrization equally applies to *intersections of V with concentric spheres* $H_t=S^{n-1}(t)\subset\mathbb{R}^n$, thus deriving the sharp isoperimetric inequality in $\mathbb{R}^n$ from those in the $(n-1)$-spheres (rather than in $\mathbb{R}^{n-1}$).

In fact, the spherical Schwartz symmetrization effortlessly extends to all complete simply connected spaces X_κ with constant sectional curvature κ, i.e. in spheres and hyperbolic spaces and yields

the isoperimetric extremality of balls in all X_κ.

Moreover, this applies to domains V in the balls $B^n_\kappa=B^n_\kappa(R)\subset X_\kappa$ and shows that

among all domains $V\subset B^n$, the ones with minimal $vol_{n-1}(\partial(V))$,(here ∂V denotes the topological boundary of V in the ball) *are those, where this boundary $\partial(V)\subset B^n$. is an umbilic hypersurface*[20] *normal to ∂B.*

Formal Schwartz. Let us apply the above argument to domains in a cylindrically split Riemannian manifold,

$$V\subset X=X_0\times\mathbb{R}$$

[20]???

and observe that this leads to a lower bound on $vol_{n-1}\partial(V)$ in terms of $vol_n(V)$ provided the "corresponding" bounds hold for domains

$$W_t = V \cap X_t \subset X_t = X \times \{t\} = X_0,\ t \in \mathbb{R},$$

where "corresponding" means "some bounds" on $vol_{n-1}(W_t)$ in terms of $vol_{n-2}(\partial W_t)$.

For instance, if *all $W \subset X_0$ satisfy the $(n-1)$-dimensional Euclidean isoperimetric inequality,*

$$vol_{n-1}(W)/vol_{n-2}(\partial W)^{n-1/n-2} \leq vol_{n-1}(B^{n-1})/vol_{n-2}(S^{n-2})^{n-1/n-2},$$

where $B^{n-1} = B^{n-1}(1) \subset \mathbb{R}^{n-1}$ is the unit ball and $S^{n-2} = \partial B^{n-1}$ is the unit sphere, *then all $V \subset X_0 \times \mathbb{R}$ also satisfy the n-dimensional Euclidean isoperimetric inequality,*

$$vol_n(V)/vol_{n-1}(\partial V)^{n/n-1} \leq vol_n(B^n)/vol_{n-1}(S^{n-1})^{n/n-1}.$$

More generally, a minor elaboration of this argument (left to the reader) shows that

if $X = X_1 \times X_2$, where $X_i = X_i^{n_i}$, $i = 1,2$, are Riemannian manifolds, such that

all $V_i \subset X_i$, $i = 1,2$, satisfy the n_i-dimensional Euclidean isoperimetric inequalities, then

all $V \subset X$ satisfy the (n_1+n_2)-dimensional Euclidean isoperimetric inequality.[21]

2.8 "Isoperimetric" Proof of Sobolev's bounds on the integrals $\int \|f\|^p$ by $\int \|df\|^q$

Let X be a Riemannian n-manifold without a boundary and $f(x) \geq 0$ be a Lipschitz function on X with a compact support. Let $B(t) = f^{-1}(t) \subset X$ be the t-levels of f and let $A(t) = f^{-1}[t,\infty) \subset X$ be the compact domains in X bounded by $B(t)$.

***2.6.A.** **M**az'ya-**C**heeger Conditional Inequality. If*

[isop]$_\nu$ $\qquad vol_n(A(t)) \leq c(vol_{n-1}(B(t)))^\nu$ *for some $\nu \geq 1$ and all $t \geq 0$,*

(e.g. $\nu = n/n-1$ as in the Euclidean isoperimetric inequality) then

[$\|...\|_{pq|\nu}$] $$\|f\|_{L_p} = \sqrt[p]{\int f(x)^p dx} \leq \frac{pc}{\nu}\|df\|_{L_q} = \frac{pc}{\nu} c \sqrt[q]{\int \|df(x)\|^q dx}$$

for all $p \geq 1$ and

$$\frac{1}{p} = \frac{1}{q} - \frac{\nu-1}{\nu}$$

Proof for $q = 1$ and $p = \nu$. Here this inequality reads:

[$\|...\|_\nu$] $$\|f\|_{L_\nu} = \sqrt[\nu]{\int f(x)^\nu dx} \leq c\|df\|_{L_1} \leq c\int \|df(x)\|dx,$$

[21]See `https://math.williams.edu/symmetrization/#:~:text=Steiner%20and%20Schwarz%20symmetrization%20can,ball%20of%20the%20same%20volume` and p.p. 204-214 in`https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/waists.pdf` for this and similar inequalities of this kinds for more general fibrations of **m**etric **m**easure spaces X.

where it is immediate if f is the characteristic function $\chi = \chi_A$ of a smooth domain $A \subset X$, since $\|f\|_{L_p} = vol_n(V)$ for all p and $\|df\|_{L_1} = vol_{n-1}\partial C$, where, either the integral $\int_X d\chi(x)dx$ is understood in the distribution sense, or, more geometrically, as the limit of smooth or Lipshitz approximations of χ, of f, e.g. by the trapezoidal functions

$$\chi_{A\setminus\varepsilon}(x) = \max\left(0, 1 - \frac{1}{\varepsilon}dist(x, A)\right),\ \varepsilon \to 0.$$

Thus, [$\|...\|_\nu$] is the same as [isop]$_\nu$ for $q = 1$.

Next, if $f = \sum_i \alpha_i \chi_{A_i}$ then

$$\|f\|_{L_\nu} \le \sum_i |\alpha_i| \|\chi_{A_i}\|_{L_\nu}$$

by convexity of the norm $\|...\|_{L_\nu}$, and if $\alpha_i \ge 0$, then

$$\|df\|_{L_1} = \sum_i |\alpha_i| \|\chi_{A_i}\|_{L_1}.[22]$$

It follows that [isop]$_\nu$ does imply [$\|...\|_\nu$] for such $f = \sum_i \alpha_i \chi(A_i)$.

Then we approximate our general positive Lipschitz function $f(x)$ by such sums, where one takes $A(t_i) = f^{-1}[t_1, \infty)$, (or smooth approximations to $A_(t_i)$, see remark ??? below) (see Lemma??? below), and thus conclude he proof for $q = 1$.

The proof for $q \ge 1$. The above applied to $f^{\frac{p}{\nu}}$ and the Gölder (r, q)-inequality, *where $r(\frac{p}{\nu} - 1) = p$, $q = \left(1 - \frac{1}{r}\right)^{-1}$ yield:*

$$\|f\|_p^{\frac{p}{\nu}} = \|f^{\frac{p}{\nu}}\|_\nu \le \|df^{\frac{\nu}{p}}\|_1 = \frac{cp}{\nu}\|f^{\frac{p}{\nu}-1}df\|_1 \le \frac{cp}{\nu}\|f^{\frac{p}{\nu}-1}\|_r \|df\|_q = \frac{cp}{\nu}\|f\|_p^{\frac{p}{r}}\|df\|_q;$$

thus

$$\|f\|_p = \|f\|_p^{\frac{p}{\nu}-\frac{r}{p}} \le \frac{cp}{\nu}\|df\|_q.$$

QED.

On Sharp Sobolev. If $q = 1$ the inequality [$\|...\|\nu$] = [$\|...\|_\nu, 1|\nu$] is sharp but it is only exceptionally sharp with our constant pc/nu for $q > 1$ and $p > \nu$, since the extremal functions f_{extr} are not like χ_X in general, but they are associated to smooth solutions of certain Bessel-like (ordinary) differential equations, (see ???? below), from which sharp constants can be derived.

***2.6.B.** L_2**-Example.** If $\nu = 1$ and $p = q = 2$, then the above inequality for $c = 1$, which says that*

$$\|f\|_{L_2} \le 2\|df\|_{L_2},$$

is sharp for the cylinder $X = \mathbb{R}_+ \times S^1$ with the the (hyperbolic) metric $dx^2 = dt^2 + e^{-2t}ds^2$ (compare with ???).

***2.6.C.** Obvious Approximation Lemma. Let $\phi(x)$ be a smooth function on a Riemannin manifold, e.g. on the Euclidean n-space, with compact support X. Then there exist decreasing families of smooth bounded domains*

$$X \supset X_1 \supset X_2 \subset .... \supset X_k$$

[22] One may be justifiably worried with possible intersections of boundaries ∂A_i of different subsets A_i but this introduces no correction terms at least for smooth ∂A_i; besides our $\partial A_i \partial A_i = B(t_i) = f^{-1}(t_i)$ do not intersect anyway.

, such that (multiples of) the sums of trapezoidal functions

$$f_{k,\varepsilon}(x) = \frac{\sup_x \phi(x)}{k} \sum_1^k \chi_{X_i \backslash \varepsilon}(x)$$

uniformly, hence L_p for all p converge to f for $k \to \infty$ and $\varepsilon \to 0$ and the differentials of these sums L_1 converge to df,

$$df_{k,\varepsilon}(x) \to_{L_1} df$$

$\bullet_{\Subset}$ *The set X_i is contained in the interior of X_{i+1} for all i,*

$$X_i \Subset X_{i+1}, i = ... - 2, -1, 01.2, ...$$

2.6.D. *Remark: Sard Theorem and Lipschitz ϕ. ???*
2.6.D. *Sharp Sobolev.*

2.8.1 Cavalieri Principle, Coarea Formula, Pushforward Measures and Formal Symmetrization

. Let $\phi : x \mapsto t \in \mathbb{R}$ be a smooth function on $\mathbb{R}^n$, let

$$A_\phi(t) = \{x \in \mathbb{R}^n\}_{\phi(x) \le t}$$

be the sublevels of ϕ and

$$B_\phi(t) = \phi^{-1}(t) \subset \partial A_\phi(t)$$

be the levels of ϕ.

Then

A $$\int_{\mathbb{R}} vol(A_\phi(t))dt = \int_{\mathbb{R}^n} \phi(x)dx$$

and

B $$\int_{\mathbb{R}} vol_{n-1}(B_\phi(t))dt = \int_{\mathbb{R}^n} |d\phi(x)|dx.$$

To see this, check A and B for the function which is equal to $x \mapsto ||x||$ in an annulus $r \le ||x|| \le R$ and which is zero outside this annulus and then extend this to all ϕ by linearity of integrals and differentials.[23]

More generally, and equally obviously, all continuous (or just Borel) functions $\psi(x)$ with compact supports on an arbitrary Riemannian manifold $X = (X, g)$,[24] *e.g. on $X = (\mathbb{R}^n, g_{Eucl} = \sum_i dx_i^2)$*

[***coar***] $$\int_X \psi(x)\|d\phi(x)\|dx = \int_{t=-\infty}^{t=+\infty} dt \int_{B_\phi(t)} \psi db_t$$

for all smooth functions $\phi = \phi(x)$ on X.

[23]Alternatively, refer to *Cavalieri/Fubini* for A and to *the coarea formula* for B.
[24]The Riemannin metric g on X is assumed continuous, although bounded Borel measurable will do here.

This, in terms of the ϕ-pushforward of the Riemannin measure dx, reads:

$$\phi_*(dx) = \mu(t)dt \text{ for } \mu(t) = \int_{B(t)} \|d\phi\|^{-1} db_t$$

Thus, for instance

$$\int_X \phi(x)^p = \int_{-\infty}^{\infty} t^p \mu(t)dt$$

and

$$\int_X \|d\phi(x)\|dx = \int_{-\infty}^{\infty} b(t)dt \text{ for } b(t) = vol_{n-1}(B_t)$$

Let us assume (to avoid irrelevant terminological complications) that the support S of the function $b(t)$ is a union of disjoint intervals and replace the t-parameter in S by s, such that $ds = b^{-1}(t)dt$

Formal Symmetrization. From now on, we think of $S = S(X) = S_\phi(X)$ as an oriented 1-dimensional Riemannian manifold with the metric ds^2, where the function $t(s)$ is viewed as a kind of symmetrization of $\phi(x)$, now denoted $\underline{\phi}(s)$.

Observe that ds^2 depends only on the partition of X to the levels of the function $\phi(x)$ (but not on the values of ϕ on these levels) and that there is a natural map, say $\sigma : X \to S$, such that, for all segments $S_0 \subset S$.

$$vol_n(\sigma^{-1}(S_0)) = \int_{S_0} \underline{b}(s)ds,$$

where $\underline{b}(s) = b(\underline{\phi}(s)) = vol_{n-1}(\sigma^{-1}(s))$.

Also observe that

$[Symm]$ $$\int_X \phi^p(x)dx = \int_S \underline{\phi}^p(s)b(s)ds,$$

$[Symm]_d$ $$\int_X \|d\phi(x)\|dx = \int_S \|d\underline{\phi}(s)\|b(s)ds$$

and

$[Symm]_d^p$ $$\int_X \|d\phi(x)\|^p dx \ge \int_S \|d\underline{\phi}(s)\|^p b(s)ds \text{ for } p \ge 1,$$

where the later follows from $[Symm]_d$ and convexity of the function $z \mapsto z^p$, $p \ge 1$.

Symmetrization and Comparison Inequalities . Let $Y = (Y, h)$ be a Riemannian m-manifold, e.g. $m = n-1$, or $m = 1$. with $vol_m(Y) = 1$ and let $\underline{X} = Y \times S$ with the (warped product) metric $\underline{g} = ds^2 + b(s)^{\frac{2}{m}}h$, i.e. such that $vol_m(Y, b(s)^{\frac{2}{n-1}}h) = b(s) = vol_{n-1}(\sigma^{-1}(s))$.

Then

$$S_{\underline{\phi}(s)}(\underline{X} = Y \times S) = S_\phi(x)(X)$$

and if Y is a homogeneous manifold, e.g. the sphere, this is commonly called a "symmetrization of" X.

2.9 Needle Decomposition

2.10 Metric Measure Spaces, Moment Maps, Maxwell Law, and Gaussian Isoperimetry

All of the above symmetrizations equally applies to Riemannin manifolds with (more ore less) arbitrary measures μ on them, where the Riemannin metric is used for evaluation of the norms of differentials $\|df\|$ while integrals are taken with respect to μ rather then with the Riemannin measure.

Thus symmetrizartion keeps us within he same the category of metric measure spaces.

But if you want you can return, at least for smooth measures $\mu(x)p(x)dx$, to the pure metric Riemannian category by passing from $(X, g, \mu$ to the warped product of X with the circle $X \times S^1$ with the metric $dx^2 + p(x)^{2/n}ds^2$.

2.11 Needle Decomposition

2.12 Cabré's ABP-Proof of the Classical Isoperimetric Inequality

[25]

Logic of the Proof. Assume without loss of generality that the boundary of a smooth bounded domain $X \subset \mathbb{R}^n$ has the same $(n-1)$-volume as the unit sphere,

$$vol_{n-1}(\partial X) = vol_{n-1}(S^{n-1}),$$

and let $f(x)$ be a smooth function, such that

$$\Delta(f) = a \text{ and } df(\nu) = 1.$$

Then the proof would trivially follow from the inequaliy

$$a^{n-1}vol_{n-1}(\partial X) = a^n vol(X) \geq n^n vol(B^n(1)) = n^{n-1}vol_{n-1}(S^{n-1}) = n^{n-1}vol_{n-1}(\partial X)$$

where $\geq$ is proven below by constructing a map from a part of X onto $B^n(1)$ with Jacobian $\leq a^n/n^n$.

Let $X \subset L = \mathbb{R}^n$ be a bounded domain with a smooth boundary ∂X and let $f : X \to \mathbb{R}$ be a smooth function.

Let $L' = \mathbb{R}^n$ be the linear dual of L, let us identify the tangent spaces $T_x(L)$, $x \in L \supset X$ with L and let $\overline{df} : X \to L'$ be the map, which thus corresponds to the differential $df : x \mapsto T_x(L)$.

ABP[26]-Lemma. Let $X_{\smile} \subset X$ be the subset, where he function f is locally convex, i.e. where the Hessian of f, (full second differential) $\{d^2f/\partial x_i \partial x_j\}$ of f is semipositive definite.

If the normal derivative of f on he boundary of$X$ is bounded from below by a positive constant R,

$$df(\nu) \geq R,$$

[25] https://pdfs.semanticscholar.org/0b0f/91abb26f8ae7c6d304f0881f646d28cabf7a.pdf

[26] Aleksandrov–Bakelman–Pucci.

where $\nu : \partial X \to T(X)$ is the *outward looking* unit normal vector field, then the image $\overline{d}(X_\smile) \subset L' = \mathbb{R}^n$ contain the R-ball inL',

$$\overline{df}(X_\smile) \supset B^n(R) \subset L' = \mathbb{R}^n.$$

Let $S' = S^{n-1}(1) \subset L' = \mathbb{R}^n$ be the unit sphere in the dual space to L and let $R(s')$ be the infimum

and let $T_+(\partial X) \subset T(\partial X)$ be the set of the *supporting hyperplanes to* X that are the hyperplanes tangent to ∂X at the intersection points of ∂X with the boundary of the convex hull $conv(X) = conv(\partial X)$.

Let $S' = S^{n-1}(1) \subset L' = \mathbb{R}^n$ be the unit sphere in the dual space to L and ${}_{s'}T \subset L$ denote the supporting hyperplanes to X parallel to the hyperplanes at the origin in L, which correspond to $s' \in S'$.

let $R_f(s')_f$, $s' \in S'$, be the maximum of the values of the differential $d(f)$ at the outward looking unit normal vector field to the boundary of X at the points, where the supporting hyperplane is tangent to ∂X, i.e. at $x \in \partial X \cap_{s'} T$.[27]

Let $R_f \geq 0$ and let $U'(R_f) \subset L'$ be the set of the vectors $r(s')s' \in L'$ for all $s' \in S'$ and $r(s') \leq R_f(s')$. (if R is constant this is the R-ball.)

???

ABP Corollary. The Laplacian of f satisfies

ABP $$\int_X |\Delta(f)|^n dx \geq \int_{X_\smile} |\Delta(f)|^n dx \geq n^n vol(B^n(R)).$$

Proof. The arithmetic/geometric inequality applied to the (real positive) eigenvalues of the Hessian of f at he points in $X_\smile$ shows that the Jacobian of he map $\overline{df} : X_\smile \to B^n(R)$ satisfies

$$Jac(\overline{df}) \leq (\Delta(f))^n / n^n,$$

while the integral of this Jacobian over $X_\smile$ is bounded from below by the integral over our R-ball of the multiplicity of the map $\overline{df}$,

$$\int_{X_\smile} |Jac(f)|^n dx \geq \int_{B^n(R)} card(\overline{df}^{-1}(l'))dl', \ l' \in B^n(R) \subset L',$$

where $card(\overline{df}^{-1}(l')) \geq 1$ on $B^n(R)$ by the ABP-Lemma.

