

Revision I: Cubes \rightarrow Cubes and Averaging Sets.

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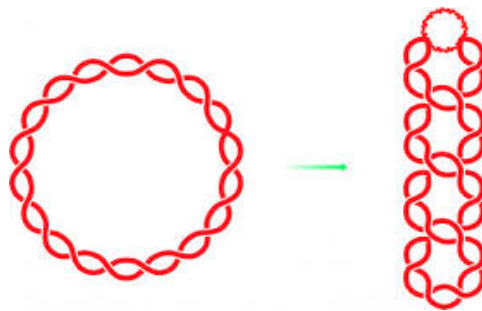


Figure 1: Topology and probability in action: linking and supercoiling of bacterial DNA.

TOPOLOGY OF THE n -CUBE. $\square^n = [-1, 1]^n$. (Probabilistic perspective on the cube, *the law of large number and the Shannon inequality* will be in another lecture.)

A. If a *continuous map between cubes*

$$f : \square^n \rightarrow \square^n,$$

that is an n -tuple of continuous functions $f_i(x_j)$, $-1 \leq x_j \leq 1$, $i, j = 1, \dots, n$,

$$f_i : \square^n \rightarrow [-1, 1], i = 1, \dots, n,$$

sends each $(n-1)$ -face $\partial_{i\pm} \subset \partial \square^n$ to the corresponding face $\partial_{i\pm} \square^n$, i.e.

$$f_i : (\dots \pm 1_i \dots) \mapsto \pm 1,$$

then f is *onto*, the equation $f(x) = y$ has a solution for all $y = (y_1, \dots, y_n) \in \square^n$.

Equivalent formulation.

B. If closed subsets $X_i \subset \square^n$, $i = 1, \dots, n$, separate the pairs of the opposite $(n-1)$ -faces $\partial_{i\pm} \subset \partial \square^n$, then the intersection $\bigcap_i X_i$ is non-empty.

B \implies **A**. To solve $f(x) = (y_1, \dots, y_n)$ let X_i be the set of $x \in \square^n$, where $f_i(x) = y_i$.

A \implies **B**. Given a closed separating $X_i \subset \square^n$, let $f_i(X)$ be a function with the zero set X_i and such that $f_i(\mp 1) = \mp 1$.

C. We shall proof A and B. by a homology theoretic argument in lecture??? which, at least in the A-form, applies to face respecting maps between general polyhedral spaces.

D. Besicovitch-Derrick Distance/Volume Inequality.¹ Let $\tilde{\square}^n$ be the cube with a Riemannian metric g on it, e.g. it is a Euclidean domain homeomorphic to the cube.

If the g -distances between the opposite $(n-1)$ -faces of $\tilde{\square}^n$ are $\geq d_i, i = 1, \dots, n$, then the g -volume of $\tilde{\square}^n$ is bounded from below by the product of d_i

$$\text{vol}(\tilde{\square}^n) \geq d_1 \cdot \dots \cdot d_n.$$

(The g -distance between two subsets is the infimum of the g -lengths of the curves joining these subsets.)

Proof. Let $\delta_{i\pm}(x)$ be the distances from $x \in \tilde{\square}^n$ to the pairs of the i -th faces of the cube and let $f_i(x) = \min(\delta_{i+}(x), \delta_{i-}(x))$.

The resulting map $f = f_1, \dots, f_n$, sends $\tilde{\square}^n$ onto the solid $[0, d_1] \times \dots \times [0, d_n]$, since, by the construction, faces go to faces.

Since f_i are distance functions, they are almost everywhere differentiable with unit gradients, i.e. $\|df_i\| = 1$, and, by the (obvious) Hadamard inequality, the Jacobian of f is almost everywhere ≤ 1 . (If you don't like "almost everywhere" approximate f_i by smooth functions with $\|grad\| \leq 1 + \varepsilon$ and let $\varepsilon \rightarrow 0$.)

Thus $\text{vol}(\tilde{\square}^n) \leq \text{vol}(\times_i [-d_i])$. QED.

