Probability/Topology – Synopsis of lecture $2\frac{1}{2}$

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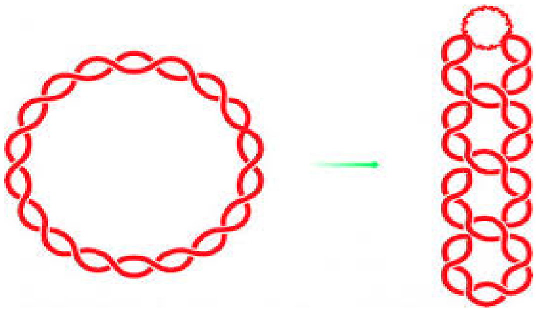
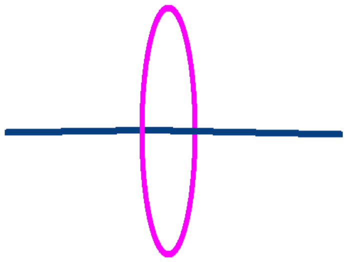


Figure 1: Bacterial DNA: Linking and supercoiling .

It is obvious one can't unlink links, e.g. unlink the "vertical" m-sphere S^m from the infinite "hoizontal" $(n-1)$-plane in the $(m+n)$-space $\mathbb{R}^{m+n} = \mathbb{R}^{m+`1} \times \mathbb{R}^{n-1}$ for "vertical" sphere $S^m \subset \mathbb{R}^{m+`1}$ and the "horizontal" $\mathbb{R}^{n-1} \subset \mathbb{R}^{(m+1)+(n-1)}$.

This obvious "unlinking property"



for $n-1=0$ (almost obviously) implies non-obvious

Browder's fixed point theorem.

every continuous map f for the unit $(m+1)$-ball $B^{m+1} \supset S^m$ to itself has a fixed point.

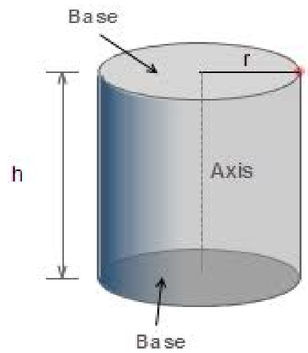
$$f(x) = x.$$

TOPOLOGY OF THE n-CUBE. $\square^n = [-1,1]^n$. (Cube also is a "probabilistic" object: *the law of large numbers, the Shannon inequality.*)

1. If a continuous map n-cube $\xrightarrow{f}$ n-cube, $f = \{f_i(x_j)\}$, $-1 \leq x_j \leq 1$, $i, j. = 1, ..., n$,

sends each $(n-1)$-face $\partial_{i\pm} \subset \partial\square^n$ *to itself*, $f_i : (.... \pm 1_i ....) \mapsto \pm 1$, then f is *onto*, e.g. the equation $f(x) = 0$ *has a solution.*

2. If $X_i \subset \square^n$, $i = 1, ..., n$, *separate* the pairs of the opposite $(n-1)$-faces $\partial_{i\pm} \subset \partial\square^n$, then the intersection $\bigcap_i X_i$ is *non-empty.*



3. Generalization, to $X_i^{n-m_i} \subset \square^n = \square^{m_1} \times ... \times \square^{m_i} \times ... \times \square^{m_k}$, $\sum_i m_i = n$, where "separate" re-

placed by "linked with" in the same way as the axis of the 3d cylinder is linked with its side boundary,

3 If a continuous maps $f : \mathbb{R}^n \to \mathbb{R}^n$, satisfy $||f(x), x|| < ||x||$ for $||x|| = 1$, then the equation $f(x) = 0$ for has a solution.

(Averaging set. A generalization of mean values and spherical designs, Seymour-Zaslavsky.)

Given finitely many continuous functions f_i, $i \in I$, on the unit interval, there exists a *finite* subset $S \subset [0,1]$, such that

$\frac{1}{card(S)} \sum_{s \in S} f_i(s) = \int_0^1 f_i(t)dt$, (Caratheodory theorem about convex hulls.)