———————-

Linear PDE-Recollection. (Neumann Boundary problem.) Let $a(x)$ and $b(y)$ be smooth functions in $x \in X$ and $y \in \partial X$. If

[GF] $$\int_X a(x)dx = \int_Y b(y)dy$$

(this is Green's formula), then there exists a smooth function $f(x)$, such that the Laplacian of f and the normal derivatives of f satisfy

$$\Delta f(x) = a(x) \text{ and } df(\nu(y)) = b(y).$$

Proof of the Isoperimetric Inequality. Let us assume without loss of generality that the boundary of X has the same $(n-1)$-volume as the unit sphere,

$$vol_{n-1}(\partial X) = vol_{n-1}(S^{n-1}),$$

[27] A supporting hyperplane to an X can be tangent to ∂X at several points x.

let $b = 1$ and let $a > 0$ be a constant, such that

$$[a/1] \qquad a \cdot vol(X) = vol_{n-1}(\partial X).$$

Let $f(x)$ be a smooth function, such that

$$\Delta(f) = a \text{ and } df(\nu) = 1.$$

Then the above ABP inequality reads

$$a^{n-1}vol_{n-1}(\partial X) = a^n vol(X) \geq n^n vol(B^n(1)) = n^{n-1}vol_{n-1}(S^{n-1}) = n^{n-1}vol_{n-1}(\partial X)$$

Hence $a \geq n$ and $[a/1]$ shows that

$$vol(X) \leq \frac{1}{n}vol_{n-1}(\partial X).$$

QED.

Question. Is there a natural Borel (measurable) correspondence

$$X \times S^{n-1} \leftrightarrow \partial X \times B^n$$

which would geometrically implement the inequality

$$vol_{2n-1}(X \times S^{n-1}) \leq vol_{2n-1}(\partial X \times B^n)?$$

Or, maybe a natural family of similar correspondences between powers of these sets

$$(X \times S^{n-1})^N \leftrightarrow (\partial X \times B^n)^N$$

$N = 1.2, ...,$ which implements the inequality

$$\lim_{N\to\infty} \sqrt[N]{vol_{N(2n-1)}(X \times S^{n-1})^N} \leq \lim_{N\to\infty} \sqrt[N]{vol_{N(2n-1)}(\partial X \times B^n)^N}$$

2.13 Dimension 2: Steiner, Santalo, Cabre, Wirtinger...

3 Laplace Operators on Riemannian manifolds X and Eigen Values $\lambda_1(X)$ and $\lambda_2(X)$.

Let X be a smooth Riemannian n-manifold, e.g. a domain in $\mathbb{R}^n$ or a smooth closed hypersurface, such as the n-sphere $S^n \subset \mathbb{R}^{n+1}$.

Recall that the Laplace operator on X is defined as

$$\Delta f(x) = \mathsf{div}\,\mathsf{grad}\, f(x) = \sum_{i=1}^{n} \partial_i^2 f(x)$$

where $\partial_i \int_x(X)$ are orthonormal tangent vectors at x and ∂_i^2 is the second derivatives along the geodesic issuing from x in the ∂_i-direction, where this is independent of a choice of orthonormal vectors by the Pythagorean theorem.

??## *Exercises. (a) Show that $\frac{1}{n}\Delta f(x)$ equals the spherical average of the second derivatives of f along geodesics issuing from x,*

$$\frac{1}{n}\Delta f(x) = \frac{1}{vol(S^{n-1})}\int_{S_x^{n-1}} \partial_s^2 f(x) ds$$

where $S^{n-1}_x \subset T_x(X)$ is the tangent unit sphere of X at x,

(b) Express the integrals $\int_X \|\Delta f(x)\|^p$ by lifting $f(x)$ to a function $f(\tau)$ on the unit tangent bundle $UT(X)$, twice differentiating f along the orbits of the geodesic flow and integrating $\|\partial^2_\tau f(\tau)\}^p$ over $UT(X)$ with the Liouville measure (which is invariant under the geodesic flow) .

(c) $f : X \to \mathbb{R}^N$ be an isometric embedding (immersion will do) and show then the Laplace operator coordinate-wise applied to $f = (f_1, ..., f_N)$ as $\Delta f = (\Delta f_1, ..., \Delta f_N)$ is equal the mean curvature vector field of $X \hookrightarrow \mathbb{R}^N$ interpreted as a map $X \to \mathbb{R}^N$ (by identifying the tangent spaces $\mathbb{T}_y(\mathbb{R}^N)$ with $\mathbb{R}^N$ for all $y \in \mathbb{R}^N$),

$$\Delta f = \mathsf{mean.curv}(X \hookrightarrow \mathbb{R}^N),$$

where the mean curvature vector of X at $x \in X$ is the n-times spherical average of the Euclidean curvature vectors of geodesic in X issuing from x.

For instance, the mean curvature of the sphere $S^n(R) \subset \mathbb{R}^{n+1}$ is $\frac{n}{R}\nu(s)$, where $nu(s) \in T_s(\mathbb{R}^{n+1})$ is the outward looking unit normal vector to the sphere at $s \in S^n(R)$.

(d) Using this definition of the mean curvature, show that it is equal to the gradient of the function $X \mapsto vol_n(X)$ on n-submanifolds in $\mathbb{R}^n$.

Observe that Δ is a negative operator as

$$\int_X \langle \Delta f(x), f(x) \rangle dx = -\int_X \|df\|^2$$

on closed manifold by the Green's formula (integration by parts) and the same is true for compact manifolds with boundaries for functions f if either f or df vanish on ∂X.

Poincaré inequality concerns the smallest non-zero *eigenvalue of the operator $-\Delta = -\Delta_X$ on the space of smooth L_2-functions f on X [28] with either* Dirichlet or Neumann boundary conditions, *that is either for functions which* vanish on the boundary *∂X or with the gradient $\mathsf{grad} f$ on the boundary ∂X* normal *to ∂X.*

If all connected components of X are compact with nonempty boundaries then there is no zero eigenvalue: *harmonic function zero on the boundary* vanish.

Thus the first non-vanishing Dirichlet eigenvalue is the smallest one, denoted $\lambda_1(X)$.

But since constant functions satishy Neumann's condition, the first non-vanishing Neumann's eigenvalue on a compact connected manifold with or without boundary is actually the second smallest one, which we denote $\lambda_2(X)$. [29]

Examples. (a) *The segment $[a, b]$ has $\lambda_1 = 0$,*

$$\lambda_2[a, b] = \frac{\pi^2}{(b-a)^2},$$

since the only bounded eigen functions of $\Delta = \frac{d^2}{dt^2}$ on the line are sin *and* cos [30] *and, for all i,*

$$\lambda_i[a, b] = \frac{(i-1)^2\pi^2}{(b-a)^2}$$

[28] "L_2" means that $\int_X f^2(s)dx < \infty$.

[29] According to our notation, compact disconnected manifolds X have Neumann's $\lambda_2 = 0$.

[30] Probably there is a direct proof of the inequality $\int_{-\pi/2}^{\pi/2} f^2(t)dt \le \int_{-\pi/2}^{\pi/2} (f'(t))^2 dt$ for functions f such that $\int_{-\pi/2}^{\pi/2} f(t)dt = 0$, but I couldn't find it.

(b) ***??? Spectra of Riemannian Products*** *One knows that the eigenvalues of* $-\Delta_{X\times Y}$ *are the sums* $\lambda_i(X)+\lambda_j(Y)$.

In fact, this follows from general properties of elliptic selfadjoint operators(See ??? below for a direct proof).

(c) The eigenvalues of the cube $[0,\pi]^n$ *are the sums* $i_1^2+i_2^2+...+i_n^2$. *Thus, there are roughly* $R^{n/2}$ *eigenvalues* $\lambda_i \leq R$.

Variational Principle and Green's Formula. *The (infinite dimensional in the present case) linear algebra tells you that the eigen values* $\lambda_1,\lambda_2,,,,\lambda_i,...$ *of* Δ *be they Dirichlet's or Neumann's ones, are equal to the critical values of the quadratic function*

$$f\mapsto -\langle f,\Delta f\rangle = \int_X f(x)\Delta f dx$$

on the unit sphere in he Hilbert space of L_2*-functions* $f(x)$*,where the corresponding eigen functions* ϕ_i*, or spaces if these for multiple* λ_i*, are mutually orthogonal.*

Green's Formula. "Integration by Parts" *shows that he above quadratic function is equal to the Dirichlet (energy) functional,*

$$-\langle f,\Delta f\rangle = \|df\|_{L_2} = \int_X \|df\|^2 dx$$

for smooth functions f *on compact manifolds, such that either* f *vanishes on the boundary* ∂X *or the normal derivative of* f *on* ∂f *vanish.*

(In the case of disconnected boundary, one may have f *vanishing on some components of* ∂X *and the normal derivative of* f *vanishing on the remaining components.)*

Thus

$$\lambda_1(X) = \inf_{f|\partial f=0} \frac{\|df\|_{L_2}^2}{\|f\|_{L_2}}$$

and

$$\lambda_2(X) = \inf_{\int f=0} \frac{\int_X \|df(x)\|^2 dx}{\int_X \|f(x)\|^2 dx}$$

???*Remarks* (a) *Since the orthogonality to constants condition* $\int_x f(x)dx = 0$ *(easily) implies that*

$$\int f^2(x)dx = \frac{1}{2vol(X)}\int_{X\times X} |f(x_1-f(x_2)|^2 dx_1dx_2$$

one can define λ_2*, as*

$$\lambda_2(X) = \frac{1}{2vol(X)} \inf_f \frac{\int_{X\times X} |f(x_1-f(x_2)|^2 dx_1dx_2}{\int_X \|f(x)\|^2 dx},$$

where this definition makes sense for maps f *froX to an arbitrary Riemannian (and Finsler) manifold with* $|a-b|$ *understood as* $dist(a,b)$

(b) The definition of λ_i *via the Dirichlet functional* $\|df\|_{L_2}^2$ *without a direct reference to the Laplace operator makes sense for manifolds* X *with bounded measurable Riemannin metrics and it is quasi-invariant under bi-Lipschitz maps:*

if $\Phi : X_1 \to X_2$ is a l-Lipschitz (i.e. $\|d\Phi\| \le l$) homeomorphism, where $\Phi^{-1} : X_2 \to X_1$ is also l-Lipschitz, then, clearly, both Dirichlet and Neumann eigenvalues saisfy:

$$\frac{1}{l^{2n-1}}\lambda_i(X_1) \le \lambda_2(X_2) \le l^{2n-1}\lambda_i(X_1)$$

for all $i = 1, 2, \ldots$.

Exercises. (a) *Show that Dirichlet's eigen values of subdomains $X_0 \subset X$ satisfy $\lambda_i(X_0) \ge \lambda_i(X)$*

(a) Let $\Phi : X_1 \to X_2$ be an l-Lipschitz map, where the (compact Riemannian) manifolds X_1 and X_2 may have different dimensions and let the pushforward of the Riemannian measure dx_1 be

$$\Phi_*(dx_1) = \delta(x_2) \cdot dx_2$$

for some positive function $\delta(x_2) > 0$ on X_2. (This makes $dim(X_1) \ge dim(X_2)$.) Show that the Neumann's eigen values satisfy

$$\lambda_i(X_1) \le l^2 \frac{\sup_{x \in X_2} \delta(x_2)}{\inf_{x \in X_2} \delta(x_2)} \cdot \lambda_2(X_2).$$

(b) Show that, for all pairs of compact Riemannian manifolds X_1 and X_2 where $dim(X_1) \ge dim(X_2)$ (e.g where X_1 is the n-cube $[0,1]^n$ and $X = X^n$ is arbitrary) there exist positive constants $l > 0$ and $\delta > 0$ depending on X_1 and X_2 and l-Lipschitz maps,

$$\Phi : X_1 \to X,$$

such that the pushforward measures $\Phi_*(dx_1)$ are constant *δ-multiples of dx.*

Remark. Arbitrarily large balls $B(R)$ in hyperbolic spaces may have arbitrarily large Dirichlet's λ_1.

Less obviously, one can show, using Riemann surface expanders, *that all compact manifolds X of dimension $n \ge 3$ (4 maybe) admit arbitrarily large Riemannian metrics with arbitrary large Neumann's λ_2.*

(c) Derive the above formula for the eigenvalues of $\Delta_{X \times Y}$ from the variational principle.

(d) Combine this with (b) for $X_1 = [0, \pi]^n$ and show that Neumann's eigenvalues of compact connected manifolds X satisfy

$$\lambda_i \ge const_X(i^{n/2} - 1).$$

(e) Derive a similar bund from this for Dirichlet's eigenvalues of compact connected n-manifolds with boundaries by applying (d) to a connected n-manifold X_+, which contains two copies of X e.g. to the double of X obtained by gluing two copies of X across their boundaries.

(f) Let X be a compact manifold with a boundary and show that he first Dirichlet's eigenvalue of X is equal to the

supremum of numbers λ, such that X admits a smooth positive function $f(x) > 0$, such that

$$-\Delta f(x) \ge \lambda f(x),$$

that is

$$\lambda_1(X) = \sup_{f>0} \inf_{x \in X} \frac{-\Delta f(x)}{f(x)}.$$

Hint. *Use the maximum principle for the first Dirichlet eigenfunction f_1,*

(g) Divide a closed Riemannin n-manifold X into two domains X_- and X_+ with common smooth boundary $Y = X_- \cap X_+ \subset X$, let $\lambda_{max}(Y) = \max(\lambda_1(X_, \lambda_1(X_+)$ and show that

$$\lambda_2(X) = \inf_{Y \subset X} \lambda_{max}(Y),$$

where "$\sup$" is taken over all smooth closed cooriented hypersurfaces $Y \subset X$.

Laplacian on Riemannin Metric Measure Spaces. *This Δ is defined on $X = (X, g, \mu)$ as*

$$\Delta(f) = \mathsf{div}_\mu \mathsf{grad}_g(f),$$

where themu-divergence of the vector field τ on X is the ratio of the τ-Lie derivative of μ by μ, which makes sense for $\mu(x) = p(x)dx$ where $p(x) > 0$. In this case the μ-Laplacian at least for smooth p has the same properties as the pure Riemannian one which can be seen, for instance, by looking at the Riemannian Laplacian on the S^1-invarian functions on the warped product $X \times S^1$ with the metric $g_X + p^{2/n} ds^2$.

3.1 Spectra of Cubes, Spheres, Balls and Hyperbolic Cusps

3.1.1 Rayleigh–Faber–Krahn Inequality

rewrite

Let $f : X \to \mathbb{R}_+$ be a non-negative measurable function on a measure space $X = (X, \mu)$ e.g. on $X = \mathbb{R}^n$, which is supported on a subset with finite measure.

Recall that the f -pushforward of μ is the measure on $\mathbb{R}_+$, such that

$$\mu(S) = \mu f^{-1}(S)$$

for all Borel subsets (segments $[a, b]$ will do] will do) $S \subset \mathbb{R}_+$.

Define the $O(n)$-invariant model $\underline{f}(\underline{x}) = \underline{f}_{O(n)}(\|\underline{x}\|)$ of f as the radial function on $\mathbb{R}^n$, such that the $\underline{f}$-pushforward of the Euclidean measure $d\underline{x}$ is equal to the f -pushforward of μ, that is such that the Euclidean volumes of the balls $B^n(\underline{a}) \subset \mathbb{R}^n$ for all $\underline{a} \geq 0$, where the function $\underline{f}$ is $\leq \underline{a}$, are equal to the μ-measures of the corresponding subsets in X, i.e. where $f(x) \leq \underline{a}$,

Remark. *In the case of the Faber-Krahn theorem, the functions f and $\underline{f}$ are defined on the same space $\mathbb{R}^n$, but one also may compare a function f on an arbitrary n-dimensional Riemannin manifold with its $O(n)$-symmetric model on $\mathbb{R}^n$ as well, more generally, on another n-manifold with $O(n)$-symmetry, e.g. the sphere S^n or the hyperbolic n-space.*

Now, similarly to the above proof of the Sobolev&... inequality, the Euclidean isoperimetric inequality

$$vol_n(X)/vol_{n-1}(\partial(X)^{n/n-1}) \leq vol_n(B^n)/vol_{n-1}(S^{n-1})^{n/n-1})$$

*and the coarea formula **[coar]** imply that the differentials of Euclidean $O(n)$-symmetric models $\underline{f}(x)$ of Lipschitz functions f on $X = \mathbb{R}^n$ with bounded supports satisfy the following*

$\|\mathbf{df}\|_{\mathbf{L_p}}$-Symmetrization Inequality.

$$\|df\|_{L_p} = \sqrt[p]{\int_{\mathbb{R}^n} \|df(x)\|^p dx} \leq \|d\underline{f}\|_{L_p} = \sqrt[p]{\int_{\mathbb{R}^n} \|d\underline{f}(x)\|^p}$$

for all $p \geq 1$.

It follows that the infimum of the ratio

$$\gamma_{p,q}(X) = (\|df\|_{L_p}/|f\|_{L_q}, p, q > 1$$

over all Lipschitz functions on $\mathbb{R}^n$ *with supports in a given bounded domain* $X \subset \mathbb{R}^n$ *is bounded from below by* $\gamma_{p,q}(B^n(a))$, *where* $B^n(a) \subset \mathbb{R}^n$ *is the ball with the volume equal to that of* X.

This, for $p = q = 2$ *is called the* ***Faber-Krahn inequality****, which, as explained below, shows that*

among all bounded domains in $\mathbb{R}^n$ *with a given volume, balls have the highest bottom oscillation frequency.*

3.1.2 Bochner Formula and Lichnerowicz λ_2-Inequality

3.2 ???

More specifically the "spherical Poincaré" says that

$$\lambda_2(S^n(1)) = n$$

where the corresponding eigen functions are the linear ones. In fact a straightforward computation (see section ??? below) shows that the Euclidean coordinates x_i *regarded as functions* $x_i(s)$ *on*

$$S^n = \{x_0, ..., x_n\}_{\sum_i x_i^2 = 1}$$

satisfy

$$\Delta_{S^n} x_i(s) = x_i(s) \cdot mean.curv(S^n, s) = -n x_i(s),$$

which agrees with the identities

$$\int_{S^n} \sum_i \|x_i(s)\|^2 ds = \int_{S^n} 1 ds = vol(S^n)$$

and

$$\int_{S^n} \sum_i \|dx_i\|^2 = \int_{S^n} n ds = n \cdot vol(S^n),$$

Bochner Formula

$$\frac{1}{2}\Delta\|df\|^2 = \|\mathsf{Hess}\, f\|^2 + \langle d\Delta f, df\rangle + Ricci(df, df)$$

$$0 = \int_X \|\mathsf{Hess}\, f\|^2 - \lambda\langle df, df\rangle + Ricci(df, df)$$

$\|\mathsf{Hess}\, f\|^2 \geq \frac{1}{n}\|\Delta f\|^2$

$Ricci_{S^n} = (n-1)g_{S^n}$

and Lichnerowicz λ_1*-theorem*

λ_1 *for convex domains by needle integration and by needle decomposition.*

3.3 Cheeger's constant and Cheeger's λ_2 inequality

change!