E. "Segments" and "Cubes" . A compact connected metric space S with two distinguished points $\tilde{0}, \tilde{1} \in S$ is called a "segment", where $\tilde{0}$, and $\tilde{1}$ are regarded as "vertices".

The product $\tilde{\square}^n = \times_i^n [S_i, \tilde{0}_i, \tilde{1}_i], i = 1, \dots, n$, is called the n -"cube" on the vertex set $\times_i \{\tilde{0}_i, \tilde{1}_i\}$.

The (K, ν) -face S_ν^K in such a "cube" for $K \subset \{1, \dots, N\}$ and $\nu \in \times_{i \notin K} \{\tilde{0}_i, \tilde{1}_i\}$ is

$$S_\nu^K = \times_{i \in K} (S_i, \tilde{0}_i, \tilde{1}_i) \times \nu \subset \tilde{\square}^N.$$

F. If a continuous map from an N -"cube" to the true N -cube,

$$f : \tilde{\square}^n = \times_1^N [S_i, \tilde{0}_i, \tilde{1}_i] \rightarrow [0, 1]^N,$$

sends each face from the "cube" to the corresponding one in the cube, then the map f is onto.

Proof. Join $\tilde{0}_i$ with $\tilde{1}_i$ by a chain of N_i consecutively mutually ε -close points in S_i , replace S_i by the unit segment $[0, 1]$ divided into $N_i + 1$ equal subsegments and reduce **F** to **A**, where $S_i = [0, 1]$, with $\varepsilon \rightarrow 0$.

Speaking formally, let $\sigma_{i,\varepsilon} : \{0, 1, \dots, N_i\} \rightarrow S_i, \varepsilon > 0$ be maps such that $\sigma_i(0) = \tilde{0}_i, \sigma_i(N_i) = \tilde{1}_i$, and $\text{dist}(j, j+1) \leq \varepsilon$ for all $i \in \{0, 1, \dots, N_i\}$ and all i , let

$$\Sigma_\varepsilon = \times_i^n \sigma_i : \times_1^n \{0, 1, \dots, N_i\} \rightarrow \tilde{\square}^n$$

and

$$\Phi_\varepsilon = f \circ \Sigma_\varepsilon : \times_1^n \{0, 1, \dots, N_i\} \rightarrow [0, 1]^n.$$

¹A Volume-diameter inequality for n -cubes, William R. Derrick, Journal d'Analyse Mathématique volume 22, pages 1-36 (1969)

Identify the sets $\{0, 1, \dots, N_i\}$ with the subsets $\{\frac{j}{N_i}\}_{j=1, \dots, N_i} \subset [0, 1]$ and extend the map Φ_ϵ to a continuous map $\Psi_\epsilon : [0, 1]^n \rightarrow [0, 1]^n$, which is obtained by consecutive peacewise linear interpolation with conical extension of maps from the boundaries of faces of small cubes. to these faces.

Since the maps Ψ_ϵ are onto, the maps Φ_ϵ have ϵ -dense images in $[0, 1]^n$, where $\epsilon \rightarrow 0$ for $\epsilon \rightarrow 0$ and the onto property of f follows with $\epsilon \rightarrow 0$.²

G. ϵ -Corollary. If a continuous map

$$f : \tilde{\square}^n = \prod_{i=1}^N [S_i, \tilde{0}_i, \tilde{1}_i] \rightarrow \mathbb{R}^n \supset [0, 1]^N,$$

sends each face from the "cube" ϵ -close to the corresponding face of $[0, 1]^n$, then the image f contains all points in $[0, 1]^n$, which lie ϵ -far from the boundary $\partial[0, 1]^n$.

Proof. Let $dist(z_0, \partial[0, 1]^n) > \epsilon$ and let $\phi_0 : [0, 1]^n \rightarrow [0, 1]^n$ be a continuous map, such that $\phi_0(z) = z$ on the boundary of the cube and in a small neighbourhood of z_0 and which sends the ϵ -neighbourhoods of the faces of $[0, 1]^n$ to these very faces.