Borsuk Ulam :

1. *Topological ("Graph")Coloring*

Theorem,

Let the n-sphere be covered by $n+1$ open subsets. $S^n = U_1 \cup \ldots \cup U_1 \cup \ldots \cup U_{n+1}$. Then some U_i contains a pair of opposite points s and $-s$.

2. *Homological Intersection Theorem,*

Let $X_i^{m_i} \subset \mathbb{R}^n = S^n/\mathbb{Z}_2 = \mathbb{R}^{n+1}/\mathbb{R}_+^\times$ be smooth (piecewise smooth) submanifolds such that $\sum_i m_i =$. If a "generic" $(n - m_i)$-plane intersects X_i at *odd number of points*, then the intersection $\bigcap_i X_i$ is *non-empty/.*

3. Onto Theorem, Continuous $f : S^n \to S^n$*, such that* $f(x_{\neq} f(-x)$*, e.g.* $f(-x) = -f(x)$ *are onto.*

*(*Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in

Combinatorics and Geometry, Jiri Matousek*)*

(The Borsuk-Ulam Theorem and Bisection of Necklaces Alon-West)

Every interval k-coloring has a bisection of size at most k. (Real moment map from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the simplex $\Delta^n \subset \mathbb{R}^{n+1}$ for $\mathcal{M}_{\mathbb{R}} : (x_0, x_1, ..., x_n) \mapsto (x_0^2, x_1^2, ..., x_n^2)$.)

Serre Finiteness theorem. The space of proper maps $\mathbb{R}^{m+N} \to \mathbb{R}^{m+N}$ has finitely many components except for two:

$N = 0$

m is even and $N = 2k - 1$.

Category of Smooth Manifolds

An X is a smooth n-manifold, if it is is "locally indistinguishable" from $\mathbb{R}^n$ in-so-far as C^∞-differentiable maps between domains in the Euclidean spaces are concerned.

To make sense of this, let $\mathcal{F}$ be a localizable and composable class of maps between open subsets in Euclidean spaces $f : \mathbb{R}^n \supset U_1 \to U_2 \subset \mathbb{R}^m$ $f = (f_1, ..., f_m)$ which contains the identity maps.

An $\mathcal{F}$-*structure* on a topological, e.g. metric space X is a localizable and composable class of maps between open subsets from X and these in the Euclidean spaces. (Such an X is a new "book" added to the "library" $\mathcal{U}$, which is written in the same words as $\mathcal{U}$ and follows the same $\mathcal{F}$-grammar.)

Then maps between two such $\mathcal{F}$-spaces $X_1 \to X_2$ are $\mathcal{F}$ if for al $U_1 \to X$ and $Y \to U_2$, $U_i \subset \mathbb{R}^{n_i}$, the composed maps $U_1 \to U_2$ for $U_1 \to X_1 \to X_2 \to U_2$ are from $\mathcal{F}$. A bijective correspondence $X_1 \leftrightarrow X_2$ is a $\mathcal{F}$-*isomorphism* if it is $mathcalF$ in both directions.

Examples. Subspaces and factor spaces.

If $\mathcal{F}$ is the class of C^∞-differentiable maps, then an $\mathcal{F}$-space is a C^∞-manifold if it is locally isomorphic to some $\mathbb{R}^n$. (Smooth manifolds added to the Euclidean library are like imaginable books written in English words and sentences used by 3-years old children of American mathematicians.)

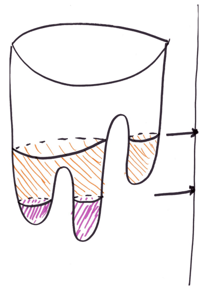
Examples. Submanifolds in Euclidean spaces, e.g. the space of orthornomal n-frames $Fr_{O(n)}\mathbb{R}^N \subset \mathbb{R}^{Nn}$ Homogeneous quotient spaces, such as the Grassmannian $Gr_N(\mathbb{R}^{n+N})$.

locally(!) given by (generic) equations.

All closed smooth n-manifolds X are pullbacks of the Grassmannians $X_0 = Gr_N(\mathbb{R}^{n+N})$ in the canonical vector bundle $V \supset X_0$ of rank N under generic smooth proper maps $\mathbb{R}^{n+N} \supset U \to V$ (or from $S^{n+N} = \mathbb{R}^{n+N}_\bullet \to V_\bullet$.) If $N = 1$ these are levels sets of generic points of smooth (proper) functions $\mathbb{R}^{n+1} \to \mathbb{R}$.