Let X be Riemannian n-manifold(possibly with a boundary and let $\mathsf{che}(X)$ *be the supremum of the numbers $c > 0$,such that all smooth domains $V \subset X$ with $vol(V) \leq \frac{1}{2}vol(X)$ satisfy*

$$vol - n - 1(\partial V) \geq c \cdot vol_n(V).$$

3.3.A. Cheeger Inequality *[Che1969]. Let a smooth function on X satisfy one of the following conditions*

$\bullet_1$ *the volume of the support of f is finite and it is smaller than the volume of its complement, i.e. $vol_n(\mathsf{supp} f \leq \frac{1}{2}vol(X)$;*

$\bullet_2$ *the function f is orthogonal to constants,*

$$\int_X f(x)dx = 0.$$

Then

$$[\mathsf{che}]_{1/4} \qquad \int_X \|df\|^2 dx \geq \frac{\mathsf{che}^2}{4}\int_X f^2 dx.$$

Proof. *The $\bullet_1$ case in the form*

$$\|f\|_{L_2} = \sqrt{\int_X f(x)^2 dx} \leq \frac{\mathsf{che}}{2}\|df\|_{L_2} = \sqrt{\int_X \|df(x)\|^2 dx}$$

follows from he Maz'ya-Cheeger conditional inequality (L_2-Example???) while $\bullet_2$ reduces to $\bullet_1$ by the following.

$[\lambda_1 \implies \lambda_2]$*-Lemma. If the first Dirichlet eigenvalues of all smooth domains $V \subset X$ with $vol(V) \leq \frac{1}{2}vol(X)$ satisfy $\lambda_(V) \geq c$, for some $c = c(X) > 0$, then the second Neumann eigenvalue of X is also $\geq c$.*

In fact, let f be a non-zero Neumann's λ-eigen-function in X, let $Y \subset X$ be the zero set of f,which divide X into two reagins, X_+ and X_1 with common boundary Y.

Then

$$\int_{X_\mp} df^2(x)dx = \int_{X_\mp} f(x)\Delta f = \lambda \int_{X_\mp} f^2 dx$$

by the Green's formula, since, in both regions, f vanishes on Y and its gradient is normal to he remaining boundaries of $X_\mp$.

Since the minimum of the ratio

$$\min_f \frac{\int_X df^2}{\int f^2}$$

on functions in X normal to the constants is achieved on a non-zero λ_2-eigen-function f_2,an application of $\bullet_1$ to the smallest of the two $X_\mp$ yields $\bullet_2$.

3.3.B. *On Logic of Proofs of $\bullet_1$ and $\bullet_2$. The reduction of $\bullet_2$ to $\bullet_1$ appears less elementary than the proof of $\bullet_1$.*

For example, if X is a polyhedral domain and f is a piecewise linear function, the proof of the $\bullet_1$-case of $[\mathsf{che}]_{1/4}$ *in section ??? can be rendered in a purely algebraic (first order) language, namely that of ordered real fields and piecewise affine (or semialgebraic if you wish) function, while the reduction $\bullet_1 \implies \bullet_2$*

depends on eigenfunction that are minima of the Dirichlet functional in the space all(!) functions.

Besides, while the proof of $\bullet_1$ given in ??? depends only on the isoperimetric properties of the domains $A(t)$) bounded by the t-levels $B(t) = f^{-1}(t)$ of our function f, while, in the $\bullet_2$ case, one needs this for all domains in X with volumes $\le 1/2vol(X)$.

However, if one doesn't care for sharpness of constants one can proceed as follows.

function on a compact Riemannian n-manifold, possibly with a boundary, and let, as earlier,

$$B(t) = f^{-1}(t) = \partial A(t) \subset X, \text{ for } A(t) = f^{-1}[t, \infty).$$

Let, for some $t_0 \ge 0$, all domains $A(t) \subset X$ with $t \ge_0$ satisfy the same isoperimetric inequality as in 2.6.A:

$$vol_n(A(t)) \le c \cdot (vol_{n-1} B(t))^\nu \text{ for some } \nu \ge 1.$$

Then the L_p nom of f is bounded in terms of its L_1 norm, the L_p-norm of the differential of f, where, as in and he volume v_0 of $A(t_0)$, where

$$\frac{1}{p} = \frac{1}{q} - \frac{\nu - 1}{\nu},$$

as follows

$$\|f\|_p \le \frac{pc}{\nu} \|df\|_{L_q} + v_0^{\frac{1}{p}-1} \|f\|_1$$

Indeed, this follows from the Maz'ya-Cheeger inequality 2.6.A applied to the function $h(t) = f(x) - t_0$ restricted to $A(t_0)$, where h is positive and vanishes on $B(0) = \partial A(t)$, and where, clearly, $\|df\|_{L_q} = \|dh\|_{L_q}$, and

$$\|f\|_p \le \|h\|_p + t_0 \sqrt[p]{v_0} \le \|h\|_p + v_0^{\frac{1}{p}-1} \|f\|_1.$$

Next, let $\int f(x)dx = 0$ as in $\bullet_2$ and let X_- and X_+ be the negative/positive regions of f of volumes v_- and v_-, and let $f_-(x) = \min(f(x), 0)$. Then

$$\|f\|_p \le \frac{pc}{\nu} \|df\|_{L_q} + \left(\frac{v_-}{v_0}\right)^{1-\frac{1}{p}} \|f_-\|_p.$$

Finally, let $v_\le v_0$ and let the domains $A_(t) = f^{-1}(-\infty, t]$ for $t \le 0$ satisfy the isoperimetric inequality

$$vol_n(A^-(t) \le c(vol_{n-1}(\partial A^-(t))^\nu,$$

then

$$\|f\|_p \le \left(\frac{pc}{\nu}\left(1 + \frac{1}{2}\right)^{1-\frac{1}{p}}\right) \|df\|_q.$$

For instance, if $\nu = 1$ and $p = q = 2$, then

$$\|f\|_2 \le 2c\sqrt{\frac{3}{2}} \|df\|_2$$

This proof uses the isoperimetric inequality only for the levels of f, but the factor $\sqrt{\frac{3}{2}} > 1$ makes it non-sharp.

To make it sharp, let $S = S_f(X) = (S, ds^2, b(t)ds^2)$ be the formal symmetrization of (X, f) (see section??), that is a one dimensional Riemannian manifold $S = (S, ds^2)$ isometric to ($\mathbb{R}$ with a measure $b(t)ds$.

The above argument applies to $(Sds^2, b(s)ds)$ – in fact, it applies to all metric measure spaces and yields Cheeger's inequality for $(S, \underline{f}$, that is

but by applying Cheeger's argument to the formal symmetrisation of f... ???

on sharpness on hyperboluc cisps ec,

evaluation of Cheeger Constant [che] *for specific manifolds convex sets, hyperbolic spaces, hyperbolic balls, hyperbolic quotients.*

[Che1969].

4 Minkowski and Brunn

Divide and rule.*(Hadwiger-Ohmann?) cut Let two subsets $X, Y \subset \mathbb{R}^n$ be divided by a hyperplane into $X_- \sqcup X_+ = X$ and $Y_- \sqcup Y_+ = Y$, then the Minkowski sums of corresponding "half-subsets" $X_- + Y_-$ and $X_- + Y_+$ in $\mathbb{R}^n$ do not intersect – they lie in different half-spaces of $\mathbb{R}^n$ divided by H – and, since*

$$X + Y \supset (X_- + Y_-) \cup (X_+ + Y_+),$$

$$vol(X + Y) \geq vol(X_- + Y_-) + vol(X_+ + Y_+).$$

Therefore, if H equidivides both sets,

$$vol(X_-) = vol(X_+) = \frac{1}{2}vol(X) \text{ and } vol(Y_-) = vol(Y_+) = \frac{1}{2}vol(Y),$$

Then the Minkowski $\frown^{1/n}$-inequalities for the pairs $(X_\mp, Y_\mp)$

$$vol(X_- + Y_-)^{1/n} \geq vol(X_-)^{1/n} + vol(Y_-)^{1/n},$$

and

$$vol(X_+ + Y_+)^{1/n} \geq vol(X_+)^{1/n} + vol(Y_+)^{1/n},$$

imply this inequality for (X, Y)

$$vol(X + Y)^{1/n} \geq \left(vol(X_- + Y_-) + vol(X_+ + Y_+)\right)^{1/n} \geq$$

$$\left(\left(vol(X_-)^{1/n} + vol(Y_-))^{1/n}\right)^n + \left(vol(X_+)^{1/n} + vol(Y_+)^{1/n}\right)^n\right)^{1/n} =$$

$$vol(X)^{1/n} + vol(Y)^{1/n}$$

Since all sets in $\mathbb{R}^n$ can be translated to a position where they are equidivided by a given hyperplane $H \subset \mathbb{R}^n$, the $\frown^{1/n}$-inequality for (X, Y) reduces to those for two pairs of twice "thinner" sets $(X_\mp, Y_\mp)$.

Then by equidividing further and further we reduce $\frown^{1/n}$ for (X, Y) to inequalities for arbitrary thin sets, i.e. to intersections of X and Y with regions $H^{n-1} \times [0, \varepsilon] \subset \mathbb{R}^n$ between pairs of mutually ε-close hyperplanes.

If X is a smooth or piecewise domain and H is transversal to ∂X then

$$vol_n(X \cap H^{n-1} \times [0,\varepsilon]) = \epsilon \cdot vol_n(X \cap H^{n-1}) + o(\varepsilon),$$ [31]

i.e.

$$|vol_n(X \cap H^{n-1} \times [0,\varepsilon]) - \epsilon \cdot vol_n(X \cap H^{n-1}|/\varepsilon \to 0 \text{ for } \varepsilon \to 0.$$

Moreover, the intersection $X \cap H^{n-1} \times [0,\varepsilon] \subset \mathbb{R}^n$ is "physically" $o(\varepsilon)$-close to the product

$$(X \cap H^{n-1}) \times [0,\varepsilon] \subset \mathbb{R}^n,$$

that is the volume of the symmetric difference between the two subsets is $o(\varepsilon)$.

It follows, in the limit for $\varepsilon \to 0$, that the $\frown^{1/n}$-inequality for (X,Y) reduces to $\frown^{1/n}$ for pairs of products $X' \times [0,\varepsilon]$ $Y' \times [0,\delta]$ for $X', Y'\mathbb{R}^{n-1}$, which is obviously equivalent to the $\frown^{1/(n-1)}$-inequality for (X',Y')

Then he Minkowski $\frown^{1/n}$-inequality for piecewise smooth domains follows by induction on n and, if you care, this inequality for general measurable subsets follows by approximation.

Isoperimetric Corollary. Isoperimetric inequality for smooth and piecewise smooth domains $X \subset \mathbb{R}^n$ follows from $\frown^{1/n}$ applied to X and the ε-ball $B^n(\varepsilon)$ for $\varepsilon \to 0$.)

Indeed, since

$$vol_m(X + B^n(\varepsilon)) = vol_n(X) + \varepsilon vol_{n-1}(\partial X) + o(\varepsilon),$$

the $\frown^{1/n}$ implies that

$$(vol_n(X) + \varepsilon vol_{n-1}(\partial X))^{1/n} + o(\varepsilon) \geq vol_m(X)^{1/n} + \varepsilon vol_n(B^n(1))^{1/n} \implies$$

$$vol_n(X) + \varepsilon vol_{n-1}(\partial X) \geq vol_n(X) + nvol(X)^{(n-1)/n} \varepsilon vol_{n-1}(B^n(1))^{1/n} + o(\varepsilon) \implies$$

$$vol_{n-1}(\partial X) \geq nvol(X)^{(n-1)/n} vol_{n-1}(B^n(1))^{1/n} \implies$$

$$\frac{vol_{n-1}(\partial X)}{vol(X)^{(n-1)/n}} \geq n \cdot vol_{n-1}(B^n(1))^{1/n},$$

which is recognizable for the constant-wise sharp isoperimetric inequality, as it becomes equality for $X = B^n(1)$:

$$\frac{vol_{n-1}(S^n(1))}{vol(B^n(1))^{(n-1)/n}} = n \cdot vol_{n-1}(B^n(1))^{1/n}.$$

(This proof, unbelievably primitive, sharply contrasts with the elegance of the ABP-proof, but, at the bottom, the two arguments rely on the same principle.)

Exercises. (a) Prove $\frown^{1/n}$ for connected solvable simpy connected n-dimensional unimodular[32] *Lie groups G. Hint. Use a normal codimension one subgroup in G for H.*[33]

(b) Show that If the Haar measures of subset X and Y in a compact topological group G satisfy $mes(X) + mes(Y) > mes(G)$, then $X \cdot Y = G$.

[31] Here $\mathbb{R}^n$ is represented as $H^{n-1} \times \mathbb{R}$.

[32] The Haar measure is biinvariant.

[33] See `https://link.springer.com/article/10.1007/s00039-023-00647-6` for such an inequality for more general locally compact groups.

Remark . *(a) There are similar results for infinite discrete groups G, e.g. Mann's theorem for $G = \mathbb{Z}$, but the proofs of these are more subtle. may (b) If G is an infinite, e.g discrete non-abelian group then it may satisfies much stronger isoperimetric and Minkowski type inequalities for sufficiently large and/or dense subsets as we shall see in section ???*

Projection Concavity Theorem. *A measure $\mu = \mu(x)dx$ on $\mathbb{R}^n$ with Borel measurable density $\mu(x)$, $x \in X$, is called $*^{1/\lambda}$-concave if the function $\mu(x)^{1/\lambda}$ is concave on the support of μ.*

BRUNN'S THEOREM. Let μ be a $*^{1/\lambda}$-concave measure on $\mathbb{R}^n$ with convex support $V \subset \mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a surjective linear map. Then the pushforward measure $\pi_*\mu$ on $\mathbb{R}^{n-k}$ is $*^{1/\lambda+k}$-concave.

Example. If $V \subset \mathbb{R}^n$ is a convex subset and $\mu(x)$ is constant on V, this says that the function

$$y \mapsto (vol_{n-1}(\pi^{-1}(y))^{\frac{1}{n-k}}, y \in \mathbb{R}^{n-k},$$

is concave on the image $\pi(V) \subset \mathbb{R}^{n-k}$.

4.1 Reparamerization, Knöte Map and Prekopa-Leindler inequality

4.2 Mass Transportation, Brenier Maps...

4.3 Alexandrov-Fenchel and Hodge Inequalities

4.4 Minkowski Inequality in Arakelov Geometry

5 Filling Inequalities

5.1 Non-Sharp Federer Fleming Filling Inequality in $\mathbb{R}^N$

???A. Euclidean Filling-by-Collapsing with Codimension one Theorem. Let $Y = Y^{n-1} \subset \mathbb{R}^N$ be a smooth submanifold .

Then there exist constants C_N and D_N and a smooth homotopy $\Phi_t(y)$ of the embedding $Y \hookrightarrow \mathbb{R}^N$, that is a smooth map

$$\Phi : Y \times [0,1] \to \mathbb{R}^N, \ \Phi_{y,0} = y,$$

with the following properties.

$\bullet_{\partial_t}$ *the partial t-derivative. of Φ is bounded by:*

$$\|\partial_t \Phi\| \le D_N vol_n(X^{\frac{1}{n-1}});$$

$\bullet_{\|d\Phi\|^{n-1}}$ *The L_{n-1} norm of the differential of the map Φ on the subset $Y_\star =$* supp $\bigwedge^n df \subset Y$*, where $rank(\Phi) = n$ is C_n-boundd*

$$\int_{Y_\star} \|df(y)\|^{n-1} dy \le C_N^{n-1}.$$

Consequently, the n-dimensional volume of the map Φ[34] satisfies

$\bullet_{vol}$ $$vol_n(\Phi) \le D_N C_N^{n-1} vol_{n-1}(Y).$$

[34]This volume may be, a priori, greater than that of the image of Φ as we count the of the volume of the image with multiplicity.

$\bullet_{n-2}$ *The image of* $\Phi_{t=1}$ *is a piecewise smooth subset in* $\mathbb{R}^N$ *of dimension* $(n-2)$,

$$dim(\Phi_1(Y)) = n-2.$$

Remark. *If* $N = n$ *and* $Y \subset \mathbb{R}^n$ *is closed hypersurface, then the above implies a non-sharp isoperimetric inequality, since the domain* X *bounded by* Y *is, because of* $\bullet_{n-2}$*, contained in the image of the (cylinder) map* Φ.

(A natural candidate for a collapsing ϕ*-like map. from the boundary* ∂X *of a closed hypersurface* $X \subset \mathbb{R}^n$ *is the canonical retraction of* X *to the cut locus of* $Y = \partial X$ *in* X*, but there is no apparent simple geometric map, which would also satisfy* $\bullet_{\|df\|^{n-1}}$ *and or* $\bullet_{n-2}$*.)*

In fact, $\bullet_{vol}$ *and* $\bullet_{n-2}$ *show that closed submanifolds* $Y_{n-1} \subset \mathbb{R}^N$ *(this effortlessly extends to quite general* $(n-1)$*-cycles) bound "cylindrical"* n*-chains with volumes bounded by* $const_N vol_{n-1} Y^{n/n-1}$.

Proof of ???A. Since $N > n-1$*, the integral* $\int \frac{1}{\|x\|^{n-1}} dx$ *converges at zero in* $\mathbb{R}^N$*, the mean of the* $dist^{n-1}$*-function in the ball* $B^N(r) \subset \mathbb{R}^N$ *satisfies,*

$$\frac{1}{vol_N(B^N(r))} \int_{B^N(r)} \|x - x_0\|^{-(n-1)} dx \le const_N / r^n$$

for all $x_0 \in \mathbb{R}^N$.

It follows, that for all submanifolds (not necessarily of dimension $n-1$*) in* $\mathbb{R}^N$*, there exist points* $x \in B^N(r)$*, such that*

$$\int_Y \|y - x\|^{-(n-1)} dy \le \frac{const_N vol(Y)}{r^N}.$$

Therefore, the radial projection ψ_x*,* $x \in B^N(r)$*, of* $Y = Y^{n-1}$ *from such a point* x *to the boundary of a convex subset* $V \supset B^N(2r)$ *satisfies:*

$$vol_{n-1}(\psi_x(Y \cap V)) \le \int_{Y \cap V} \|d\psi(y)\|^{n-1} dy \le \frac{diam(V) \cdot const'_N vol_{n-1}(Y \cap V)}{r^N}.$$ [35]

Now, let us partition $\mathbb{R}^N \supset Y$ *into parallel translates of the cube* $[0,d]^N \subset \mathbb{R}^N$*, where* d *is much larger than* $vol_{n-1}(Y)^{1/n-1}$*, say*

$$d = (10N)^{10N^2} vol_{n-1}(Y)^{1/n-1}.$$

Then radially project the intersections of Y *with these cubes, say* $Y \cap ([0,d]^N + x_i$*, where* x_i *are vectors in the lattice* $d\mathbb{Z}^N \mathbb{R}^N$*, to the boundaries of these cubes with a controlled increase of their volumes, say by factors* $(2N)^N$.

Then apply the same to the images of these maps intersected with the$(N-1$*-faces of the cubes and continue until you land up in the* $n-1$*-faces.*

Since the cubes were chosen so large, none of these maps is onto, hence their imagers can be radially projected to the $(n-2)$*-dimensional boundaries of these faces.*

Then composition of all these radial projections naturally included into a homotopy of maps satisfies $\bullet_{\partial_t}$ $\bullet_{\|d\Phi\|^{n-1}}$ $\bullet_{vol}$ *and* $\bullet_{n-2}$*. QED.*

[35] Even if the boundary ∂V is non-smooth, this makes sense since the map ϕ_x is Lipschitz.