Then **F** applies to the composed map $\phi_0 \circ f : \tilde{\square}^n \rightarrow [0, 1]^n$ and **G** follows.

H. The convex hull of a subset $X \subset \mathbb{R}^n$ is the set of all convex combinations

$$z = \sum_{j=1}^N p_j x_j, \quad x_j \in X, p_j \geq 0, \sum_j p_j = 1,$$

where, this is called *Caratheodory theorem*,

if $z = \sum_{j=1}^N p_j x_j$, then there exists a subset $K \subset J = \{1, \dots, N\}$ of cardinality $n + 1$, such that $z = \sum_{k=1}^{n+1} q_k x_k$, for some $q_k \geq 0, \sum_k q_k = 1$.

In fact, the convex polyhedron $conv\{x_j\}$ can be (obviously) subdivided into simplices with vertices in $\{x_j\}$.

A point z in the convex hull of $X \subset \mathbb{R}^n$ is called *X-rational* if it is equal to a convex combination of points from X with *rational weights*,

$$[p_j] \quad z = \sum_{j=1}^N p_j x_j, \quad x_j \in X,$$

where $p_i \geq 0$ are rational numbers, such that $\sum_j p_i = 1$.

Equivalently, *X-rational* points $z \in conv(X)$ are *centers of mass* of finite multisets³ from X ,

$$[1/M] \quad z = \frac{1}{M} \sum_{k=1}^M x_k,$$

where $[p_j] \implies [1/M]$ for M equal the common denominator of the numbers p_j .

I. SZ Theorem.⁴ If a compact subset $X \subset \mathbb{R}^n$ contains $2n$ point $\underline{x}_i, \underline{y}_i \in X$, $i = 1, \dots, n$, such that the n vectors $\underline{x}_i - \underline{y}_i \in \mathbb{R}^n$ are *linearly independent* and such

²This argument in homological terms proves continuity of the *Čech cohomology*.

³A multiset is an image of a map $I \rightarrow X$, written as $\{x_i\} \subset X, i \in I, x_i \in X$.

⁴Seymour, P. D. and Zaskavsky, T., *Averaging set. A generalization of mean values and spherical designs*, Adv. Math. 52 (1984), 213-246.

that \underline{x}_i and \underline{y}_i lie in the same connected component of X for all $i = 1, \dots, n$, then all points in the interior of the convex hull of X , are X -rational.

Proof. Since rational numbers are dense in \mathbb{R} the X -rational points are dense in the convex hull of X and it suffices to show that the "rational interior" of the convex hull $\text{conv}(X)$ is non-empty: $\text{conv}(X)$ contains a ball of positive radius, say $B_{\underline{z}}^n(\underline{\delta}) \subset \text{conv}(X)$, $\underline{z} \in X$, $\underline{\delta} > 0$, such that all points in this ball X -rational

In fact, the existence of an X -rational ball $\underline{B} = B_{\underline{z}}^n(\underline{\delta})$ implies the existence of rational δ -balls around all points $z \in \text{conv}(S)$, $B = \bar{B}_z(\delta)$, where δ is bounded from below essentially by the distance from z to the boundary of $\text{conv}(X)$, namely

$$\delta \geq \frac{(\underline{\delta} \cdot \text{dist}(z, \partial \text{conv}(X)))}{2 \text{diam}(X)}.$$

Indeed, extend the straight segment between \underline{z} and z to the boundary of the ball $B_z(d)$, $d = \text{dist}(z, \partial \text{conv}(X))$, let

$$[z_0, \underline{z}] \subset \text{conv}(X)$$

be the extended segment with $z_0 \in \partial B_z(d)$ and with $z \in [z_0, \underline{z}]$, where $\|z - z_0\| = d$.