Examples. Submanifolds in Euclidean spaces locally(!) given by (generic) equations.

Homogeneous quotient spaces



What are non=generic manifolds? (Simons-Federer theorem)

Figure 2: Zermelo Choice Problem

Two solutions :

I. 1/2 probabiliy

II Mëbius Strip

Triangulated spaces.

Algorithms and Counting: (How many triangulation spheres have?)

Almost definition: Homology classes $[C] \in H_i(X)$, classes of "compact oriented i-submanifolds $C \subset X$ with singularities of *codimension two*".

But:

closed self-intersecting curves in surfaces, and/or the double covering map $S^1 \to S^1$.

$C \subset X$ may have singularities of codimension one, and, besides orientation, a locally constant integer valued function on the non-singular locus of C.

dimension on closed subsets in smooth manifolds: of monotonicity, locality and max-additivity, i.e. $dim(A \cup B) = \max(dim(A), dim(B))$. is monotone decresaig under *generic* smooth maps of compact subsets A, i.e. $dim(f(A)) \leq dim(A)$ and if $f : X^{m+n} \to Y^n$ is a *generic* map, then $f^{-1}(A) \leq dim(A) + m$.

"generic dimension" is the *minimal* function with these properties which coincides with the ordinary dimension on smooth compact submanifolds. no problems if we do not take limits of maps.

[smooth generic & piecewise linear$\rightsquigarrow$ generic piecewise smooth $\rightsquigarrow$ stratawise smooth]

An i-cycle $C \subset X$ is a closed subset in X of dimension i with a $\mathbb{Z}$-*multiplicity function* on C with the following set decomposition of C.

$$C = C_{reg} \cup C_{\times} \cup C_{sing},$$

such that

• C_{sing} is a closed subset of dimension$\leq$ $i - 2$.

• C_{reg} is an open and dense subset in C and it is a smooth i-submanifold in X.

$C_\times \cup C_{sing}$ is a closed subset of dimension $\leq i-1$. Locally, at every point, $x \in C_\times$ the union $C_{reg} \cup C_\times$ is diffeomorphic to a collection of smooth copies of $\mathbb{R}^i_+$ in X, called *branches*, meeting along their $\mathbb{R}^{i-1}$-boundaries where the basic example is the union of hypersurfaces in general position.

• The *$\mathbb{Z}$-multiplicity structure*, is given by an orientation of C_{reg} and a locally constant *multiplicity/weight* $\mathbb{Z}$-function on C_{reg}, (where for $i = 0$ there is only this function and no orientation) such that the sum of these oriented multiplicities over the branches of C at each point $x \in C_\times$ equals zero.

Every C can be modified to C' with

empty $C'_\times$ and if $codim(C) \geq 1$, i.e. $dim(X) > dim(C)$, also with weights = ± 1.

Double circle $2S^1$ can be separated in two ways.

If $2l$ oriented branches of C_{reg} with multiplicities 1 meet at $C_\times$, divide them into l pairs with the partners having *opposite* orientations, keep these partners attached as they meet along $C_\times$ and separate them from the other pairs.

(The separation of branches is, say with the total weight $2l$, can be performed in $l!$ different ways: parasitic structure)

A closed oriented n-manifold itself makes an n-cycle which represents the *fundamental class* $[X] \in H_n(X)$. Other n-cycles are integer combinations of the oriented connected components of X.

$(i+1)$-*Plaque* D with a boundary $\partial(D) \subset D$ is the same as a cycle, except that there is a subset $\partial(D)_\times \subset D_\times$, where the sums of oriented weights do not cancel, where the closure of $\partial(D)_\times$ equals $\partial(D) \subset D$ and where $dim(\partial(D) \smallsetminus \partial(D)_\times) \leq i-2$

Two opposite canonical induced orientations on the boundary $C = \partial D$.

Plaque can be "subdivided" $D_1 = D_2$. $D = 0$ if the weight function on D_{reg} equals zero.

$-D$ the plaque with the either minus weight function or with the opposite orientation.