Questions. (a) What are the optimal constants C_N and/or D_N in $\bullet_{\partial_t}$, $\bullet_{\|d\Phi\|^{n-1}}$ and $\bullet_{vol}$? Do they depend on n rather then on N? (Compare with ???C below).

(b) Does ???A hold for submanifolds Y in complete simply connected spaces with $sect.curv \leq 0$?

Does ???A hold with *logarithmic bound* on $\|\partial_t\|$ and on $\|d\Phi\|_{L_{n-1}}$ in symmetric spaces with $sect.curv \leq 0$ and with $rank \leq n-1 = dim(Y)$?

Does ???A hold in complete N-dimensional manifolds with $sect.curv \geq 0$ (and singular Alexandrov spaces) with volume growth $constR^N$?

5.1.1 Slicing by Parallel "Planes

. Let us show that an isoperimetric (filling) inequality for maps – we allow non-embeddings $Y \hookrightarrow Z$ but keep notation as if these are embedding – for manifolds (or cycles) of dimensions $n-1$ and $n-2$ in a Riemannian manifold Z, e.g. $Z = \mathbb{R}^{N-1}$, implies a similar inequality for maps $Y^{n-1} \to Z \times \mathbb{R}$.

In fact, given $Y \hookrightarrow Z \times \mathbb{R}$, the mean/mean value argument, implies, as earlier, that, for all d there a "greed of parallel Z-planes within distance d one from another", that is a subset $Z' = Z \times (d\mathbb{Z} + t)$. for some $t \in \mathbb{R}$, such that

$$vol_{n-2}(Y \cap Z') \leq vol_{n-1}(Y)/d.$$

Let $d = vol_{n-1}(Y)^{1/n-1}$, let is fill in all intersections $Y \cap Z \times \{di + t\}$, $i \in \mathbb{Z}$ by $(n-1)$-chains in the "planes" $Z_i = Z \times \{di + t\}$ and thus decompose the $(n-1)$-cycle (represented by) Y into the sum $Y = +_i Y_i$, where each Y_i is contained in the "d-band between two "planes",

$$Y_i \subset Z \times [di + t, d(i+1)] + t.$$

Then, fill in all Y_i by firstly normally projecting them to $Z_i = Z \times \{di + t\}$ and then filling them in these $Z_i = Z$.

Example. Let $Z = \mathbb{R}^2$ and $Y \subset Z \times \mathbb{R} = \mathbb{R}^3$ be a surface with unit area. Then $d = 1$ and the total sum of the planar domains in all Z_i encompassed by their intersections with Y is $< 1/2$ by the (rough) 2-dimensional isoperimetric inequality. Then we project each Y_i to Z_i which needs $d\cdot$-area of filling 3-volume, while nothing is added in Z_i since $dim(Z_i) = dimY_i$ in the present case.

Thus we conclude that the domain $X \subset \mathbb{R}^3$ bounded by Y satisfies

$vol_3(X) < C_\bullet area(Y)^{2/3}$ with $C_\bullet = 2$ instead of $C_3 = \frac{1}{6\sqrt{\pi}} = \frac{4\pi/3}{(4\pi)^{3/2}}$ of the sharp inequality.

Question. Does the Euclidean type filling inequality hold for $(n-1)$-cycles Y in product manifolds $Z_1 \times Z_2$, where Z_1 and Z_2 are complete contractible Riemannian manifolds, such that $sect.curv(Z_1) \leq 0$, $sect.curv(Z_2 \geq 0$ and where Z_2 has (at least) $const_2 R^{dim(Z_2)}$-volume growth?

Remark. The (n_1-1)-cycles $Y \subset Z_1$. are known (see [???] and section ???) to bound chains X with

$$vol_{n_1}(X) \leq const_{n_1} vol_{n_1-1}(Y)^{\frac{n_1}{n_1-1}},\ n_1 = 2, 3, ...$$

and $Y^{n_2-1} \subset Z_2$ bound X with

$$vol_{n_2}(X) \leq \frac{const_{n_2}}{const_2^{n_2/N}} vol_{n-1}(Y)^{\frac{n_2}{n_2-1}},\ n_2 = 2, 3, ...,$$

(see section ??? in [???]).

Such inequalities with $n_1 = dimZ_1$ and $n_2 = dimZ_2$ imply the corresponding Euclidean-type isoperimetric inequality for hypersurfaces in $Z_1 \times Z_2$ by the (formal) Schwarz symmetrization argument (see section ??? below, [grigorian???95], section 9 in [waists???2003], [morgan????2006] and references therein) but a similar product property is unlikely to hold, in general, for cycles of higher codimension.

However, this may work in the (annoyingly eclectic) $[curv \leq 0] \times [curv \geq 0]$-case.

5.2 Filling Lipschitz Cycles in Riemannian Manifolds

Let X be a Riemannian manifold and

$$C_* = (C_*(X, \mathbb{F}, \partial_*) = (\{\partial_i : C_i \to C_{i-1}\}_{i=0,1,\ldots,n=dim(X)}$$

be the complex of Lipschitz singular chains: the i-chains are finite sums $c = \sum_j f_j\sigma_j$, where $\sigma_j : \Delta^i \to X$ are Lipschitz maps of the standard i-simplex to X and $f_i \in \mathbb{F}$.

If $\mathbb{F}$ comes with a norm (e.g. $\mathbb{F}$ equals $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), then each chain $c \in C^i$ is given the Riemannian i-volume norm, $vol_i(c) = ||c||_{vol_i} = \sum_i ||f_j|| vol_i(\sigma_j)$.

Clearly $||\partial_i|| = \infty$ for all $i \geq 2$ (and also for $i = 1$ if the norm in $\mathbb{F}$ is unbounded).

The corresponding minimal norm on the homology $H_i(X) = Ker\partial_i / Im\partial_{i+1}$, that is the infimum of the volumes of the cycles $c \in Ker\partial_{i=1}$ representing an $h \in H_i(X)$ is called the volume or the mass norm,

$$vol_i(h) = inf_{[c]=h} vol_i(a).$$

If X is compact, or, more generally, admits a co-compact isometry group, then one easily sees that this norm does not vanish: $\inf_{h \neq 0} vol_i(h) > 0$, for $h \neq 0$.

The i-th $\mathbb{F}$-systole of X is then defined as

$$syst_i(X) = \inf_{h \neq 0} vol_i(h), \text{ where } h \in H_i(X).$$

5.2.1 Riemannian Federer-Fleming

A minor modification of Federer-Fleming's "filling-by-collapsing" argument from section ??? yields. the following general filling inequality for "small" Lipschitz chains in all Riemannian manifolds with bounded Lipschitz geometries.

Let X be a Riemannian manifold and

$$C_* = (C_*(X, \mathbb{F}, \partial_*) = (\{\partial_i : C_i \to C_{i-1}\}_{i=0,1,\ldots,N=dim(X)}$$

be the complex of Lipschitz singular chains: the i-chains are finite sums $c = \sum_j f_j\sigma_j$, where $\sigma_j : \Delta^i \to X$ are Lipschitz maps of the standard i-simplex to X and $f_i \in \mathbb{F}$.

If $\mathbb{F}$ comes with a norm (e.g. $\mathbb{F}$ equals $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), then each chain $c \in C^i$ is given the Riemannian i-volume norm, $vol_i(c) = ||c||_{vol_i} = \sum_i ||f_j|| vol_i(\sigma_j)$.

Clearly $||\partial_i|| = \infty$ for all $i \geq 2$ (and also for $i = 1$ if the norm in $\mathbb{F}$ is unbounded).

The corresponding minimal norm on the homology $H_i(X) = Ker\partial_i/Im\partial_{i+1}$, that is the infimum of the volumes of the cycles $c \in Ker\partial_{i=1}$ representing an $h \in H_i(X)$ is called the volume or the mass norm,

$$vol_i(h) = inf_{[c]=h} vol_i(a).$$

An N-dimensional Riemannian manifold (possibly with a boundary) X is ***bLg*** *(**b**ounded **L**ipschitz **g**eometry) if there exit numbers* $r_0 > 0$ and $\lambda > 0$, *such that all ball $B(r) \subset X$, $r \le r_0$ admit λ-bi-Lipshitz embeddings $B(r) \to \mathbb{R}^N$.*

For instance compact manifolds and those admitting co-compact isometry groups are ***bLg***

Exercise. Show that the mass/volume norm does not vanish in the ***bLg*** *manifolds:* $\inf_{h\neq 0} vol_i(h) > 0$, *for $h \neq 0$.*

??? A. Theorem *Let X be an N-dimensional Riemannian* ***bLg*** *manifold. Then there exists a positive constant $\beta_0 = \beta_0(X) > 0$, such that if $n - 1 \ge 1$, then every $(n-1)$-cycle b in X with $||b|| \le \beta_0$ satisfies*

$$||b||_{fil} \le cons||b||^{\frac{n}{n-1}}, \text{ where } const = const(N, r_0, \lambda),$$

where the n-fillings c_ε, $varepsilon > 0$ of a $(n-1)$cycle b, i.e. chains ∂c_ε, such that $\partial c_\varepsilon = b$ and $\|c\| \le cons||b||^{\frac{n}{n-1}} = \varepsilon$, $\varepsilon \to 0$, are (implemented by) the cylinderes of Lipschitz map Φ_ε from the support of b to X, such that $dist(x, \Phi_\varepsilon(x) \le const + \varepsilon$ for all x in the support of b and such that the rank of the differential of Φ_ε is almost everywhere $\le n-2$.

If, moreover, X is compact, then all b homologous to zero satisfy

$$||b||_{fil} \le const_X ||b||.$$

mostly move to ???**From const(N) *to* const(n)**. *A Riemannin manifold X is* **ulLc**(***u**niformly **l**ocally **L**ipschitz **c**ontractible) if there exist positive constants* constants ,$r_0 > 0$ and $\lambda \ge 0$ *such that all balls $B(r) = B_x(r) \subset X$, $x \in X$, with $r \le r_0$ admit λ-Lipschitz homotopies H_t, $t \in [0,1]$ to points in X:*

these homotopies are λ-Lipschitz maps $H : B(r) \times [0,1] \to X$, such that $H_0 : B(r) \to X$ are equal to the original embeddings $B(r) \hookrightarrow X$ and H_1 are constant maps, $B_x(r) \to x'(x) = H_1(B_x(r))$.

??? B. Theorem. *Let X be an UulLc Riemannian manifold. Then there exists a positive constant $\beta_0 = \beta_0(X) > 0$, such that if $n - 1 \ge 1$, then every $(n-1)$-cycle b in X with $||b|| \le \beta_0$ satisfies*

$$||b||_{fil} \le cons||b||^{\frac{n}{n-1}}, \text{ where } const = const(n, r_0, \lambda),$$

If, moreover, X is compact, then all b homologous to zero satisfy

$$||b||_{fil} \le const_X ||b||.$$

Remark. Unlike the above theorem ???A, it is unclear if there are n-fillings c of b, which can be implemented by the cylinderes of $dim(b)$-controllably Lipschitz homotopies Φ_t with $dist(x, \Phi_\varepsilon(x)) \le const(n)\|b\|^{1/n-1}$ and with the ranks of the differentials of Φ_1 almost everywhere bounded by $n-2$.

5.3 Vitali Decomposition of Submanifolds and Measures into ε-Round Peacies

???A. Vitali Covering Lemma. Let a metric space X, e.g. $X \subset \mathbb{R}^n$, be covered by finitely many subsets $B_i \subset X$, $i \in I$, of diameters δ_i (e.g by balls of radii $r_i = \delta_i/2$). Then there exists a subset $J \subset I$, such that the sets B_j, $j \in J$, *do not pairwise intersect* and such that the *closed δ_i-neighbourhoods*[36] $U_{\delta_j}(B_j)$ (e.g. the concentric balls $B(3r_j)$ in the case of r_j-balls) *cover X*.

Proof. Let B_1 be the subset with the largest diameter, let B_2 be the largest subset which doesn't intersect B_1, let B_3 be the largest one, which doesn't intersect $B_1 \cup B_2$, and, in general, let B_{j+1} be the largest subset, among B_i, which don't intersect the union $B_1 \cup ... \cup B_j$.

Since each B_i intersects some B_j with $\delta_j = diam(B_i) \geq diam(B_i)$

$$U_{\delta_j}(B_j) \supset B_i$$

and the proof follows. ,

Corollary "Round" (Quasi) Decomposition of Measures and Submanifolds.

Let V be a metric space, e.g. $V \subset \mathbb{R}^N$ with the Euclidean metric (distance function)) and μ be a Borel measure in v, e.g. the n-volume measure on a smooth n-dimensional submanifold in $V \subset \mathbb{R}^N$ and let $\upsilon(r)$, $r \geq 0$, be a positive monotone increasing function, e.g. $\upsilon(r) = \varepsilon r^n$.

A Borel (e.g. closed) subset $B \subset V$ is called υ-round with respect to μ if

$$\mu(V) \geq \upsilon(diam(V)).$$

More specifically an n-dimensional submanifold $V \subset X$ is called ε-round if

$$vol_n(V) \geq \varepsilon \cdot diam(V)^n$$

For instance the Euclidean n-balls $B^n(r)$ in $\mathbb{R}^n$ are ε_n-round with $\varepsilon_n > n^{-n}$, while the cylinders $B^{n-1}(r) \times [0, R]$ are $\varepsilon(n, r, R)$-round with $\varepsilon(n, r, R) \to 0$ for $R/r \to \infty$ as well as for $R/r \to 0$.

***???B.** Let V be covered by finitely many υ-round subsets $B_i \subset V$, $i \in I$. If*

$$\upsilon(3r) \leq C\upsilon(r) \text{ for all } r > 0 \text{ and some } C > 0,$$

(e.g. $\upsilon(r) = \varepsilon r^n$, where $C = 3^n$). Then there exit finitely many disjoint υ-round subsets $B_j^+ \subset V$, $j \in J$, which contain a "substantial amount" of the measure of V, that is

$$\mu\left(\bigcup_{j\in J} B_j^+\right) \geq \frac{\mu(V)}{C}.$$

Proof. Let $\rho_i^+ \geq 0$ be the maximum of the numbers $\rho \geq 0$ such that the ρ-neighbourhood $U_\rho(B_i) \subset V$ is υ-round [37] *and apply Vitali 's lemma to the covering of V by $B_i^+ = U_{\rho_i^+}(B_i)$, $i \in I$.* [38] *This [argument, I guess, is used everywhere in analysis; I learned it from the geometric paper [We????]*

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[36] ???

[37] This ... taken with respect to the. restriction of the metetric (distance function) in CV indused from CX

[38] (

???C. *Let X be a Riemannian Manifold $V \subset X$ a compact n-dimensional submanifold with a (possibly empty) boundary ∂V let $\mu = dv$ be the Riemannian measure of V, let $v(r) = \varepsilon r^n$ and let $B_i \subset X$ be a covering of V by v-round subsets.*

Then there exit finitely many disjoint $v/9^n$-round n-dimensional submanifolds $B_k^{++} \subset V$, $k \in K$, such that

$$\mu\left(\bigcup_{K \in J} B_j^{++}\right) = vol_n\left(\bigcup_{k \in K} B_j^{++}\right) \geq \frac{\mu(V)}{9^n} = \frac{vol_n(V)}{9^n}$$

and such that the $(n-1)$-dimensional volumes off the boundaries of $B^{++}k$ in V satisfy

$$vol_{n-1}(\partial_V B_k^{++}) \leq \frac{2^n vol_n(B_k^{++})}{diam(B_k^{++})}.$$

Proof. Let $B_j^+ \subset V$ be as above, apply Vitali's lemma to the covering of V by $U_j = U_{d_j}(B_j^+)$, for $d_j = diam(B_j^+)$ and let $U_k = U_{d_k}(B_k^+) \subset V$ be Vitali's disjont subsets, which, observe, cover at least $vol_n(V)/3^n$.

Then let $\rho_k(v)$ be the distance functions on U_k,

$$\rho_k(v) = dist(v, B_k^+),$$

let $S_k^{++} = \rho^{-1}(r_k) \subset U_k$ be the level of the function ρ_k, which minimizes $vol_{n-1}(\rho^{-1}(r))$ for $0 \leq r \leq d_k = diam(B_k^+)$ and observe that the subsets

$$B_k^{++} = \rho_k^{-1}[0, r_k] \subset U_k \subset V, k \in K \subset J,$$

satisfy the requirements of ???C by. the coarea inequality.

5.3.1 From const(N) to const(n) by Cutting off Bubbles on Narrow Necks.

Let Z be a Riemannian manifold and A, B, d be positive numbers, such that

$\bullet_A$ $(n-2)$-cycles Y' in a Riemannian manifold Z bound (n-1)-chains X' with volumes

$[Vol_{n-2}]$ $$vol(X') \leq A \cdot vol_{n-1}(Y')^{\frac{n-1}{n-2}};$$

$\bullet_B$ $(n-1)$-cycles Y with dimeters D, bounds chains X in Z with

$[D \cdot vol_{n-1}]$ $$vol_n(X) \leq BD \cdot vol(Y)^{n/n-1};$$

$\bullet_d$ $(n-1)$-cycles Y in the δ-balls $B_z(\delta) \subset Z$ for $\delta \leq d$ bound n-chains $X \subset B_z(\delta)$ of volumes

$[\delta \cdot Vol_{n-1}]$ $$vol_n(X) \leq B\delta \cdot vol(Y)^{n/n-1}$$

and $(n-2)$-cycles $Y' \subset B_z(\delta)$ bound $(n-1)$-chains $X' \subset B_z(\delta)$ of volumes

$[\delta \cdot Vol_{n-2}]$ $$vol_{n-1}(X) \leq B\delta \cdot vol(Y)^{n-1/n-2}$$

Then there exists a constant C, which depends only on n, A, B (but not on d), such that $(n-1)$-cycles in Z bound chains X with

$$vol_n(X) \leq vol_{n-1}(Y)^{n/n-1}.$$

Remark. The condition $\bullet_d$ with some $d > 0$ is satisfied by all compact manifolds Z and also non-compact complete ones with locally bounded geometries.

Idea of the Proof. Property $\bullet_A$ allows a decomposition of Y into a sum $Y = Y_1 = Y_D + Y_2$, such that $\bullet_B$ applies to ("connected components" of) Y_D and where $vol(Y_2) \leq (1-\varepsilon)vol(Y_1)$. Then this applies to Y_2, etc and eventually reduces the problem to cycles of volumes $<< d^{n-1}$, where $\bullet_d$ leads to termination of the iteration process.

(See `https://arxiv.org/abs/math/0703889`,
`https://arxiv.org/abs/math/0306089`,
`https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/fillingRiemannianManifolds.pdf`
for several versions of this inequality.)

Questions.

(Hypebolic case? sharp constant $\pi/2$?

2. proof induction with hyperplanes.

