0.1. Brunn-Minkowski inequality. Recall that the Minkowski sum \( X+Y \) of subsets \( X \) and \( Y \) in the Euclidean space \( \mathbb{R}^n \) is the set of the sums \( x+y \in \mathbb{R}^n \) for all \( x \in X \) and \( y \in Y \). An equivalent definition is

\[
X+Y = \bigcup_{y \in Y} (X+y)
\]

where \( X+y \) denotes the \( y \)-translate of \( X \) which is the same thing as the sum of \( X \) with the one-point set \( \{y\} \). Note that \( X+Y = Y+X \) as \( x+y = y+x \) in \( \mathbb{R}^n \).

0.1 A. Example. Let \( X_\varepsilon \) be the \( \varepsilon \)-ball in \( \mathbb{R}^n \) around the origin. Then, by the second definition, \( X_\varepsilon + y \) equals the union of the \( \varepsilon \)-balls in \( \mathbb{R}^n \) with centers in \( Y \) which is customary called the \( \varepsilon \)-neighborhood of \( Y \).

0.1 B. Brunn-Minkowski theorem. The \( n \)-dimensional volume (i.e. Lebesgue’s measure) of \( X+Y \) is bounded from below by

\[
[\text{Vol}(X+Y)]^{1/n} \geq (\text{Vol} X)^{1/n} + (\text{Vol} Y)^{1/n}.
\]

(*)

Remarks and corollaries.0.1 B1. We are most interested here in the classical case of (*) where \( X \) and \( Y \) are bounded convex subsets in \( \mathbb{R}^n \). Yet, (*) remains valid for arbitrary (measurable) subsets \( X \) and \( Y \) in \( \mathbb{R}^n \) (see 3.1.).

0.1 B2. Let \( X \) and \( Y \) be rectangular solids with mutually parallel edges of lengths \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \). Say,
\[ X = [a_0, a_1] \times \cdots \times [a_{n-1}, a_n], \]
\[ Y = [b_0, b_1] \times \cdots \times [b_{n-1}, b_n]. \]

Then
\[ X + Y = [a_0 b_0 + b_1] \times \cdots \times [a_{n-1} b_{n-1} + b_n] \]

and (*) reduces to the following well-known algebraic inequality,
\[
\left( \frac{n}{\pi} \left( \sum_{i=1}^{n} a_i \right) \right)^{1/n} \geq \left( \frac{n}{\pi} \left( \sum_{i=1}^{n} a_i^2 \right) \right)^{1/n} + \left( \frac{n}{\pi} \left( \sum_{i=1}^{n} b_i^2 \right) \right)^{1/n} \quad (*)
\]

0.1 B\(_3\). Let \( Y \) have smooth boundary \( \partial Y \) and take the \( \varepsilon \)-ball \( B_\varepsilon \) in \( \mathbb{R}^n \) for \( X \). Then, one easily sees (compare 0.1.A) that the \( (n-1) \)-dimensional volume of \( \partial Y \) satisfies
\[
\text{Vol} \ \partial Y = \lim \varepsilon \to 0 \varepsilon^{(n-1)} (\text{Vol} (Y + B_\varepsilon) - \text{Vol} Y).
\]

Thus, (*) yields the Euclidean isoperimetric inequality,
\[
\text{Vol} \ \partial Y \geq C_n (\text{Vol} Y)^{n/(n-1)}
\]

where \( C_n \) denotes the \((n-1)\)-dimensional volume of the boundary sphere of the ball \( B \subset \mathbb{R}^n \) normalized by the condition \( \text{Vol} B = 1 \).

0.2. Hodge-Teissier-Hovanski inequality. Consider the Cartesian product of two complex projective spaces \( P_1 \times P_2 \) with the standard metric and let \( V \) be a complex algebraic subvariety in \( P_1 \times P_2 \) of complex dimension \( n \). (The reader unfamiliar with this terminology is addressed to section 3.3.). Denote by \( V_1 \subset P_1 \) and \( V_2 \subset P_2 \) the projections of \( V \) to \( P_1 \) and to \( P_2 \).

0.2.A. Algebraic Brunn-Minkowski. If \( V \) is irreducible (see 3.3.), then the \( 2n \)-dimensional volumes of \( V, V_1 \) and \( V_2 \) satisfy
\[
(\text{Vol} \ V)^{1/n} \geq (\text{Vol} \ V_1)^{1/n} + (\text{Vol} \ V_2)^{1/n}.
\]

(*)
**Remark and Corollaries.** 0.2. A1. If \( n = 1 \), then \((+)\) is trivial. In fact one has equality in this case.

0.2. A2. If \( n = 2 \) then \((+)\) is equivalent to the *Hodge index theorem* (see 3.3.). Note that \((+)\) may easily fail if \( V \) is reducible. For example, take

\[
V = (V_1 \times v_2) \cup (v_2 \times V_2)
\]

for \( V_i \subset P_i \) and \( V_i \subset P_i \) for \( i = 1, 2 \). Then

\[
(V \text{Vol})^{1/n} = (V_1 \text{Vol} + V_2 \text{Vol})^{1/n} < (V_1)^{1/n} + (V_2)^{1/n}.
\]