Let $z'_0 \in \text{conv}(X)$ be an X -rational point ϵ -close to z_0 for

$$\epsilon \leq \frac{\text{dist}(z'_0, \underline{z})}{10 \text{dist}(z, \partial \text{conv}(X))}.$$

Now, the ball $B = B_z(\delta)$ for

$$\delta = \frac{\underline{\delta} \cdot \text{dist}(z'_0, \underline{z})}{2 \text{dist}(z, \partial \text{conv}(X))} - \epsilon$$

is the required X -rational one, since all points in it are convex combinations $Nz'_0 + (1 - N)b$, $b \in \underline{B}$, for an integer N , such that

$$\left| N - \frac{\text{dist}(z'_0, \underline{z})}{\text{dist}(z, \partial \text{conv}(X))} \right| \leq 1.$$

With the above understood, the proof of the theorem reduces to the following.

J. Lemma. Let $\square^n \subset \mathbb{R}^n$ be the Minkovski mean of the straight segments $[\underline{x}_i, \underline{y}_i] \subset \mathbb{R}^n$, that is the set of the averages

$$\frac{1}{n} \sum_i z_i, \quad z_i \in [\underline{x}_i, \underline{y}_i] \subset \text{conv}(X).$$

Then all points in the interior of \square^n are X -rational.

Proof. Let us show the existence of subsets, or rather multisets, in the connected components $S_i \subset X$ of $\underline{x}_i \in X$,

$$\{x_{i,j}\} \subset S_i, \quad i = 1, \dots, n, \quad j = 1, \dots, N,$$

such that all interior points $z \in \text{int}(\square^n)$ are representable as

$$z = \frac{1}{nN} \sum_{i,j} x_{ij}$$

for sufficiently large $N = N(z)$.

Definition of "Chain Segment". Given a "segment" $[S, \tilde{0}, \tilde{1}]$ let the $N * S$ -chain in the N -"cube" $[S, \tilde{0}, \tilde{1}]^N$ be the union of the N consecutive "edges" E_j in this cube, which join the diagonally opposite "vertices" $(\underbrace{\tilde{0}, \dots, \tilde{0}}_N)$ and $(\underbrace{\tilde{1}, \dots, \tilde{1}}_N)$,

$$E_j = \{\underbrace{\tilde{0}, \dots, \tilde{0}}_{j-1}, s, \underbrace{\tilde{1}, \dots, \tilde{1}}_{N-j}\}_{s \in S} \subset [S, \tilde{0}, \tilde{1}]^N,$$

where this chain $[N * S] = \bigcup_{i=1}^N E_i$ is itself a "segment" with the "vertices" $(\underbrace{\tilde{0}, \dots, \tilde{0}}_N)$ and $(\underbrace{\tilde{1}, \dots, \tilde{1}}_N)$.

Let $\phi : S \rightarrow \mathbb{R}^n$ be a continuous map and let

$$N * \phi : [N * S] \rightarrow \mathbb{R}^n$$

send $(s_1, \dots, s_N) \in N * S \subset S^N$ to the *center of mass of the N image points* $\phi(s_j) \in \mathbb{R}^n$, $j = 1, \dots, N$,

$$N * \phi : (s_1, \dots, s_N) \mapsto \frac{1}{N} \sum_1^N \phi(s_j).$$

Clearly, the "division points" from the chain, that are

$$\{\underbrace{\tilde{0}, \dots, \tilde{0}}_j, \underbrace{\tilde{1}, \dots, \tilde{1}}_{N-j}\},$$

lands in the segment $[\phi(\tilde{0}), \phi(\tilde{1})] \subset \mathbb{R}^n$, such that

- these points divide this segment into N equal subsegments,
- the image of the j -th copy of S in $N * S$ goes to the δ -neighbourhood of the j -th subsegment in $[\phi(\tilde{0}), \phi(\tilde{1})]$, where δ is small when N is much greater than the diameter of the ϕ -image of S in \mathbb{R}^n :

$$\delta \leq \frac{\text{diam}(\phi(S))}{\sqrt{N}}.$$

Next let $S_i \subset X$ be the common connected components of $\underline{x}_i, \underline{y}_i \in X$, where we set $\tilde{0}_i = \underline{x}_i$ and $\tilde{1}_i = \underline{y}_i$, and let $N * S_i \subset S_i^N$ be their chain "segments" $N * S_i$. Map $\times_1^n [N * S_i] \rightarrow \text{conv}(X)$ by

$$\Phi_N : s_{i,j} \mapsto \frac{1}{n} \sum_{i=1}^n \frac{1}{N} \sum_{j=1}^N s_{i,j}$$

where, by the above, with ϕ_i being the imbeddings $S_i \hookrightarrow \mathbb{R}^n$,