3. Induction with spheres, Banach spaces

5.4 Dehn-Levy-Almgren Local-to-Global Argument

Here we are mostly concerned with explicite bounds on the constants $const_n$ and $const_X$ for "simple" manifolds and we start with

Sharp evaluation of $||\partial_{i+1}^{-1}||_{rand}$ for round spheres. If X is the round Euclidean sphere S^n with the $O(n+1)$-invariant i-volumes normalized so that the equatorial spheres $S^i \subset S^n$ have volume 1, then the average $(i+1)$-volume of the geodesic cones from the points $s \in S^N$ over an equator $S^i \subset S^n$ (with respect to every probability measure on S^n), obviously, equals 1/2. Since the group $O(n+1)$ is transitive on the set of tangent i-planes in S^n, the equality $||S^i||_{rand} = vol(S^i)/2$ for the averages with respect to the $O(n+1)$-*invariant* measure on S^n implies that $||c||_{rand} = vol(c)/2$ for all i-chains c; hence

$$||\partial_{i+1}^{-1}||_{rand}(\beta) = 1/2 \text{ for all } i = 1, 2, ..., n-1 \text{ and } \beta \geq 0.$$

Notice that the resulting bound $||\partial_{i+1}^{-1}||_{fil}(\beta) \leq 1/2$ is *not sharp* unless $\beta = 1$, but, as one everybody would guess,

If X is either $\mathbb{R}^n$ or S^n, then round (umbilical) i-spheres of volume v_0 (filled-in by flat $(i+1)$-discs) have maximal filling volumes (i.e. $||...||_{fil}$) among all i-cycles b in X, with $vol(b) = v_0$.

If $X = \mathbb{R}^n$ this is due to Almgren [**?**] and the spherical case was reduced to $\mathbb{R}^{n+1} \supset S^n$ by Bruce Kleiner (private communication.)

Remarks. If $X = S^n$, this leaves open the sharp bound on $||\partial_{i+1}^{-1}||_{fil}(\beta)$ for $\beta > 1$ that may depend on n (if $n > i+1$) and on the coefficient field $\mathbb{F}$ in a rather complicated manner.

Also, the "filling extremality" of round i-spheres (filled -in by flat $(i+1)$-discs) remains unproven in the hyperbolic spaces X; but the Almgren-Levy argument provides rather sharp bounds on the filling volume of cycles in manifolds with *lower* bounds on curvatures (see below).

Local-to-Global Variational Principle: $||...||_{fil} \leq ||...||_{fil}^{loc}$. Let (C_*, ∂_*) be normed chain complex, $h \in H_i(C_*)$ a homology class and let $B = B_i(h) \subset C_i$ be the space of i-cycles in the class of h with the filling metric $dist_Z(b_1,b_2) = ||b_1 - b_2||_{fil}$. Define the supremum norm of the "downstream gradient" of the function $b \mapsto ||b||$ on B as follows,

$$|| \downarrow b||_{sup} = \limsup_{||c|| \to 0} \frac{||b|| - ||b + \partial_{i+1}(c)||}{||c||} \text{ for } c \in C_{i+1} \setminus \{0\}.$$

Observe that this norm on smooth submanifolds Y representing cycles b in Riemanniann manifolds X equals the supremum of the norm of the mean curvatures of Y, denoted $sup_y ||M(Y)||_y$.

Let $m(v) = \inf_{||b||=v} || \downarrow b||_{sup}$, define

$$||\partial_{i+1}^{-1}||_{fil}^{loc}(\beta) = \int_0^\beta v \cdot m^{-1}(v) dv$$

and say that ∂_{i+1} satisfies *local-to-global principle* if

$$||\partial_{i+1}^{-1}||_{fil} \le ||\partial_{i+1}^{-1}||_{fil}^{loc}.$$

This would hold if we had a gradient flow $b(v)$ in B parameterized by $v = ||b(v)||$ starting from b with $||b|| = \beta$ and terminating with $b = 0$. One can not, in general, guarantee such flows; yet,

if C_* is the complex of Lipschitz chains in a smooth Riemannian manifold X with (possibly empty) i-mean convex boundary ∂X (i.e. where the traces of the second fundamental form are non-negative on all tangent i-planes in ∂X), then

$$||\partial_{i+1}^{-1}||_{fil} \le ||\partial_{i+1}^{-1}||_{fil}^{loc}$$

for all $i = 1, 2, \ldots$.

In fact, there is the following better bound. Call a compact i-dimensional subvariety (rectifiable set) Y in X *quasiregular* if the subset $reg(Y) \subset Y$ of C^2-smooth points has full i-measure in Y and such that the function $d_x(y) = dist_X(x,y)$ assumes its minimum in Y at a *regular* point $y \in reg(Y) \subset Y$ for almost all $x \in X$. Set

$$m_{reg}(v) = \inf_{|Y|=v} \sup_y ||M(Y)||_y \text{ (this is } \ge m(v))$$

where the supremum is taken over all compact *quasiregular* i-dimensional subvarietyies Y in X with $|Y| =_{def} vol_i(Y) = v$, where $||M||_y$ denotes the norms of the mean curvature vectors at regular points $y \in reg(Y)$ and where the supremum is taken over all $y \in reg(Y)$. Define

$$||\partial_{i+1}^{-1}||_{fil}^{reg}(\beta) = \int_0^\beta v \cdot m_{reg}^{-1}(v) dv \text{ (this is } \le ||\partial_{i+1}^{-1}||_{fil}^{loc}(\beta))$$

and conclude, appealing to the geometric mesaure theory, to the (intuitively obvious but technically non-trivial)

Local-to-Global Inequality. ————————-

$$||\partial_{i+1}^{-1}||_{fil} \le ||\partial_{i+1}^{-1}||_{fil}^{reg}$$

for all $i = 1, 2, \ldots$.

Remark. The local-to-global principle is ubiquitous in the geometric measure theory, albeit it is rarely stated explicitly (see [?] and references therein). It holds for *complete non-compact* X with "decent" behavior at infinity e.g. for ε-*locally contractible* X with some $\varepsilon > 0$ where every ball of radius ε is contractible in the concentric unit ball. (This principle seems to hold for many classes of non-Riemannian X, e.g. for Alexandrov spaces with a lower bound on curvatures and for smooth strictly locally convex Finsler spaces.)

Example: Dehn's Lemma. Let X admit a family of properly immersed cooriented smooth hypersurfaces S_r, $r > 0$, such that

(a) *the i-mean curvatures $M_{i-1}(S_r)$* of all S_r, i.e. the traces of the restrictions of the second fundamental form of S_r to all tangent i-planes to S_r, are bounded from below by a positive constant m_0.

(b) There exists a locally compact space $\tilde{X}$, a proper continuous map $p : \tilde{X} \to X$ and a continuous function $f : \tilde{X} \to \mathbb{R}_+$, such that

(b_1) $p^{-1}(S_r) = f^{-1}(r)$ for all $r > 0$;

(b_2) the map p properly embeds the 0-level $f^{-1}(0) \subset \tilde{X}$ to X, where the image is a rectifiable set and where either $dim(f^{-1}(0)) \le i - 1$ (e.g. $f^{-1}(0)$ is empty) or $f^{-1}(0)$ is contractible of dimension i.

Then every quasiregular Y, that is not contained in $p((f^{-1}(0))$, (obviously) has $\sup_y \|M(Y,y)\| \ge m_0$; hence,

$$\|\partial_{i+1}^{-1}\|_{fil}(\beta) \le m_0^{-1}$$

for all $\beta \ge 0$.

Example. The concentric r-spheres S_r in the hyperbolic n-space X, and in every complete simply connected manifold of curvature ≤ -1, have $M_i(S_r) \ge i$; thus, $\|\partial_{i+1}^{-1}\|_{fil}(\beta) \le i^{-1}$ for all $i \ge 1$ in these X.

Remark. The Dehn inequality is never sharp (at least in the natural examples) and the true value of $\|\partial_i^{-1}\|_{fil}$ remains unknown in most cases, even in the hyperbolic n-space for $3 \le i \le n-1$.

5.5 Tube Formulas and Generalized Levy-Almgren Inequalities move tubes to an appendix

Given a submanifold $Y \subset X$, possibly with a boundary ∂Y denote by $U_R^\perp(Y) \subset X$ the subset of those $x \in X$ for which $dist_X(x,Y) \le R$ and such that all distance minimizing segments $[x,y] \subset X$ (of lengths $= dist(x,Y)$ have their Y-ends y in the interior $Y \setminus \partial Y$ of Y.

Observe that if Y has no boundary, then $U_R^\perp(Y)$ equals the R-neighbourhood $U_R(Y)$ and, thus, $vol_n(U_R^\perp(Y)) = vol_n U_R(Y) \ge vol_n(U_R(y))$ for $n = dim(X)$ and all $y \in Y$ (where $U_R(y)$ is the r-ball around y).

If $X = X^n(\kappa)$ is the complete simply connected n-dimensional manifold of constant sectional curvature κ and $Y = Y^i(M)$ is a round (umbilical) i-sphere of mean curvature m, let $Vl_i(R;,m,\kappa) = vol_n(U_R(Y))/vol(Y)$.

The Hermann Weyl tube formula implies that every quasiregular $Y = Y^i \subset X^n(\kappa)$ with $\sup_{y \in reg(Y)} \|M(Y)\|_y \le m$ has

$$vol_n(U_R(Y) \le Vl_i(R,m,\kappa) \cdot vol_i(Y).$$

If $\kappa = 1$ and $R = 2$ this, combined with the local-to-global-principle, immediately yields the Almgren-Kleiner result on filling extremality of round subspheres in S^n and a similar filling inequality in $X = S^n/G$ for finite isometry groups G (fixed points are allowed) of order $|G|$:

the filling norm in X is bounded by that in S^n as follows:

$$||\partial_X^{-1}||_{fil}(\beta) \leq ||\partial_{S^n}^{-1}||_{fil}(|G|\cdot\beta) \textit{ for all } \beta \leq vol_i(S^i)/|G|.$$

If $\kappa = 0$ this yields, with $R \to \infty$, Almgren's sharp filling inquality in $\mathbb{R}^n$ and similar inequalities in the quotient spaces $\mathbb{R}^n/G$.

Question. Does this bound on $|b||_{fil}$ remain valid if $vol_{i-1}(b)$ is substituted by the measure of the set $A_{cros}(b) \subset Af_{n-i+1}(\mathbb{R}^n)$ of the $(n-i+1)$-dimensional affine subspaces g in $\mathbb{R}^n$ that intersect b? (See the last three lines in 5.7 of [?] for such bound for hypersurfaces.).

Let X be an n-dimensional Riemannian manifold, and $V_\varepsilon = V_\varepsilon^i \subset X$ an ε-germ of a smooth i-submanifold at a point $v_0 \in X$.

The normal R-tube around V_ε, denoted $V_\varepsilon \dot{+} R \subset X$, is the union of the geodesic segments $\gamma = [v, x] \subset X$ normal to V_ε, such that, every γ has $lenght(\gamma) \leq R$ and such that no point in γ is focal with respect to V_ε, (which is essentially equivalent to $dist(x,)V_\varepsilon = dist(x, v) = lenght(\gamma)$).

If $\varepsilon \to 0$, then the volume of the tube depends on the second jet of V_ε at v_0,

$$vol_n(V_\varepsilon \dot{+} R) = vol_i(V_\varepsilon) \cdot VL_i(X, R, v_0, \tau_0, K_0) + o(vol_i(V_\varepsilon))$$

where τ is the tangent space to V_ε at v_0 and K_0 the second fundamental form of V_ε at v_0.

Denote by $MVL_i(X, R, m)$ the supremum of $VL_i(X, R, u_0\tau_0, K_0)$ over all K_0 with the norm of the trace (mean curvature) bounded by m, over all tangent i-planes τ at v_0 and all $v_0 \in X$.

For instance, if X is a complete simply connected space of curvature κ then $MVL_i(X, R, m)$ equals the above $Vl_i(R, m, \kappa)$, since the supremum of $VL_i(X, R, u_0\tau_0, K_0)$ is asssumed on umbilic submanifolds V_ε.

Clearly, every qusiregular Y with mean curvature bounded by m satisfies

$$vol_i(Y)/vol_n(X) \geq MVL_i(X, R, m)$$

for all $R \geq diam(X)$ and so an upper bound on MVL can be used in a conjunction with the local-to global-principle same way as the Weyl tube formula.

On the other hand, the function MVL_i can be evaluated in a variety of cases. This provides lower bounds on the volumes of quasiregular $Y \subset X$ in terms of the the mean curvature, namely $sup_y||M(Y)|_y$. However, the such bounds are sharp only in rather special cases.

Examples. (A) If X is a symmetric space (where the equation for Jacoby fields along geodesics γ in X satisfy a linear ODE-system with *constant* coefficients) it satisfies a Weil type formula and MVL is, in principle computable. The resulting lower bound on $vol(Y)$ by $\sup_y ||M(Y)|_y$ is sharp for manifolds of constant curvature.

Also, if $X = \mathbb{C}P^k$ is a complex projective space and Y is minimal, i.e. $M(Y) = 0$, then the above indicated bound is sharp: every $2j$-diminsional quasiregular $Y \subset \mathbb{C}P^k$ has $vol_{2j}(Y) \geq vol_{2j}(\mathbb{C}P^j)$.

However, the corresponding sharp bound is unknown for odd dimensional Y. For instance, if $Y \subset \mathbb{C}P^k$ is a hypersurface, $dim(Y) = 2k-1$, one expects that its volume is bounded from below by the volume of some *homogeneous* $Y_0 \subset \mathbb{C}P^k$ with $||M(Y_0)|| = \sup_y ||M(Y)|_y$, where "homogeneous" means that the isometry group of $\mathbb{C}P^k$ preserving Y_0 is transitive. on Y_0.

On the other hand, if, for instance, $M(Y) = 0$ (i.e. M is minimal), then a potential Y_0 with minimal volume guarantied by the tube formula would be a totally geodesic submanifold. But there is no odd dimensional totally geodesic $Y_0 \subset \mathbb{C}P^k$ for $dim(Y_0) > 1$.

It follows by compactness argument, that, there is a non-zero correction term to the lower bound on $vol_i(Y)$ impruving the bound with the tube formula, but it still leaves far from a sharp bound.

Also, one does not know if domains $U \subset \mathbb{C}P^k$ solving the isoperimetric problem have homogeneous boundaries, unless they have (very) small volumes.

(B) If $Ricci(X) \geq (n-1)\kappa$ and $i = n-1$, then

$$MVL_i(X, R, m) \leq Vl_i(R, m, \kappa),$$

by the Paul Levy tube bound.

(C) If X is a Riemannian manifold where the sectional curvatures are bounded from below by κ, then

$$MVL_i(X, R, m) \leq Vl_i(R, m.\kappa),$$

for all i by Buyalo-Heintze-Karcher comparison theorem [**?**], [**?**]

(D) If $Ricci(X) \geq (n-1)/\rho^2 > 0$, then, $MVL_i(X, R, m) \leq MVL_i(X, \pi\rho, m)$ for all R. and if, moreover, $curv(X) \geq \kappa$, then $MVL_i(X, R, m) \leq Vl_i(\pi\rho, m\kappa)$.

These, together with the local-to-global principle, provide bounds on the filling volumes in X, e.g. as follows.

Let $X = X^n$ be a complete no-compact Riemannian manifold with sectional curvature≥ 0 and let the R-balls around some (and, hence each) point satisfy,

$$\limsup_{R\to\infty} vol_n(B(R; X))/R^n \geq c \cdot vol_n(B(1; \mathbb{R}^n)).$$

Then, for each $i = 1, 2, ..., dim(X) - 1$, every i-cycle of volume $c \cdot \alpha$ bounds an $(i+1)$-chain of volume $c \cdot \beta$ where $\beta = \beta(\alpha))$ equals the volume of the $(i+1)$-dimensional Euclidiean ball B with $vol_i(\partial B) = \alpha$. Furthermore, if $i = dim(X) - 1$, then the condition $curv \geq 0$ can be relaxed to $Ricci \geq 0$.

Remarks and Questions. (a) If $X = \mathbb{R}^n$ this reduces to Almgren's inequlity.

(b) If The sectional curvatures of X are bounded from below by 1, then the corresponding filling inequality generalizes that of Kleiner for S^n, where the case $i = n-1$ and $Ricci \geq n-1$ goes back to Paul Levy.

(c) Does every quasiregular Y^i in a complete simply connected manifold X with non-positive curvature (or in any $CAT(0)$ space for this matter) and with $Ricci_{i+1}(X) \leq -i$ have $\sup_{y\in reg(Y)} ||M(Y)||_y$ greater or equal than the mean curvature of a round (umbilical) sphere S^i with $vol_i(S^i) = vol_i(Y)$ in the hyperbolic space of constant curvature -1? (The lower bound on $\sup_{y\in reg(Y)} ||M(Y)||_y$ and the issuing bound the filling inequality issuing from Weyl's formula are non-sharp in the hyperbolic spaces of constant curvature $\kappa < 0$.)

(b) Can one "hybridize" Dehn's and Almgren's inequalities, e.g. for Cartesian products of manifolds of positive and of negative curvatures?

(c) If $X = X^\infty$ is an infinite dimensional Riemannian manyfold that densely and isometrically contains an increasing union of finite dimensional submanifolds, $X^\infty \supset ... \supset X^{n+N} \supset ... \supset X^n$, such that all X^{n+N}, $N = 1,2,...$, have $\|\partial_i^{-1}\|_{fil}(\beta_0) \leq \delta_0$, for some $i(<\infty)$, then, obviously, X^∞ also has $\|\partial_i^{-1}\|_{fil}(\beta_0) \leq \delta_0$. This applies, for example, to the Hilbert space $\mathbb{R}^\infty$, to the Hilbertian sphere $S^\infty \subset \mathbb{R}^{\infty+1}$ and to other infinite dimensional symmetric spaces of "compact type", where the argument depends on the N-asymptotic of the $(n+N)$-volumes of X^{n+N}.

Is there a dimension free proof applicable to more general X^∞ (e.g. to S^∞ divided by an infinite discrete isometry group)?

(d) Is there a (sufficiently) *sharp* generalization of Almgren's filling bound in $\mathbb{R}^n$ to non-Euclidean Banach-Minkowski spaces X^n in the spirit of the Brunn-Minkowski inequality (corresponding to $i = n-1$)? Are there such inequalities in the metric spheres in these X^n and other flag (e.g. Grassmannian) manifolds?

(e) Does the variational method apply to $\Delta(V)$ and similar measurable complexes, and improve the bound $\|(\partial^i)^{-1}\|_{fil} \leq 1$?

(f) Is there an algebaric/topological version of $\|...\|_{fil}^{loc}$ in the context of our chpter 4?

6 Isoperometry on Submanifolds

6.1 n-Divergence, Mean Curvature, Minimal Surfaces and Allard-Michael&Simon Inequality

???A. *The n-Divergence* $\mathsf{div}^{[n]}(\tau) = \mathsf{div}^{[n]}(\tau, V)$ of a vector field τ in a Riemannian manifold X is the rate of increase (decrease) of the volumes of n-submanifolds $V = V^n \subset X$ under the flow generated by τ.

That is, if the the-ε-initial τ-flow moves

$$V \overset{\varepsilon\tau}{\mapsto} (1+\varepsilon\tau)(V) \subset X.$$

then

$$\mathsf{div}^{[n]}(\tau, V) = \partial_\tau vol_n(V) = \lim_{\varepsilon\to 0} \frac{vol_n((1+\varepsilon\tau)(V)) - vol_n(V)}{\varepsilon},$$

where, for smooth fields τ, $\mathsf{div}^{[n]}(-\tau) = -\mathsf{div}^{[n]}(\tau, V)$ by the basic rules of the calculus.

