Probability/Topology – Synopsis of lecture $2\frac{1}{2}$

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Borsuk Ulam 1: Topological Graph Color Number is $n+2$.

Let the n-sphere be covered by $n+1$ open subsets. $S^n = U_1 \cup ... \cup U_1 \cup ... \cup U_{n+1}$. Then *some of U_i contains a pair of opposite points s and $-s \subset S^n$.*

Borsuk Ulam 2: Homological Intersection.

Let $X_i^{m_i} \subset S^n$ be n closed $\mp$symmetric subsets such that each X_i *separates all pairs of symmetric points* in the sphere, i.e. the complements $S^n \setminus X_i$ contain no connected symmetric subsets. Then the *intersection $\bigcap_1^n X_i$ is non-empty.*

Borsuk Ulam 3: Onto Theorem. Continuous $f : S^n \to S^n$, such that $f(x) \neq f(-x)$, e.g. $f(-x) = -f(x)$, are onto.

Generlization to Pseudomanifolds. By, definition, the only 1-dimensional pseudomanifolds are disjoint union of lines ($\mathbb{R}$(and circles.

The n-dimensional (topological) pseudomanifolds are the spaces, which are locally homeomorphic to cones over compact $(n-1)$-dimensional pseudomanifolds/[1]

If X is an n-dimensional pseudomanifold, then Continuous maps $f : S^n \to X$, such that $f(x) \neq f(-x)$ for all $x \in X$, then f is onto.

(Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry, Jiri Matousek)

(The Borsuk-Ulam Theorem and Bisection of Necklaces Alon-West)

Every interval n-coloring has a bisection of size at most n.

That is, given continuous functions $f_1(t), ..., f_j(t), , ..f_n(t)$, $t \in [0,1]$, there exist n points $t_1 < ... < t_i < .. < t_n \in [0,1]$ of and a partition of the set of the $n+1$

[1] A cone over X is the cylinder $X \times [0,1]$ with the base $X \times 0$ collapsed to a point.

segments $S_0 = [0, t_1], S_1 = [t_1, t_2], ..., S_n = [t_n, 1]$ into two subsets , say I_+ and I_-, such that the integrals of the functions f_j over the unions of these intervals, called

$$S_+ = \bigcup_{i \in I_+} S_i \text{ and } S_- = \bigcup_{i \in I_-} S_i$$

satisfy

$$\int_{S_+} f_j(t)dt = \int_{S_-} f_j(t)dt, j = 1, ..., n.$$

Proof: Let $\mathcal{M} = \mathcal{M}_{\mathbb{R}}$ be the (real) *moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ defined by $\sum x_i^2 = 1$ to the simplex $\Delta^n \subset \mathbb{R}^{n+1}$ defined by $\sum x_i = 1, x_i \geq 0$,,

$$\mathcal{M} : (x_0, x_1, ..., x_n) \mapsto (x_0^2, x_1^2, ..., x_n^2).$$

and

$$\overline{\mathcal{M}}_i : S^n \to [0, 1], i = 1, ..., m, \text{ for } \overline{\mathcal{M}}_i : (x_0, x_1, ..., x_n) \mapsto \sum_{j=0}^{i-1} x_j^2.$$

Thus, each $s = (x_0, ..., x_n) \subset S^n$ defines a partition of $[0, 1]$ into subsets $S_+(s)$ and $S_-(s)$ by the rule: a point $t \in [0, 1]$ from the interval $\underline{\mathcal{M}}_{i-1}(s) \leq t \leq \underline{\mathcal{M}}_i(s)$ is in S_+ if $x_i \geq 0$ and it is in S_- for $x_i \leq 0$.

Since the functions $F_{j\mp}(s) = \int_{S_\mp(s)} f_j(t)$ are *continuous* (check it!), the Borsuk-Ulam 3 applies to the map $F = (F_1, ..., F_n) : S^n \to \mathbb{R}^n$ and since this map can't be onto (the sphere is compact and the Euclidean space is non-compact), there exist two opposite points in the sphere,

$$s_+ = (x_{0+}, ..., x_{n+}) \text{ and } s_- = (x_{0-}, ..., -x_{n-}) = (-x_{0+}, ..., -x_{n+}) \in S^n,$$

such that

$$\int_{S_+(s)} f_j(t) = \int_{S_-(s)} f_j(t) = 1, ..., n.$$

QED.

Remarks. (a) The the only feature of the real moment map needed for the proof is that it is invariant under the reflection group $\mathbb{Z}_2^n$ acting on the sphere and $\mathcal{M}$ is a homeomorphism of $S^n/\mathbb{Z}_2^n$ with the simplex Δ^n.

In fact, the sphere is equal to *the universal orbifold covering* of the simplex.

(b) Besides topology, the moment map also establishes a probabilistic link between the simplex and the sphere:

Fisher Theorem. *The Riemannin metric on $\Delta^n \subset \mathbb{R}^{n+1}$ defined by the minus Hessian of the entropy,*

$$g_{ij} = \partial_{ij}^2 \sum_{i=0}^{n} x_i \log x_i$$

is equal, up to a scaling, to the spherical metric (of constnt curvature) on the positive quadrant in S^n (check it!).