*the map Φ_N sends each face of the n -"cube" $\times_1^n [N * S_i]$ to the δ -neighbourhood of the corresponding face in \square^n , for*

$$\delta \leq \frac{\sum_1^n \text{diam}(\phi_i(S_i))}{\sqrt{N}}.$$

Finally, the ε -Corollary **G** applies and the proof follows.

K. From Multisets to Sets. The X -rationality of a point z implies the existence of a multiset in X with the center of mass z but the above proof allows effortless disengagement of multiple points by small perturbations. Therefore,

all points in the interior of $\text{conv}(X)$ are representable by centers of mass of (true) finite subsets in X .

L. The original formulation of I reads:

Let X be a compact connected⁵ space with a probability (total mass one) Borel measure dx , which is strictly positive on non-empty open subsets in X and let $f_i(x)$, $i = 1, \dots, n$, be continuous functions on X . Then there exists a finite subset $\Sigma \in X$ such that

$$\frac{1}{\text{card}(\Sigma)} \sum_{\sigma \in \Sigma} f_i(\sigma) = \int_X f_i(x) dx$$

for all $i = 1, \dots, n$.

Reduction L \implies **K.** Map $X \rightarrow \mathbb{R}^n$ by $x \mapsto 1(x), \dots, f(n)x$, observe that the vector

$$z = \left(\int_X f_1(x) dx, \dots, \int_X f_n(x) dx \right) \in \mathbb{R}^n$$

is the interior of $\text{conv}(X)$ due to positivity of dx . Then the subset $\Sigma \subset X$ with the center of mass z does the job.

M. Exercises. (a) Reduce the SZ-theorem for *no-compact path connected* X to the compact case.⁶

(b) Let S_i be the images of C^1 -maps $\phi_i : S_i \rightarrow \mathbb{R}^n$ of smooth connected manifolds S_i and show that the linear independence of $\underline{x}_i - \underline{y}_i$ implies that *the images of the differentials $d\phi_i : T(S_i) \rightarrow \mathbb{R}^n$ at some points $s_i \in S_i$ span \mathbb{R}^n .*

Then prove lemma **J** in this case by applying the implicit function theorem.

N, Question Let $S_i \subset \mathbb{R}^n$, $n = 1, \dots, n$, be compact connected subsets (e.g. the images of $[0, 1]$ under continuous maps), which contain pairs of points $x_i, y_i \in S_i$ with linearly independent $x_i - y_i$. Is then the interior of the Minkovski mean (or the sum if you wish) non-empty. (Looks easy but I couldn't figure it out.)

O. Hilbert's Rationality. Hilbert in his solution of the Waring problem⁷ uses and proves (but doesn't formulate) **I** in the case, where rational points are *dense* in X and where this is done for images of spheres S^l in \mathbb{R}^n under polynomial maps with rational coefficients.⁸

Thus, this is small step in Hilbert's (arithmetic) argument, he constructs what is now-a-days called spherical designs⁹ $\Sigma \subset S^l$, where all points $\sigma \in \Sigma$ are rational, and where this rationality is most essential in the following steps of Hilbert's proof.

⁵In the Seymour- Zaslavsky paper X is assumed path connected but not necessarily compact.

⁶I am not certain of this with "connected" instead of "path connected".

⁷For all $p = 2, 3, \dots$, there exists a constant $N = N(p)$, such that every positive integer x is the sum $x = \sum_1^M y_i^p$ for positive integers y_i and $M \leq N$.

⁸Hilbert, D., *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n -ter Potenzen (Waring'sches Problem)* Math. Ann. 67 (1909), 281-300.

⁹See *Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs*, Yuri I. Lyubich & Leonid N. Vaserstein Geometriae Dedicata volume 47, pages 327-362 (1993).