???B. *Examples.* (a) If $n = N$, this is the *ordinary divergence*,

$$\mathsf{div}^{[\mathsf{n}]}(\tau) = \mathsf{div}(\tau),$$

(b) if $X = \mathbb{R}^N$ and τ is the *radial field* $x \mapsto \tau_x \in T_x(\mathbb{R}^N)$, $x \in \mathbb{R}^n$,[39] that is

$$\tau = \mathsf{grad}\|x\|^2,$$

then

$$\mathsf{div}^{[n]}(\tau) = n,$$

[39] Here τ_x is the tangent vector corresponding to the point (Euclidean vector) $x \in \mathbb{R}^n$ under the obvious identification $T_x(\mathbb{R}^N) = \mathbb{R}^N$.

since the multiplicatively written one-parameter group, generated by this field is the group of homotheties, $x \mapsto tx$, which expands the volumes of all n-submanifolds by the factor nt.

(c) Let X be a Riemannian manifold, where the inverse exponential map $\exp_{x_0}^{-1} : X \to T_{x_0}(X)$ is a homeomorphism onto a (star convex) domain in the tangent space $T_{x_0}(X)(=\mathbb{R}^N, N = dimX,)$,.i.e. all $x \in X$ are joint with $x_0 \in X$ by a unique geodesic segment $[x_0, x] \subset X$.

Let $\tau_0(x)$ be the field tangent to $[x_0, x]$ at x with norm $\|\tau(x)\| = length[x_0, x]$, that is τ_0 is equal to the gradient of the squared distance function to x_0,

$$\tau_0 = \mathsf{grad}\, dist(x, x_0)^2.$$

.If $sect.curv(X) \leq 0$, then

$$\mathsf{div}^{[n]}(\tau_0) \geq n$$

and if $sect.curv(X) \leq -1$ then the corresponding unit field $\tau_\bullet = \tau_0/\|\tau_0\|$ satisfies a similar inequality

$$\mathsf{div}^{[n]}(\tau_\bullet) \geq n.$$

???C. First variation of the n-Volume. Let X be a Riemannian manifold and $V \subset X$ an n-dimensional submanifold with a boundary $S = \partial V$, e.g. a curve or surface in the Euclidean space, and let $\tau : V \to T_V(X)$ be an X-tangent field along V. Then the τ-derivative of the volume of V is equal to the sum of two terms:

$$\partial_\tau vol_n(V) = \int_S \langle \tau(s), \nu(s) \rangle ds + H^*(V, \tau),$$

where:

$\bullet_S$ the field ν $\nu : S \to T_S(V) \ominus T(S)$ is the external looking unit normal field to S in V, thus, $\langle \tau(s), \nu(s) \rangle$ is the (signed) length of the normal projection of τ to V along S;

$\bullet_H$ the second term $H^*(V, \tau) = mean.curb(V)$ is the *mean curvature* of V, that is a cotangent vector filed along V,

$$mean.curv = H^* : V \to T^*_V(X),$$

which is the differential of the n-volume function(al) on the space of submanifolds in X obtained by *normal deformations*[40] of V, which is customary represented by the scalar product of tangent fields τ with a normal (also called mean curvature) field

$$H : V \to T^\perp(V) = T_V(X) \ominus T(V),$$

which represents the "normal gradient" of the volume function. Thus, the value of H^* on the tangent fields $\tau : V \to T_V(X)$ is:

$$H^*(V, \tau) = \int_V H^*(v) dv = \int_V \langle \tau, H(v) \rangle dv^{41}$$

[40]This means "along vector fields normal to V".

[41]A priori, the mean curvature is a vector valued measure on V but it is represented by a (co)vector valued function on V, i.e. a section of the bundle $T_V(X) \to V$, in the present case of a Riemannian X and smooth V and τ.

1d Example. If $V^1 \hookrightarrow \mathbb{R}^N$ is a curve parametrized by the ark length t, then the mean curvature vector $H(v)$ is equal to the second derivative of $v(t)$

$$H(t) = \frac{d^2v(t)}{dt^2}.$$

"From-1-to-n" Example. If $V_i^1 \subset V = V^n \subset \mathbb{R}^N$, $i = 1, ..., n$, are mutually normal geodesic (with respect to the Riemannian metric induced from $\mathbb{R}^N \supset V$) lines in V at $v \in V$, then

$$H(V, v) = \sum_{i=1}^{n} H(V_i^1, v).$$

"Spherical" sub-Example. The (normal) mean curvature field of the sphere $S^n(R) \subset \mathbb{R}^{n+1} \subset \mathbb{R}^N$ is contained in (tangent to) $\mathbb{R}^{n+1} \supset S^{n-1}$) and the norm of this field is everywhere

$$\|mean.curv(S^n(R), s)\| = n/R$$

"Closed" Example. Let $V \subset B^N(R) \subset \mathbb{R}^N$ be a closed (i.e. compact without boundary) n-submanifold in the R-ball. Then (the mean value of) the norm of the mean curvature of V is bounded from below by that of the sphere $S^n(R) = \partial B^{n+1}(R)$,

$$\int_V R\|mean.curv(V, v)\|dv \geq n \cdot vol_n(V).$$

In fact,

$$\partial_\tau vol_n(V) \leq \int_V \|\tau\| \cdot mean.curv(V, v)\|dv$$

for all fields τ, and the proof follows by applying this to the above radial field $\tau(x) = \mathsf{grad}\|x\|^2$.

??? D. *Example "with a Boundary".* Let $V \subset B^N(R) \subset \mathbb{R}^N$ be a compact manifold with a boundary $S = \partial V$. then

$$\int_V R\|mean.curv(V, v)\|dv + R \cdot vol_{n-1}(S) \geq n \cdot vol_n(V).$$

Proof. Argue as above with $\tau = \|x\|^2$ and observe that the $\int_S$-term in the first variation formula ($\bullet_S$ in ???C) is bounded by

$$\int_S \langle \tau(s), \nu(s) \rangle ds \leq R \cdot vol_{n-1}(S).$$

Minimality: *Definition/Exercise.* A smooth submanifold $V \subset X$ is called *minimal* if one of the following four *equivalent* conditions is satisfied.

$\bullet_\nu$ $\partial_\nu vol_n(V) = 0$ for all fields ν normal to V;

$\bullet_\tau$ $\partial_\tau vol_n(V) = 0$ for all fields τ with compact supports away from the boundary of V, $supp(\tau) \subset \smallsetminus \partial V$.

$\bullet_H$ the mean curvature of V is zero;

$\bullet_{min}$ $V$is locally volume minimizing: all points $v \in V \smallsetminus \partial V$ admit neighbouthoods, $U = U(v) \subset V$, such that all n-submanifolds $U' \subset X$ with $\partial U' = \partial U$ have larger volumes than U,

$$vol_n(U') \geq vol_n(U),$$

where this inequality is strict unless $U' = U$.

Miranda-Allard-Michael-Simon Inequality. *Let X be a Riemannian manifold and let $\delta > 0$ and $R_0 \geq 0$ be constants, such that all balls $B_x^N(r)$, $x \in X$, $r < R_0$ admit vector fields $\tau = \tau_{x,r}$, such that*

$$div^{[n]}(\tau) \leq -\delta$$

and the norms of τ on the boundaries of these balls are bounded by r. (E.g. $X = \mathbb{R}^N$,, $R_0 = \infty$, and $\delta = n$.)

Let $V = V^n$ be a smooth compact n-dimensional submanifold with a (possibly empty) boundary.

Then either $vol_n(V) \geq R_0^n/\delta^n$ or

$$\int_V \|mean.curv(V,v)\| dv + vol_{n-1}(S) \geq const_n \cdot vol_n(V)^{n-1/n}$$

for some *strictly positive const* = $const(n, \delta) > 0$.

Proof. ???

6.2 Mass Transportation: Castillon-Brendle Inequality

7 Bolzmann-Gibbs-Shannon Entropic Inequalities

7.1 Hölder Inequality via Tensorisation.

We introduce below the *Gibbs tensorisation trick* and then use it for the proof of the *Shannon inequalities* relating the entropy of a measure and its pushforards under the maps (partitions) in a given family.

Hölder Inequality. *The log of the integral*

$$\int_X \prod_{i \in I} f_i(x)^{\beta_i} dx$$

is a convex function of $\overline{\beta} = \{\beta_i\} \in \mathbb{R}^I$ for arbitrary positive functions f_i on X.

Proof. The inequality

$$log\left(\int_X \prod_{i \in I} f_i(x)^{\alpha_i\beta_i}\right) dx \leq \sum_{i \in I} \alpha_i log\left(\int f_i(x)^{\beta_i}\right) = log\prod_{i \in I}\left(\int f_i(x)^{\beta_i}\right)^{\alpha_i}$$

for $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$ is (trivially) true if the functions $f_i(x)$ are constant on the intersection $S \subset X$ of their supports (with the equality for functions with a common support $S \subset X$, where all f_i are constant) and the general case reduces to this by the law of large numbers via the tensorisation.

This argument also shows by how much the inequality deviates from equality. *Denote $\mu_i = g_i dx$ for $g_i = f_i^{\beta_i}$ and let $\mu = (\prod_{i \in I} g_i^{\alpha_i})dx$. Then*

$$log\int \prod_{i \in I} g_i^{\alpha_i} dx \leq \sum \alpha_i ent_{\mu_i}(\mu) \leq \sum \alpha_i log(\mu_i(X)) = log\prod_{i \in I}\left(\int_X g_i dx\right)^{\alpha_i}.$$

The Hölder Inequality can be equivalently stated as follows *let ν be a measure on the linear space X, and let Y be the linear dual to X. Then the function*

$$\Psi(y) = \Psi_\nu(y) = log\left(\int_X exp\langle x,y\rangle d\nu\right)$$

is convex on Y, where the entropy of a measure with the density function $exp(\varphi(x))$, $x \in \mathbb{R}$ equals the derivative $\psi'(y=1)$ for $\psi(y) = \int_{\mathbb{R}} exp\langle x,y\rangle dx$ by the Boltzman formula.

This appears in the Gibbsian thermodynamics as the *concavity of the entropy of the ideal gas* and represents a tiny instance of Boltzmann's and Gibbs' ideas (see [?]).

Remarks.(a) The information theoretic rendition of the Gibbs argument is often presented as a chat between Alice and Bob. (See [?] and references therein.)

(b) The differential $D\Psi : Y \to X$ *injectively* sends Y to X, where

the closure of the image equals the convex hull of the support of μ.

Thus, if $X = Y = \mathbb{R}^n$, then the volume of this hull equals the integral of the determinant of the Hessian of the (convex!) function Ψ, where the $\mathbb{R}_+$-valued map

$$\Psi \mapsto M(\Psi) =_{def} \int_Y det(Hess(\Psi(y)))dy$$

obeys non-trivial convexity relations: the Minkovski inequality, $M^{\frac{1}{n}}(\Psi_1+\Psi_2) \geq M^{\frac{1}{n}}(\Psi_1) + M^{\frac{1}{n}}(\Psi_2)$, and *the Alexandrov-Fenchel-Hodge inequality.* (See [?] for a survey and references).

Let (X,λ) be a Borel measure space, where λ is regarded as a *background measure* and where we use the notation $|Y| = |Y|_\lambda = \lambda(Y)$ for all $Y \subset X$.

The basic examples are given by countable spaces (X,λ) with the *unitary* measures, where all atoms have unit weights (thus, $|Y| = card(Y)$) and by the Euclidean spaces with the Lebesgue or with the Gaussian measures λ.

Consider measures $\mu = f(x)\lambda$ for (non-strictly) positive measurable functions f on X and first define the entropy of such a μ where $f(x)$ is *constant* on its (essential) support $S = supp(f) \subset X$ by

$$ent_\lambda(\mu) = log|S| = log(\mu(S)) - \mu(S)^{-1}\int_S \log(f)d\mu,$$

where, observe, $f \equiv \mu(S)/|S|$.

Then, for a general $\mu = f\lambda$, let $|\mu|_\varepsilon = \lambda_\varepsilon(\mu)$ denote the infimum of the λ-measures of the subsets $S_\varepsilon \subset X$ with $\mu(S_\varepsilon) \geq (1-\varepsilon)\mu(X)$, where we assume that μ has finite total mass, $|\mu| =_{def} |X|_\mu = \mu(X) < \infty$.

Take the Cartesian (tensorial) powers $(X^N, \lambda^N = \lambda^{\otimes N}, \mu^N = \mu^{\otimes N})$ and with $\lambda^{\otimes N}$ for the background measures on X^N. Set

$$ent(\mu^N - [\varepsilon]) = \liminf_{N\to\infty} \frac{1}{N} log|\mu^N|_\varepsilon)$$

and

$$ent(\mu) = ent_\lambda(\mu) = \lim_{\varepsilon\to 0} ent(\mu^N - [\varepsilon]).$$

Observe that, this entropy is invariant under scaling of μ, that is $ent(c\cdot\mu) = ent(\mu)$, while $ent_{c\lambda} = ent_\lambda + log(c)|\mu|$.

If μ is a probability measure with a λ-measurable density function $f = d\mu/d\lambda$ and with the support denoted $S \subset X$, then $ent_\lambda(\mu) \le log\lambda(S)$ with equality (only) for $\mu = \lambda(S)$. On the other hand, $ent_\lambda(\mu) \ge log(\sup_{x\in S} f(x)) - 1$.

We shall use the above definition only for *log-LLN-measures* μ, i.e. where $\mu = f\lambda$ for a λ-measurable function f, such that $log(f)$ satisfies

The Law of Large Numbers. *The $\mu^{\otimes N}$ measure of the subset $Y(\varepsilon, N) \subset X^N$ of the points $y \in X^N$, where*

$$\frac{1}{N}|log(f^{\otimes N}(y)) - \int_S log(f)d\mu| \ge \varepsilon$$

satisfies

$$(LLN) \qquad \mu^{\otimes N}(Y(\varepsilon, N)) \to 0 \text{ for } N \to \infty.$$

.

One knows that (LLN) is satisfied if and only if the function $|log(f)|$ is summable on its support S, e.g. if $|log(f)|$ is bounded on S.

If μ is not *log*-LLN, one can LLN-regularize it, e.g. by cutting away the part of the suppopt of f where $|log(f)|$ approaches infinity and then define a suitable regularized entropy with such an approximation.

Cartesian Additivity of the Entropy. Observe that LLN ensures the additivity of the entropy under the Cartesian product of measure spaces and yields the celebrated

Boltzman Formula. *All log-LLN-measures μ satisfy,*

$$ent_\lambda(\mu) = log|\mu| - |\mu|^{-1}\int_S log(f)d\mu = log|\mu| - |\mu|^{-1}\int_S flog(f)d\lambda$$

(for $|\mu|$ denoting the total mass $\mu(X) = \mu(S)$).

In other words,

the μ-average of $log(f)$ plus $ent_\lambda(\mu)$ equals the log of the total mass of μ.

In particular, the entropy of a *probability* measure μ is expressed by the Boltzman integral,

$$ent(\mu) = \int_S log\frac{1}{f}d\mu = \int_S flog\frac{1}{f}d\lambda.$$

This formula is customary taken for the definition of the entropy without assuming LLN, but only the convergence of the Boltzman integral, possibly to $\pm\infty$. This definition is equivalent to the above "regularized entropy" but in all our applications we can (and do) assume that μ is *log*-LLN.

7.2 Entropic Profiles and Stable $E_\circ$ Functions of Families of Partitions.

Given a finite mass measure μ on a Borel measure space $X = (X, \lambda)$ with a family of partitions P_i, we denote by $\mu_i = \mu/P_i$ = the pushforward of μ to $X_i = X/P_i$, call this the *P_i-reduction of* μ, and write

$$ent(\mu/P_i) = ent(\mu_i) = ent_{\lambda_i}(\mu_i)$$

for the background measures λ_i in X_i.

For example, if μ equals the restriction of the background measure λ on X to a subset $Y \subset X$, then the value of the density function of μ_i with respect to the background measure λ_i on X/P_i at each point $x_i \in X/P_i$ equals the Fubini mass of the corresponding P_i-slice of Y.

Denote by μ_{x_i} $x_i \in X_i = X/P_i$ the measure $fdP(x)$ on the slice $P^{-1}(x_i)$ for the background Fubini measure $dP(x)$ on this slice and $f = d\mu/d\lambda$ and let ent_{x_i} be the entropy of μ_{x_i} with respect to $dP(x)$ on this slice. Define *the entropy of* (X,μ) *over* X_i, also denoted $ent(P_i)$ as the average

$$ent(P_i) = \mu(X)^{-1} \int_{X_i} ent_{x_i} d\mu_i.$$

It is obvious (but significant) that

Entropy is additive.

ent+ $$ent(P) + ent(\mu/P) = ent(\mu).$$

Finite Example. Let P be a partition of X, a finite set with the unitary atoms and take a subset $Y \subset X$. Denote by $|P(y)|$ the cardinality of the P-slice of Y through $y \in Y$, and observe with the Boltzman (and Shannon in the finite case) formula that

$$ent(P|Y) = log \prod_{y \in Y} |P(y)|^{\frac{|P(y)|}{|Y|}}.$$

Entropic Profile. Consider a family $\mathcal{P}$ of partions P_i, $i \in I$, of X, where we usually assume that the single slice partition, corresponding to the map of X to a single point, is among our P. Every LLN measure μ on X defines the point $e(\mu) = \{ent(P_i)\} \in \mathbb{R}^I$; the set $ENT(\mathcal{P})$ of these points for all μ is called *the entropic profile of* $\mathcal{P}$. In what follows we shall evaluate the *conical convex hull* of $ENT(\mathcal{P}) \subset \mathbb{R}^I$ in the simple cases.

The definition of the entropy and the slice removal lemma from 4.4 imply the following

Sliced Tensorisation Lemma. *Given a finite family* $\mathcal{P}$ *of partitions* P_i, $i \in I$, *of* X *and an LLN measure* μ *on* X, *there exists, for every* $\varepsilon > 0$, *an integer* $N_0 = N_0(\varepsilon,\mu,\mathcal{P})$ *and a subset* $Y = Y_N$ *in the Cartesian power* X^N, *for every* $N \geq N_0$, *such that*

$$Y \subset supp(\mu^{\otimes N}), \ \ where \ \mu^{\otimes N}(Y) \geq (1-\varepsilon)\mu^{\otimes N}(X),$$

and the Fubini measures $\phi_i^N = \lambda^{\otimes N}/\lambda_i^{\otimes N}$ *of the* P_i^N*-slices of* Y *satisfy,*

$$N(ent(P_i^N) - \varepsilon) \leq log(\phi_i^N(P_i^N(y) \cap Y)) \leq N(ent(P_i^N) + \varepsilon)$$

for all $y \in Y$ *and all* $i \in I$.