$D = D_1 + D_2$: a plaque D containing both D_1 and D_2 as its subplaques with the obvious addition rule of the weight functions.

$D_1 = D_2$ if $D_1 - D_2 = 0$.

THE SUM OF GENERIC PLAQUES IS A PLAQUE.

If $D \subset X$ is an i-plaque (i-cycle) then the image $f(D) \subset Y$ under a generic map $f : X \to Y$ is an i-plaque (i-cycle).

If $dim(Y) = i + 1$, then the self-intersection locus of the image $f(D)$ becomes a part of $f(D)_\times$ and if $dim(Y) = i+1$, then the new part the $\times$-singularity comes from $f(\partial(D))$.

the pullback $f^{-1}(D)$ of an i-plaque $D \subset Y^n$ under a generic map $f : X^{m+n} \to Y^n$ is an $(i+m)$-plaque in X^{m+n}; if D is a cycle and the map f is proper), then $f^{-1}(D)$ is cycle.

All of this extends to piecewise smooth, e.g. piecewise linear spaces.

Homology. C_1 and C_2 in X are *homologous*, $C_1 \sim C_2$, if there is an $(i+1)$-plaque D in $X \times [0,1]$, such that $\partial(D) = C_1 \times 0 - C_2 \times 1$.

For example *every contractible cycle* $C \subset X$ *is homologous to zero*, since the cone over C in $Y = X \times [0,1]$ corresponding to a smooth generic homotopy makes a plaque with its boundary equal to C.

Since small subsets in X are contractible, a cycle $C \subset X$ is homologous to zero if and only if it admits a decomposition into a sum of "arbitrarily small cycles", i.e. if, for every locally finite covering $X = \bigcup_i U_i$, there exist cycles $C_i \subset U_i$, such that $C = \sum_i C_i$.

The *homology group* $H_i(X)$ is the

Abelian group with generators $[C]$ for all i-cycles C in X and with the relations $[C_1] - [C_2] = 0$ whenever $C_1 \sim C_2$.

$H_i(X;\mathbb{Q})$: C and D come with fractional weights.

Examples. Every closed orientable n-manifold X with k connected components has $H_n(X) = \mathbb{Z}^k$, where $H_n(X)$ is generated by the fundamental classes of its components.

every closed orientable manifold X is non-contractible.

(*on-contractibility of* S^n and issuing from this the *Brouwer fixed point theorem* nearaly impossible within the world of *continuous* maps without using generic smooth or combinatorial ones, except for $n = 1$ with the covering map $\mathbb{R} \to S^1$ and for S^2

with the Hopf fibration $S^3 \to S^2$.

The catch is that the difficulty is hidden in the fact that a *generic* image of an $(n+1)$-plaque e.g. a cone over X) in $X \times [0,1]$ is again an $(n+1)$-plaqueisue.

But no problem with $H_0(X) = \mathbb{Z}^k$, where k components is the number of component.)

The spheres S^n have $H_i(S^n) = 0$ for $0 < i < n$, since the complement to a point $s_0 \in S^n$ is homeomorphic to $\mathbb{R}^n$ and a generic cycles of dimension $< n$ misses s_0, while $\mathbb{R}^n$, being contractible, has zero homologies in positive dimensions.

Continuous maps $f : X \to Y$, when generically perturbed, define homomorphisms $f_{*i} : H_i(X) \to H_i(Y)$ for $C \mapsto f(C)$ and that *homotopic maps* $f_1, f_2 : X \to Y$ *induce equal homomorphisms* $H_i(X) \to H_i(Y)$.

Indeed, the cylinders $C\times[0,1]$ generically mapped to $Y \times [0,1]$ by homotopies f_t, $t \in [0,1]$, are plaque D in our sense with $\partial(D) = f_1(C) - f_2(C)$.

It follows, that the

homology is invariant under homotopy equivalences $X \leftrightarrow Y$ *for manifolds* X, Y *as well as for triangulated spaces.*

Similarly, if $f : X^{m+n} \to Y^n$ is a proper (pullbacks of compact sets are compact) smooth generic map between *manifolds* where Y has no boundary, then the pullbacks of cycles define homomorphism, denoted, $f^! : H_i(Y) \to H_{i+m}(X)$, which is invariant under proper homotopies of maps.