Next, observe that the $E_\circ$-functions of Cartesian powers of partitions P_i of X, satisfy,

$$E_\circ(n_i^{N_1}; P_i^{N_1}) \cdot E_\circ(n_i^{N_2}; P_i^{N_2}) \geq E_\circ(n_i^{N_1+N_2}; P_i^{N_1+N_2})$$

and define

$$E_\infty(n_i; P_i) = \lim_{N\to\infty} (E_\circ(n_i^N; P_i^N))^{\frac{1}{N}}.$$

The above lemma implies the following

Shannon E_∞-Inequality. *Let $\mathcal{P} = \{P_i\}$ be a finite family of partitions on X and μ a measure of finite mass on $X = (X, \lambda)$. Then the entropies $ent(P_i) = ent(\mu) - ent(\mu/P_i)$ of P_i with respect to μ satisfy,*

$$ent(\mu) \geq log E_\infty(exp(ent(P_i)); \mathcal{P}).$$

Remark on Hölder. The tensorisation lemma also implies the Hölder version of the above inequality.

Let $f_i \geq 0$ be measurable functions on $X_i = X/P_i$, let

$$|f_i|_p = \left(\int_{supp(f_i)} f_i^p\right)^{\frac{1}{p}}$$

and let $|\Pi_{\mathcal{P}} f_i|_1$ denote the integral of the product of the pullbacks of f_i to X. Then

$$|\Pi_{\mathcal{P}} f_i|_1 \geq E_\infty(|\Pi_{\mathcal{P}} f_i|_1/|f_i|_{p_i}; \mathcal{P})$$

for all $\{p_i\} \in \mathbb{R}_+^I$.

If all P_i are single slice partitions, this reduces to the Hölder inequality from 5.1 with positive p_i (and with no entropic correction term).

7.3 Shannon and Harper Inequalities for the Coordinate Line and Plane Partitions.

Let $(X, \lambda) = \times_i(X_i, \lambda_i)$, $i = 1, 2, ..., k$. Then the partitions P_i of X into the "coordinate lines" with the slices isomorphic to X_i and corresponding to the projections $P_i : X \to X_{\hat{i}} = (X_{\hat{i}}, \lambda_{\hat{i}}) = \times_{j \in I \smallsetminus \{i\}}(X_j, \lambda_j)$ satisfy

$$Sh_1 \qquad ent(\mu) \geq \sum_i ent(P_i),$$

or, equivalently,

$$ent(\mu) \leq \frac{1}{k-1} \sum_i ent(\mu/P_i)$$

for all measures μ on X. Furthermore, the partitions P_J of X into the fibers of the projections $X = X_I \to X_{I \smallsetminus J} = \times_{i \in I \smallsetminus J} X_i$ (with "J-plane" slices representing $X_J = \times_{i \in J} X_i$) satisfy

$$Sh_\alpha \qquad ent(\mu) \geq \sum_{J \subset I} \alpha_J \cdot ent(P_J)$$

for all partitions of unity α_J of I (see 4.3).

Proof. Here, obviously, $E_\infty = E_\circ$ and the above applies.

Loomis-Whitney Inequality. This is an upper bound on $|Y| = \lambda(Y)$ for subsets $Y \subset X$ in terms of the background measures of Y/P_i, (assuming these are measurable) written as if it were a lower bound,

$$|Y| \geq \prod_i |Y| (\lambda_{\hat{i}}(Y/P_i))^{-1}.$$

This follows from the Shannon Inequality, since $ent_{\lambda_{\hat{i}}}(\mu_i)) \leq \lambda_{\hat{i}}(supp(\mu_i))$ and $ent(\mu) = log(\lambda(Y))$ for $\mu = \lambda|Y$.

Similarly one derives the *Shearer Inequality* that is the bound on $log|Y|$ by $log|Y| - log(\lambda_{\hat{J}}(Y/P_J)$ substituting $ent(P_J)$ in Sh_α. (The role of the entropy in such inequalities was pointed out to me by Noga Alon.)

If X_i are countable sets with the atoms of unit weights, then the Shannon inequality for subsets $Y \subset X = \times_i X_i$ with the restricted product unitary measures reads,

Combinatorial Shannon Inequality for the Coordinate Line Partitions. *Let $|P_i(y)|$, $y \in Y$, denote the cardinality of the P_i-slice of Y through y. Then the geometric means*

$$|MP_i| = \left(\prod_{y \in Y} |P_i(y)|\right)^{\frac{1}{|Y|}}$$

satisfy

$$\prod_i |MP_i| \leq |Y|.$$

Harper Inequality. The Shannon inequality, when applied to the vertex set X of the edge graph of a Euclidean n-cube with the edges for slices, says that

the vertex and the edge numbers of every subgraph Y in the cubical graph satisfy,

$$(4^N) \qquad N_{vert} \geq 4^{N_{edg}/N_{vert}}.$$

For example, if all vertices in Y have the valency (degree) at least d, then $|Y| \geq 2^d$.

Another corollary of the combinatorial Shannon inequality is the following (well known) relation between the three numbers: the cardinality $|Y|$, the number N of the slices of Y with respect to all P_i and the sum C of the cardinalities of all these slices.

A^B-Inequality. *Let $A = C/N$ and $B = C/|Y|$. Then*

$$|Y| \geq A^B.$$

Proof. Since the function s^s is log-convex, $log(s^s)'' = 1/s$,

$$A^B \leq \prod_S |S|^{\frac{|S|}{|Y|}} \leq |Y|,$$

where the product is taken over all slices S of the partitions P_i.

7.4 Strict Concavity of the Entropy and Refined Shannon Inequalities.

A probability measure μ on $X_1 \times X_2$ can be regarded as a family of probability measures μ_{x_1} on X_2 parametrized by $x_1 \in X_1$, where the density $f_{x_1}(x_2)$ of (almost) every measure μ_{x_1} on X_2 equals the restriction of the density of μ to $x_1 \times X_2 \subset X_1 \times X_2$ divided by $p(x_1) = \int_{X_2} f_{x_1}(x_2) d\lambda_2$.

The Shannon inequality written as $ent(\mu/P_2) \geq ent(P_1)(= ent(\mu) - ent(\ \mu/P_1))$ says that the entropy is a *concave* function on the space of probability measures on X_2, since the measure μ_2 on X_2, that is the pushforward of μ, equals the $p(x_1)$-weighted convex combination of the probability measures μ_{x_1}, while the entropy is (defined as) the corresponding convex combination of the entropies of μ_{x_1}.

In fact, the entropy is *strictly* concave as follows from the Boltzmann formula and the strict convexity of the function $t \cdot log(t)$. (This is the common way for deriving the Shannon inequality). Then the quantity $ent(\mu) - ent(P_1) - ent(P_2) \geq 0$ tells us how far μ is from *equilibrium*, i.e. a probability measure μ' on $X_1 \times X_2$, for which the probability measures μ'_{x_1} on X_2 are mutually equal for all $x_1 \in X_1$, or equivalently all μ'_{x_2} on X_1 are equal.

Here is another characteristic of (non-)equilibrium for measures μ on product spaces $X = \times_i X_i$, $i \in I$.

The index set $I \sqcup I$, (disjoint union of I with itself) and, hence, the Cartesian power X^2 of X, is naturally acted by the *Mendelian recombination group* $\mathbb{Z}_2^I = (\mathbb{Z}/2\mathbb{Z})^I$ generated by $|I|$ coordinate involutions on $I \sqcup I$ and/or on $X_i \times X_i$ for all $i \in I$. By strict convexity, a measure μ on X is at equilibrium, where (by definition if you wish) all Shannon inequalities Sh_α becomes equalities, if and only if the measure $\mu^{\otimes 2}$ on X^2 is invariant under $\mathbb{Z}_2^I$ and (where, observe, $\mu^{\otimes 2}$ is invariant under the diagonal involution on X^2 for all μ on X.)

We introduce *the entropic displacement* of $\mu^{\otimes 2}$ by z,

$$|\mu^{\otimes 2} - z(\mu^{\otimes 2})|_{ent} =_{def} ent(\frac{1}{2}(\mu^{\otimes 2} + z(\mu^{\otimes 2})) - \frac{1}{2}(ent(\mu^{\otimes 2}) + ent(z(\mu^{\otimes 2}))) \geq 0$$

and then identify involutions $z \in \mathbb{Z}_2^I$ with subsets $J \subset I$ by

$$z \leftrightarrow J = J(z) = supp(z) \subset I$$

where the *support* of z is defined by $z(i) \neq i$.

The composition of involutions corresponds to the symmetric difference of subsets that we denote $J_1 \cdot J_2 =_{def} (J_1 \cup J_2) \smallsetminus (J_1 \cap J_2)$. We also abbreviate by writing

$$|J|_{ent}(\mu) = |\mu^{\otimes 2} - z(J)(\mu^{\otimes 2})|_{ent},$$

where, observe, $|J|_{ent}(\mu) = |J^\perp|_{ent}(\mu)$, for $J^\perp = I \smallsetminus J$.

A measure μ on X satisfies the *equality*

$$ent(P_J) + ent(P_{J^\perp}) = ent(\mu)$$

if and only if $\mu^{\otimes 2}$ is $z(J)$- (or,equivalently $z(J^\perp)$)-invariant; this is also equivalent to

$$|J|_{ent}(\mu) = 0.$$

Since the entropy is strictly concave, the function $|J|_{ent}(\mu)$ of $J \subset I$ satisfies some *triangle-type inequalities*,

$$(\Delta) \qquad |J_1 \cdot J_2|_{ent}(\mu) \le \Delta_\mu(|J_1|_{ent}(\mu), |J_2|_{ent}(\mu)),$$

where $\Delta_\mu(0,0) = 0$ for all μ and $\Delta_\mu(a,b)$ is uniformly continuous in (a,b) with the *modulus of continuity* δ *depending on* μ. Moreover, δ is uniformly bounded on certain (compact in a suitable sense) classes of measures μ.

For example, if the density function f of μ satisfies

$$\int_X |log(f(x))| d\mu \le const < \infty,$$

then δ is bounded by some universal δ_{const} as a simple continuity argument shows.

This is useful, for instance, if $log(f(x)) \le 0$, e.g. if X is a discrete space with unitary atoms, where (Δ) becomes a relation between the entropies of P_J depending *only* on $ent(\mu)$,

$$(\Delta) \qquad |J_1 \cdot J_2|_{ent}(\mu) \le \Delta_{ent(\mu)}(|J_1|_{ent}(\mu), |J_2|_{ent}(\mu)),$$

for some function $\Delta_e(a,b)$ that is continuous in a, b and e and such that $\Delta_e(0,0) = 0$.

Remarks. (a) All this is, apparently, well known but I could not find a reference; nor do I know a specific sufficiently "elegant" $\Delta_e(a,b)$. I guess, there are sharp "mixed symmetric mean inequalities" for measures on $\times X_i$ similar to the classical Muirhead's inequalities, such as the mixed discriminant inequality of Alexandrov (that is $GL(k)$- rather than just S_k-symmetric).

(b) The above generalizes to the Cartesian powers X^N with the Cartesian products of I-copies of the permutation group S_N acting on it. The resulting inequalities become, in a sense, asymptotically sharp for $N \to \infty$ due to the law of large numbers (applied to convolution of measures on the spaces of measures).

A possible framework for this is suggested by the *Mendelian dynamics*

7.5 Equipartitions, Tensorization and the Hölder-Looms-Whitney-Shearer Inequality

`https://web.mit.edu/paigeb/www/994paper.pdf`

Recall the *classical Hölder inequality*

$$\int_X \prod_{i\in I} f_i(x)^{p_i\beta_i} \le \prod_{i\in I}\left(\int_X f(x)^{p_i}\right)^{\beta_i}, \beta_i \ge 0, \sum_{i\in I}\beta_i = 1,$$

which takes more familiar form for $\beta_i = 1/p_i$.

Exercise. Show that the Hölder inequality implies *log-convexity* of the function

$$\mathsf{H}(p_i) = \mathsf{H}_{\{f_i\}}(p_i) = \int_X f_i(x)^{p_i}$$

in $(p_i) \in \mathbb{R}^I$ for all positive measurable functions $f_i(x)$ on a measure space X, that is *convexity of* $\log \mathsf{H}(p_i)$.

The *Hölder-Looms-Whitney-Shearer Inequality* refines classical Hölder as follows.

Let K be a finite or countable set, let X_k, $k \in K$, be measure spaces, e.g. $X_k = \mathbb{R}$ or $X_k = \{0,1\}$, let $J_i \subset K$, be subsets indexed by a finite or countable set $I \ni i$ and let $\beta_i \geq 0$ satisfy the following *partition of unity* condition:

$$\sum_{i \in I} \beta_i 1_{J_i}(k) = 1,$$

where $1_{J_i}(k)$ are the characteristic functions of the subsets $J_k \subset K$

For instance, if all $J_i = K$, this becomes $\sum_{i \in I} \beta_i = 1$.

Let $Y = \times_{k \in K} X_k$ and let $f_i(y)$, $y = (x_k) \in X$ be positive measurable functions, such that f_i depends only on the variables x_k for $k \in J_i$.

In other words f_i is equal to the pullback of a function on $\times_{k \in J_i} X_k$ under the projection

$$Y = \bigtimes_{k \in K} X_k \to \bigtimes_{k \in J_i} X_k.$$

Then

$$[HLWS] \qquad \int_Y \prod_{i \in I} f_i^{p_i \beta_i} \leq \prod_{i \in I} \left(\int_Y f^{p_i} \right)^{\beta_i},$$

Proof. Observe that the integral $\int_Y a(y)$ is multiplicative under product of measure spaces

$$\int_{Y \times Z} a(y) b(z) = \int_Y a(y) \int_Z b(z)$$

Therefore [HLWS] for the functions $f_i(y)$ is equivalent to this inequality for the corresponding product functions

$$f_{i,N}(y_1, ..., y_N) = f_i(y_1) \times ... \times f_i(y_N)$$

and/or their geometric means $\sqrt[N]{f_{i,N}}$ on the Nth power spaces

$$Y^N = \underbrace{Y \times Y \times ... \times Y}_{N}.$$

Next, let X_k be finite sets with atoms of unit weight, observe that the general case of [GLWS] reduces to that by an obvious approximation argument. [42] and also observe. that if a function $f(y) > 0$ satisfies $\int_Y f(y) = \sum_{y \in Y} f(y) = 1$, then, *the law of large numbers* for the sums of independent random variables

$$\sum_{j=1}^{N} \log f(y_j)$$

on the power probability spaces $(Y, f(y))^N$,yields the following.

Bernoulli Approximation Theorem. *There exist* (automatically measurable under our assumptions) *subsets $V_N \subset Y^N$, such that some constant multiples of the characteristic functions of these subsets are* asymptotically equivalent *to functions f_N, according to the following definition.*

Definition. Two sequences of probability measures ϕ_N and ψ_N defined by positive functions $\phi_N(y)$ and $\psi_N(y)$ on finite sets Y_N are *asymptotically*

[42]This is unnecessary, if you are comfortable with the abstract measure theoretical terminology.

equivalent if there exist subsets $Y'_N \subset Y_N$, such that both functions $\phi_N(y)$ and $\psi_N(y)$ are strictly positive on Y'_N and

- $\phi_N(Y'_N) \to \phi_N(Y_N) = 1$ and $\psi_N(Y'_N) \to \psi_N(Y_N) = 1$ for $N \to \infty$;
- $\sup_{y \in Y'} \frac{\log|\phi(y)/\psi(y)|}{\log \mathsf{card}(Y'_N)} \to 0$ for $N \to \infty$.

CONCLUSION. [HLWS] *reduces to* [LWS], *that is where the functions f_i are characteristic functions of measurable subsets $V_i \subset Y$.*

This settles the problem for $card(K) \leq 1$ (the classical Hölder inequality) and $card(K) = 2$, where [LWS] is obvious but if $Y = \times_{k \in K} X_k$ and $card(K) \geq 3$, which, geometrically, is the most interesting case, one needs to use the law of large numbers for the second time as follows.

Lemma. Let $X_1 = (X_1, dx_1)$ and $X_2 = (X_2, dx_2)$ be measure spaces, let $V \subset X_1 \times X_2$ be a measurable subset with measure one and let χ_N be the characteristic functions of power subsets $V^N \subset X_1^N \times X_2^N$.. Then there exists subsets $V_{1,N} \subset X_1^N$ and $V_{2,N} \subset X_2^N$, such that the characteristic functions of the products

$$V_{1,N} \times V_{2,N} \subset X_1^N \times X_2^N$$

are asymptotically equivalent to χ_N.

Proof. Apply Bernoulli approximation theorem to $\phi_1 = \phi_1(y_1)$ and $\phi_2 = \phi_2(y_2)$ that are the pushforwards of the measure $\chi(x_1, x_2)dx_1dx_2$ to X_1 and to X_2 under the projections maps $X_1 \times X_2 \rightrightarrows X_1, X_2$.

Proof of [LWS]. Apply lemma to the subsets $V_i \subset Y = X_K = \times_{k \in K} X_k$ and the splittings $Y = X_{J_1} \times X_{J_2}$ for all decomposition $K = J_1 \sqcup j_2$ and thus reduce the general [LWS] to the trivial case, where all V_i are products sets, $V_i = \times_{k \in K} V_{i,k}$, $V_{i,k} \subset X_k$. Q.E.D

[LW]-Corollary (Loomis-Whitney theorem V from section 1) *The volumes of subsets $V \subset \mathbb{R}^n$ are bounded by the volumes of their n projections V_i to the coordinate hyperplanes as follows:*

$$vol_n(V) \leq \prod_{i=1}^{n} vol_{n-1}(V_i)^{1/n-1}.$$

Indeed, this is [LWS] for subsets $J_i \subset K$, which are complements to points $k \in K$.

Non-Sharp Isoperimetric Subcorollary. Let $V \subset \mathbb{R}^n$ be a domain with a smooth boundary. Then

$$vol(V) \leq \left(\frac{1}{2n} vol_{n-1}(\partial V)\right)^{n/n-1}.$$

In fact, $vol(V_i) \leq \frac{1}{2} vol_{n-1}(\partial V)$ and the arithmetic/geometric mean inequality applies.

Exercises. (a). Derive [HLWS] from the classical Hölder by induction on $card(K)$.

(b) Prove the following ***Bolloás-Thomason Box Theorem.*** Given a bounded measurable subset $V\mathbb{R}^n$, there is a rectangular parallelepiped U of the same volume as V , such that the projection of U onto any coordinate subspace is at most as large as that of the corresponding projection of V.

Apology. I couldn't find the above "Bernoulli proof" of [HLWS] in the literature and recorded it in [???] and [???]. My apologies to the person who was the first to use it.

Remark. Besides Bernoullian, there are other "equalization techniques" such as Knöte map, Brenier's solution to *Monge-Kantorovich transportation problem* in the proof of *Bracamp-Lieb refinement of the Shannon-Loomis-Whitney-Shearer inequality* (see **[?]** and references therein) and invertibility of some *Hodge operators on toric Kähler manifolds* as in the analytic rendition of Khovanski-Teissier proof of the Alexandrov-Fenhcel inequality for mixed volumes of convex sets **[?]**. It is tempting to to find "quantum counterparts" to these proofs.

Also it is desirable to find more functorial and more informative proofs of "natural" inequalities in geometric (monoidal?) categories. (See **[?]**,**[?]** for how it goes along different lines.)

7.5.1 Reverse Loomis-Whitney Inequality

Let

$$V_{i,d} \subset V \subset \mathbb{R}^n$$

be the union of straight segments I, *contained in* V,

which are *parallel to the i-th coordinate axes* in $\mathbb{R}^n$

and such that $lenght(I) = d$.

Observe that the

volume of the complement to $V_{i,d}$ *is bounded by the* $(n-1)$*-volume of the boundary of* V as follows,

[$Vol_{\smallsetminus} < d...$] $$vol(V \smallsetminus V_{i,d}) \leq d \cdot vol(\partial V).$$

Next, given numbers $d_i \geq 0$, $i = 1, ..., n$, let

$$\varepsilon_i = vol(V \smallsetminus V_{i,d_i})/vol(V)$$

and rewrite [$Vol_{\smallsetminus} < d...$] as

[$Vol < d_i/\varepsilon_i...$] $$vol(V) \leq \frac{d_i}{\varepsilon_i} vol_{n-1}(\partial V), \ i = 1, ..., n.$$

Let

$$V_{\square} = \bigcap_i V_{i,d_i} \subset V,$$

observe that

[$Vol_{\square} > ...$] $$vol(V_{\square}) \geq \left(1 - \sum_{n=1}^{n} \varepsilon_i\right) vol(V).$$

and that, by the mean value theorem, there exists an affine hyperplane $A = A^{n-1} \subset \mathbb{R}^n$ parallel to the first $n-1$ coordinate axes, such that

$$vol_{n-1}(V_{\square} \cap A) \geq \left(1 - \sum_{n=1}^{n} \varepsilon_i\right) vol_{n-1}(V \cap A).$$

Since

$$V \cap A)_{i,d_i} \supset V_{\square} \cap A \text{ i=1,...,n-1},$$

this shows that

$$vol_{n-1}(V \cap A)_{i,d_i} \geq \left(1 - \sum_{n=1}^{n} \varepsilon_i\right) vol_{n-1}(V \cap A)$$

■ *our V contains a large part of the volume, say more than one half, of a Euclidean rectangular $\times_i d_i$-solid*

$$\bigtimes_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n}, \ [a_i, b_i] \subset \mathbb{R}, \ b_i - a_i = d_i.$$

It follows, that

$$d_{i_o} = \min_i d_i \leq 2\sqrt[n]{vol(V)}$$

and $[Vol < d_i/\varepsilon_i ...]$ for $i = i_o$ shows that

$$vol(V) \leq \frac{2\sqrt[n]{vol(V)}}{\varepsilon_{i_o}} vol_{n-1}(\partial V),$$

that is the isoperimetric inequality with an albeit depending only on n, but an comfortably large constant,

$[isop : C_o]$ $$vol(V)^{n-1/n} \leq C_o vol_{n-1}(\partial V) \text{ for } C_0 = 2/\varepsilon_{i_o}.$$

Here is a justification of ■ and $[isop : C_o]$.

??? Proposition. There

Let $\square^n = \times_{i=1}^{n}[0, a_i]$ be the rectangular solid and

$V \subset \square^n$ be an open subset. and let the (topological) boundary of V in $\square^n$ is bounded by

$$vol_{n-1}(\partial V) \leq \varepsilon vol_n(V), \text{ for some } \varepsilon > 0.$$

Then V contains a product subset

$$V_\square = \bigtimes_{i=1}^{n} S_i, S_i \subset [0, a_i],$$

such that the volume of $V_\square$ is bounded from below by

$$vol_n(V_\square) \geq (1 - n\varepsilon) vol_n(V)$$

Let $\hat{V}_i \subset V$, $i = 1, ..., n$, be the union of $[0, a_i]$-segments which are fully contained in V,

$$\hat{V}_i = \{x_1, ..., x_i, .., x_n\}, \text{ such that } (x_1, ..., x'_i, .., x_n) \in V, \text{ for all } x'_i \in [0, a_i].$$

If the $(n-1)$-volume of the boundary of V in $\square^N$ is ε-small compare to the volume of V $vol_{n-1}(\partial V) \leq \varepsilon vol_n(V), \varepsilon > 0$

Second Isoperimetric Application of [LW]

Let $A, B \subset Y = \bigtimes_{k \in K} X_k$, $k \in K$, where $X_k = (X_k, \mu_k)$ are be *probability*[43] measure spaces, be measurable subsets, such that the images. of A and B under the projections

$$\pi_k : Y \to Y_k = \bigtimes_{j \in K \setminus \{k\}}$$

do not intersect Y_K for all $k \in K$.

Lemma. *If the (product probability) measure of the union of A and B in Y is bounded from below by*

$$\mu(A \cup B) \geq 1 + \delta$$

then

$$\frac{\min(\mu(A), \mu(B))}{\max((\mu(A), \mu(B))} \leq C_n \delta$$

where $C_n, \leq ???$, $n = card(K)$.

Proof.???

Corollary. *Let $U \subset [0,1]^n$ be a subset in the unit cube with $vol(U \leq \delta$. Then there exits a connected subset in the complement of U, say $V \subset [0,1]^n \setminus U$, with volume*

$$V \geq 1 - D_n \delta,$$

where D_n =??? Proof.???

Rewrite ... be disjoint as well Γ_1-disjoint subsets, i.e. there is no edge in Γ_1 between their points. Let $\mu_1^{\times}(M_>) \geq \mu_1^{\times}(M_<)$ and let $\mu_1^{\times}(M_> \cup M_<) \geq 1_\delta$ Then

$$\mu_1^{\times}(M_<) \leq C_n \delta,$$

where $C_n, \leq ???$

let the pullbacks of points by the natural maps (projections) $Y \to Y_k = \bigtimes_{j \in K \setminus \{k\}} X_j$, be called X_k-"*lines*" or just *"lines"*, and where the set of X_k-"*lines*" is naturally identified with Y_k.

Let Γ_L be be the (naturally K-colored) graph. of "lines" with the vertex set

$$L = \bigsqcup_{k \in K} Y_k, k \in K,$$

where the pairs of *intersecting* lines in Y are taken for the edges.

PreIsometric Lemma Let $K = \{1, ...n\}$, let (X_k, μ_k) be *probability*[44] measure spaces, i.e. $\mu_k(X_k) = 1$, let the product spaces Y_k be given the. corresponding

[43]This is just to simplify notation.
[44]This is just to simplify notation.

product measures, denoted $\mu_k^\times$, and let $M_k \subset Y_k$ be measurable subsets with measures

$$\mu_k^\times(M_k) > 1 - \varepsilon_k \geq 0$$

Then there exists a *connected* subgraph in Γ_L with the vertex set $M_1' \subset M_1$, where

$$\mu_1^\times(M_1') \geq 1 - A_n \sum_{k \in K} \varepsilon_k$$

for $A_n \leq ???$

Proof. Let is introduce another ($\{2, ..., n\}$-colored) graph Γ_1 now on the vertex set Y_k where two points are joint by an k-colored edge if they lie on the projection of an X_k-"line" to Y_k.

Sublemma. Let $M_>, M_< \subset Y_1$ be disjoint as well Γ_1-disjoint subsets, i.e. there is no edge in Γ_1 between their points. Let $\mu_1^\times(M_>) \geq \mu_1^\times(M_<)$ and let $\mu_1^\times(M_> \cup M_<) \geq 1_\delta$ Then

$$\mu_1^\times(M_<) \leq C_n \delta,$$

where $C_n, \leq ???$

Proof. Let

$$\pi_{1,k} : Y_1 \to Y_{1,k} \bigtimes_{j \in K \setminus 1,k} X_j, k = 2, ..., n$$

be the natural maps (projections) and let $a_k^>$ and $a_k^<$ be the measures of the images of $M_>$ and of $M_<$ under these maps.

Γ_1-disjointnes of $M_>$ and of $M_<$ says that these images are disjont and the [LW]-inequality shows that

$$\left(\sqrt[n]{\prod_{k \neq 1} a_k^>} \right)^{n/n-1} + \left(\sqrt[n]{\prod_{k \neq 1} a_k^<} \right)^{n/n-1} \geq 1 + \delta.$$

Since

$$\sqrt[n]{\prod_{k \neq 1} a_k^>} + \sqrt[n]{\prod_{k \neq 1} a_k^<} \leq 1$$

by the geometric.arithmetic mean inequality, and since $(1-t)^{n/n-1} \geq 1 - tn/n-1$, $0 \leq t \leq 1$,

$$\mu_k(M_,) \leq \left(\sqrt[n]{\prod_{k \neq 1} a_k^<} \right)^{n/n-1} \leq (n-1)\delta.$$

QED.

Let Γ be a finite edge colored graph on a set V, where the set of colors k is denoted $K \ni k$ of a finite set K, let $V_k \subset V$ be the sets of ends of k-colored edges and $V_{\hat{k}}$ be the sets of connected components of V_k. and let $\pi_{\hat{k}} : V \to V_{\hat{k}}$ be the natural (quotient) maps.

Let V and $V_{\hat{k}}$ be endowed with measures μ and $\mu_{\hat{k}}$ structures, such that the subsets $V_k subset V$ are measurable (e.g. $V_k = V$)) and the maps $\pi_{\hat{k}} : V \to V_{\hat{k}}$ are measure preserving.

Let $c^\perp = c^\perp(\Gamma$ is the infimum of the numbers $c \geq 0$ with the following property;

Given measurable subsets $U_{\hat{k}} \subset V_{\hat{k}}$ there exists a *connected* subgraph in Γ on a measurable vertex subset $U^\perp$ such that

- the $\pi_{\hat{k}}$-images of $U^\perp$ lie in the complements to $U_{\hat{k}}$,

$$\pi_{\hat{k}}(V_{\hat{k}}) \subset V_{\hat{k}} \setminus U_{\hat{k}};$$

-

$$\mu(V \setminus U^\perp) \leq c \sum_{k \in K} \mu_{\hat{k}}(U_{\hat{k}})$$

Cubical Example. Let $K = \{1, ..., n\}$, let $V = [0,1]^n$, let $p_{\hat{k}}$ be the projections of the cube to its coordinate faces $V_{\hat{k}}$,

$$p_{\hat{k}} : (x_1, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n) \mapsto (x_1, ..., x_{k-1}, x_{k+1}, ..., x_n),$$

such that the edges of the graphΓ are pair of vertices (v_1, v_2) on same coordinate lines in the cube, i.e. $p_{\hat{k}}(v_1) = p_{\hat{k}}(v_2$ for some k.

Then, given $U_k \subset V_k$, let $U^\perp$ be the union of all lines which meet through the intersection $\cap_k \pi_{\hat{k}}^{-1}(V_{hatk} \setminus U_k)$.

Thus,

$$\mu(U^\perp) \geq \mu(\bigcap_{k \in K} \pi_{\hat{k}}^{-1}(V_{\hat{k}} \setminus U_k)) \geq \sum_{k \in K} \mu(U_k)$$

and all lines in $\mu(U^\perp)$ meet other lines in the remaining $(n-1)$ directions.

The latter implies the connectednes of the subset $U^\perp$; thus, *the inequality* $c^\perp \leq 1$ *for this* Γ.

Isoperimetric Corollary. (Compare with ??? in section 3.3.) Let $Y \subset [0,1]^n$ be a hypersurface, which divides the cube into two parts, say W_1 and W_2. Then

$$\min(vol(W_1), vol(W_2)) \leq n \cdot vol_{n-1}(Y).$$

Proof. Apply the above to the projections $U_k = \pi_{\hat{k}} \subset V_{\hat{k}}$ and show that

$$\max(vol(W_1), vol(V_2)) \geq 1 - nvol_{n-1}Y.$$

7.5.2 Combinatorial Shannon and Harper Inequalities.

7.5.3 Linearized Loomis-Whitney-Shearer Inequality

Let

$$Y \subset X_K = \bigtimes_{k \in K} X_k, \ K = \{1, ..., n\},$$

be a finite set let L_k be the linear spaces of functions on X_k, let $L_0 = L_Y$ be the space of functions on Y and let

$$\Phi(l_0, l_1, ..., l_k) = \sum_{y \in Y} l_0(y) \cdot l_1(x_1) \cdot ... \cdot l_n(x_k).$$

be an $(n+1)$-linear form in the varIables l_i, $i = 0, 1, ..., n$, where every subset $J \subset K = \{1, ..., n\}$ turns this Φ into a bilinear form Φ_J between tensor product spaces, L_J and $L_{J^\perp}$,

$$L_J = \bigotimes_{i \in J} L_i \text{ and } L_{J^\perp} = L_0 \otimes (\bigotimes_{i \in K \setminus J} L_i).$$

Since $rank(\Phi_J) = card(Y_J)$, where $Y_J \subset X_J = \times_{i \in J} X_j$ is the projection of Y to X_J (check it!) the [LWS] inequality (for functions constant on their supports Y_J) says in this terms that

$$(P_\otimes) \qquad \prod_{J_i \subset K, i \in I} (rank(\Phi_{J_i}))^{\beta_i} \geq rank(Y)$$

for all partitions of unity (J_i, β_i), $i \in I$ on K.

Linearized [LWS] claims that this inequality holds true for *all* $(n+1)$-linear forms $\Phi(l_0, l_1, ..., l_N)$.

This can be reduced (this is easy) to the original combinatorial [LWS] by using a suitable basis in L_K or proven by the Bernoulli approximation argument applied to $L_K^{\otimes N}$. with $N \to \infty$ (see ???[strfucure]. [action]).

Example: The linearized Loomis-Whitney 3D-isoperimetric inequality for ranks of bilinear forms associated with a 4-linear form $\Phi = \Phi(l_0, l_1, l_2, l_3,)$ reads

$$|\Phi_{0,123}|^2 \leq |\Phi_{01,23}| \cdot |\Phi_{02,13}| \cdot |\Phi_{03,12}|$$

where $|...|$ stands for $rank(...)$.

Remark. Probably, the linear [LWS]-inequalities are the only universal relations between the ranks of Φ_J, but there are further inequalities of this type for particular polylinear forms, e.g. defined by the $\smile$-product in the cohomology algebras of certain manifolds (see [expanders, singularities]) and also in spaces of sections and (cohomologies in general) of holomorphic vector bundles such e.g. as in the Khovanski-Teissier theorem and in the Esnault-Viehweg proof of *the heneralized Dyson-Roth lemma*, but a direct link. between all such inequalities is yet to be found.

7.6 Isoperimetry in the Exterior Algebras

7.7 Strong Subadditivity of the von Neumann Quantum Entropy

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8 Fixed Points, Amenability, T-property and Isoperimetry in Groups and Algebras

von-Neumann

8.1 "Parallel" Mass Transport in Groups and Saloff-Coste bound on the Fölner-Vershik function

8.2 Kazhdan's T-Property, Margulis' Expanders, Spectral Logic, Garland theorem, High dimensional Expanders

$X \subset B^N(R) \implies Vol(X) \leq 1/nvol(\partial X)$ In fact,

$vol(X) = \int_{Y=P_\nu(X-x_0)} dy$ P_ν is the projection to the normal line to Y at y. (This integral doesn't depend on x_0.)

Average intersection of Y with the $(n-1)$ faces of an ε-cubilation of $\mathbb{R}^n$

9 Measure Concentration

9.1 Talagran Inequality

9.2 Poincare Concentration Inequalities for Mapping to Wirtinger and other Spaces

9.3 Stability of Matter

10 Waist Inequalities

11 Isoperimetry Settings and Directions of Generalizations

1. Given Euclidean vector bundles over a Riemannian manifold

$$V_0, V_1, ...V_k \to X,$$

and linear differential operators on spaces of sections $X \to V_i$.

$$D_i : C^\infty(V_0) \to C^\infty(V_i), i = 1, ..., k$$

evaluate (the size of) the set of values of the L_{p_i}-norms of these sections for given p_i,

$$\mathcal{F} = \left\{ \left(\int_X D_i f(x)^{p_i} dx \right)^{1/p_i} \right\}_{f \in C^\infty(V_0)} \subset \mathbb{R}_+^k$$

For instance, decide when $\mathcal{F} \neq \mathbb{R}_+^k$.

More generally, study possibilities for the joint distribution of $\|D_i f\|$ regarded as random variables on X.

Example

12 Isoperimetry for Families, Spectra and Morse

13 Poincare-Hahn Banach duality

14 Isoperimety Problems Inspired by Biology

14.1 Micella, Nash Blow up and and Higher Order Soap Bubbles

14.2 Viral Isoperimetry: Minimization of Information for Building the Wall around the Carrier of this Information

15 Appendices

15.1 Basics on Curvature

§2 https://link.springer.com/article/10.1007/BF02925201

§2 https://arxiv.org/pdf/1908.10612.pdf

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

15.2 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0,\varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0,\varepsilon]$.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0,\varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1,...,n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y,y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

Sign Convention. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[45]

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t{}^{46}$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t{}^{47}$$

Now comes the first formula with the Riemannian curvature in it.

15.3 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

[45] At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[46] This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

[47] This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν, τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau),\tau\rangle_g \text{ for all } \tau \in T(Y).$$

15.4 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry.[48] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape s $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [49]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu, \nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i \neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

[48] The first main formula is *Gauss' Theorema Egregium.*

[49] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $\|A^*\|^2 = \sum_i (\alpha_i^*)^2$.

15.5 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y, h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0, \varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}\ .$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[50]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y \times \mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y \times \{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

[<] $$\frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y \in Y.$$

Show that

if Y is compact, there exists a smooth positive function $\varphi(t)$, $0 \le t < \infty$, which is constant at infinity and such that the the warped product metric $g = \varphi^2 h + dt^2$ has

the same Bartnik data *as g_0, i.e.*

$$g|Y = h_0 \text{ and } mean.curv_g(Y) = mean.curv_{g_0}(Y),$$

Then show that

one *can't make* $Sc(g) \ge Sc(X')$ in general, if [<] is relaxed to the corresponding *non-strict* inequality, where an example is provided by the Bartnik data of $Y' \in X'$ itself.[51]

Vague Question. What are "simple natural" Riemannian metrics g on $X = Y \times \mathbb{R}_+$ with given Bartnik data $(Sc(Y), mean, curv(Y))$, where $Y \subset X$ is allowed *variable* mean curvature, and what are possibilities for lower bound on the scalar curvatures of such g granted $|mean.curv(Y,y)|^2/Sc(Y,y) < C$, e..g. for $C = |mean.curv(Y')|^2/Sc(Y')$ for Y' being a sphere in a space of constant curvature.

Curvature Formulas for Manifolds and Submanifolds.

[50]See §12 in [GL(complete) 1983].

[51]It follows from [Brendle-Marques(balls in S^n)N 2011] that the the cylinder $S^{n-1} \times \mathbb{R}_+$ admits a complete Riemannian metric g cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere X'_0 in the hemisphere X' in the unit n-sphere. But the non-deformation result from [Brendle-Marques(balls in S^n) 2011], suggests that this might be impossible for the Bartnik data of *small* balls in the round sphere.

15.5.1 Comparison Inequalities

15.6 Carno-Caratheodory Spaces

16 Amenability and Isoperimetry in Groups and Algebras

17 references

???See 612 Gro 1996]???

https://maa.org/sites/default/files/pdf/upload_library/22/Ford/blasjo526.pdf