The homology groups are much easier do deal with than the homotopy groups, since the definition of an i-cycle in X is purely local, while "spheres in X" can not be recognized by looking at them point by point – they are not "sums" of their parts.

Homologically speaking, a space is the sum of its parts: the locality allows an effective computation of homology of spaces X assembled of simpler pieces, such as cells, for example.

Degree of a Map. Let $f : X \to Y$ be a smooth (or piece-wise smooth) generic map between closed connected oriented equidimensional manifolds

Then the degree $deg(f)$ can be

(obviously) equivalently defined either as the image $f_*[X] \in \mathbb{Z} = H_n(Y)$ or as the $f^!$-image of the generator $[\bullet] \in H_0(Y) \in \mathbb{Z} = H_0(X)$. For, example, l-sheeted covering maps $X \to Y$ have degrees l. Similarly, one sees that

finite covering maps between arbitrary spaces are surjective on the rational homology groups.

If a compac X allowed a non-empty boundary, then f-pullback $\tilde{U}_y \subset X$ of some (small) open neighbourhood $U_y \subset Y$ of a generic point $y \in Y$ consists of finitely many connected components $\tilde{U}_i \subset \tilde{U}$, such that the map $f : \tilde{U}_i \to U_y$ is a diffeomorphism for all $\tilde{U}_i$.

Thus, every $\tilde{U}_i$ carries two orientations: one induced from X and the second from Y via f. The sum of +1 assigned to $\tilde{U}_i$ where the two orientation agree and of -1 when they disagree is called the *local degree* $deg_y(f)$.

If two generic points $y_1, y_2 \in Y$ can be joined by a path in Y which does not cross the f-image $f(\partial(X)) \subset Y$ of the boundary of X, then $deg_{y_1}(f) = deg_{y_2}(f)$ since the f-pullback of this path, (which can be assumed generic) consists, besides possible closed curves, of several segments in Y, joining ± 1-degree points in $f^{-1}(y_1) \subset \tilde{U}_{y_1} \subset X$ with ∓ 1-points in $f^{-1}(y_2) \subset \tilde{U}_{y_2}$.

The local degree does not depend on y if X has no boundary. Then, clearly, it coincides with the homologically defined degree.

The local degree is invariant under generic homotopies $F : X \times [0,1] \to$

Y, where the smooth (typically disconnected) pull-back curve $F^{-1}(y) \subset X \times [0,1]$ joins ± 1-points in $F(x,0)^{-1}(y) \subset X = X \times 0$ with ∓ 1-points in $F(x,1)^{-1}(y) \subset X = X \times 1$.

Geometric Versus Algebraic Cycles. The homology of a triangulated space is algebraically defined with $\mathbb{Z}$-$cycles$ which are $\mathbb{Z}$-chains, i.e. formal linear combinations $C_{alg} = \sum_s k_s \Delta_s^i$ of oriented i-simplices Δ_s^i with integer coefficients k_s, where, by the definition of "algebraic cycle" , these sums have zero algebraic boundaries.

This is exactly the same as our generic cycles C_{geo} in the i-skeleton X_i of X and, tautologically, $C_{alg} \overset{taut}{\mapsto} C_{geo}$ gives us a homomorphism from the algebraic homology to our geometric one.

An $(i+j)$-simplex minus its center can be radially homotoped to its boundary. Then the obvious reverse induction on skeleta of the triangulation shows that the space X minus a subset $\Sigma \subset X$ of codimension $i+1$ can be homotoped to the i-skeleton $X_i \subset X$.

Since every generic i-cycle C misses Σ it can be homotoped to X_i where the resulting map, say $f : C \to X_i$, sends C to an algebraic cycle.

Similarly, the eqivalence of the two definitions of homology is seen for all cellular spaces X with piece-wise linear attaching maps.

(The usual definition of homology of such an X amounts to working with all i-cycles contained in X_i and with $(i+1)$-plaques in X_{i+1}. In this case the group of i-cycles becomes a subspace of the group spanned by the i-cells, which shows, for example, that the rank of $H_i(X)$ does not exceed the number of i-cells in X_i.)

If X is a *non-compact* manifold, one may drop "compact" in the definition of these cycles. The resulting group is denoted $H_1(X, \partial_\infty)$. If X is compact with boundary, then this group of the interior of X is called the *relative homology group* $H_i(X, \partial(X))$. (The ordinary homology groups of this interior are canonically isomorphic to those of X.)

Intersection Ring. The intersection of cycles in general position in a smooth manifold X defines a multiplicative structure on the homology of an n-manifold X, denoted

$$[C_1]\cdot[C_2] = [C_1]\cap[C_2] = [C_1\cap C_2] \in H_{n-(i+j)}($$

for $[C_1] \in H_{n-i}(X)$ and $[C_2] \in H_{n-j}(X)$,

where $[C] \cap [C]$ is defined by intersecting $C \subset X$ with its small generic perturbation $C' \subset X$.

(Here genericity is most useful: intersection is painful for simplicial cycles confined to their respective skeleta of a triangulation. On the other hand, if X is a *not* a manifold one may adjust the definition of cycles to the local topology of the singular part of X and arrive at what is called the *intersection homology.*)

The intersection is respected by $f^!$ for proper maps f, but not for f_*. The former implies. in particular, that this product is invariant under oriented (i.e. of degrees +1) homotopy equivalences between *closed equidimensional* manifolds. (But $X \times \mathbb{R}$, which is homotopy equivalent to X has trivial intersection ring, whichever is the ring of X.)

The intersection of cycles of *odd* codimensions is *anti-commutative* and if one of the two has *even* codimension it is *commutative.*

The intersection of two cycles of complementary dimensions is a 0-cycle, the total $\mathbb{Z}$-weight of which makes sense if X is oriented; it is called *the intersection index of the cycles.*

The intersection between C_1 and C_2 equals the intersection of $C_1 \times C_2$ with the diagonal $X_{diag} \subset X \times X$.

Examples. (a) The intersection ring

of the complex projective space $\mathbb{C}P^k$ is multiplicatively generated by the homology class of the hyperplane, $[\mathbb{C}P^{k-1}] \in H_{2k-2}(\mathbb{C}P^k)$, with the only relation $[\mathbb{C}P^{k-1}]^{k+1} = 0$ and where, obviously, $[\mathbb{C}P^{k-i}]\cdot[\mathbb{C}P^{k-j}] = [\mathbb{C}P^{k-(i+j)}]$.

In fthe homology class $[\mathbb{C}P^i]$ *(additiacvely) generates* $H_i(\mathbb{C}P^k)$, which is seen by observing that $\mathbb{C}P^{i+1} \setminus \mathbb{C}P^i$, $i = 0, 1, ..., k-1$, is an open $(2i+2)$-cell, i.e. the open topological ball B^{2i+2}_{op} (where the cell attaching map $\partial(B^{2i+2}) = S^{2i+1} \to \mathbb{C}P^i$ is the quotient map $S^{2i+1} \to S^{2i+1}/\mathbb{T} = \mathbb{C}P^{i+1}$ for the obvious action of the multiplicative group $\mathbb{T}$ of the complex numbers with norm 1 on $S^{2i+1} \subset \mathbb{C}^{2i+1}$).

(b) The intersection ring of the n-torus is isomorphic to the exterior algebra on n-generators, i.e. the only relations between the multiplicative generators $h_i \in H_{n-1}(\mathbb{T}^n)$ are $h_i h_j = -h_j h_i$, where h_i are the homology classes of the n coordinate subtori $\mathbb{T}_i^{n-1} \subset \mathbb{T}^n$.

This follows from the Künneth formula below, but can be also proved directly with the obvious cell decomposition of $\mathbb{T}^n$ into 2^n cells.

The intersection ring structure immensely enriches homology. Additively, $H_* = \oplus_i H_i$ is just a graded Abelian group – the most primitive algebraic object (if finitely generated) – fully characterized by simple numerical invariants: the rank and the orders of their cyclic factors.

But the ring structure, say on H_{n-2} of an n-manifold X, for $n = 2d$ defines a symmetric d-form, on $H_{n-2} = H_{n-2}(X)$ which is, a polynomial of degree d in r variables with integer coefficients for $r = rank(H_{n-2})$. All number theory in the world can not classify these for $d \geq 3$ (to be certain, for $d \geq 4$).