0.2. A3. The inequality \((+)\) for \( n \geq 3 \) was discovered by Hovanski and Teissier. Their proof (see 3.3.) goes by induction on \( n = \dim V \) which starts with \( n = 2 \), where the inductive step for \( n \geq 3 \) is realized by intersecting \( V \) with an appropriate hypersurface \( H \) in \( P_1 \times P_2 \), and where the irreducibility of the intersection \( V \cap H \) (having the dimension by one less than \( V \)) is achieved with the *Bertini* theorem (see 3.3.). In fact, Teissier and Hovanski proved a refinement of \((+)\) which is parallel to the *Alexandrov-Fenchel inequality* for convex sets (see 1.6.). Alexandrov gave two proofs of his inequality. The first proof (see [Al]1) is combinatorial and resembles the algebra-geometric argument by Hovanski and Teissier (instead of Hodge index theorem for \( n = 2 \) Alexandrov uses a corresponding geometric inequality of Minkowski). The second proof by Alexandrov (see [Al]2) appeals to the elementary theory of second order elliptic operators. We shall see in §2 that a modern rendition of Alexandrov's proof (exterior products of differential forms instead of mixed discriminants of quadratic forms) yields the Hodge-Teissier-Hovanski inequality as readily (even faster) as it yields the Alexandrov-Fenchel inequality (for \( n = 2 \) Alexandrov's argument is essentially equivalent to Hodge's proof of his index theorem).

0.2. B. **Moment map, Legendre transform and the implication \((+) \Rightarrow (+)\).** A variety \( V \) is called *toral* if it admits an isometric (for the metric induced from \( P_1 \times P_2 \supset V \)) action of the torus \( T^\theta \). Such an action induces what is called the *moment map* \( M : V \Rightarrow \mathbb{R}^\theta \) which is defined with the induced *symplectic* (Kähler) *form* on \( V \) (see 3.2.). Similar (moment) maps, also denoted \( M \), are defined for \( V_1 \) and \( V_2 \). One shows (see 3.2.) that \( M \) preserves volumes (up to a normalizing constant) and that the image \( M(V) \) is the Minkowski sum of the moment-images of \( V_1 \) and \( V_2 \),

\[
M(V) = M(V_1) + M(V_2).
\]
Thus \((*) \Rightarrow (+)\) for toral varieties \(V\). On the other hand one knows (see 2.4. and 3.3) that for any pair of convex polyhedra \(X_1\) and \(X_2\) in \(\mathbb{R}^n\) with vertices in the integral lattice \(\mathbb{Z}^n \subset \mathbb{R}^n\), there exists a toral variety \(V\) such that \(M(V_i) = X_i\) for \(i = 1, 2\). Using this along with an approximation of convex sets by polyhedra with rational vertices one derives \((*)\) from \((+\)) for all convex subsets in \(\mathbb{R}^n\).

0.2. B1. REMARK. The correspondence between toral varieties and convex polyhedra goes back to Newton and Minding (see the discussion by Hovanski in chapter 4 of [Buz-Za]). The relation between \((+\)) and \((*)\) was discovered by Teissier and Hovanski (see [T] and [B-z]). The approach using the moment map is due to Arnold and Atiyah (see [At]2).

0.2. B2. The action of \(\mathbb{T}^n\) on \(V\) can be complexified to an action of \((\mathbb{C}^\times)^n = \mathbb{T}^n \times (\mathbb{R}^\times_+)^n\) on \(V\) (see 3.2.). Then the restriction of the moment map to the \((\mathbb{R}^\times_+)^n\)-orbits can be identified with the Legendre transform for the \(K\ddot{a}hler\) potential on \(V\) (see 3.2. and [At]2).

Note that this kind of Legendre transform is built into Alexandrov's argument as it applies to supporting functions of the convex sets in question (see [At]2).

§1. Legendre transform, mixed volumes and \(K\ddot{a}hler\) forms. Consider a \(C^1\)-function \(f\) on a linear space \(L\) and let us interprete the differential of \(f\) as a map of \(L\) into the dual space \(L'\), say \(D_f : L \rightarrow L'\) (If \(L\) is a Hilbert space one can use instead the \(\text{gradient}\) map \(L \rightarrow L\) for \(x \rightarrow \text{grad}_x f\) that some people find more geometric).

Recall that a map \(\varphi : L \rightarrow L'\) is called \textit{monotone increasing} if

\[< \varphi(x_1) - \varphi(x_2), x_1 - x_2 > \geq 0\]

for all \(x_1\) and \(x_2\) in \(L\). One calls \(\varphi\) strictly increasing if the above inequality is strict for all \(x_1\) and \(x_2 \neq x_1\). It is obvious that every strictly increasing map is one-to-one. In particular, if such a \(\varphi\) is continuous and \(L\) is finite dimensional, then \(\varphi\) is a homeomorphism. Also observe that the map \(\varphi = D_f\) is (strictly) increasing if and only if \(f\) is (strictly) convex. Thus we obtain the

1.1. Homeomorphism property. If is a strictly convex \(C^1\)-function on \(\mathbb{R}^n\) then the map \(D_f : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a homeomorphism (onto an open subset in \(\mathbb{R}^n\)).

The following property is somewhat more exciting.
1.2. Convexity theorem. If $f$ is convex then the closure of the image $D_f(\mathbb{R}^n) \subset \mathbb{R}^n$ is convex.

Proof. For an arbitrary function $f$ on $L$ denote by $L_f^+ \subset L$ the set of those linear functions $y$ on $L$ for which

$$f - y \geq \text{const} > -\infty.$$  \hspace{1cm} (1)

This means that the graph $\Gamma_f \subset L \times \mathbb{R}$ lies above that of the function $y - \text{const}$, and so $L_f^+$ contains the image $D_f(L) \subset L$ for convex functions $f$. In fact if $y = d_{x_0} f$ then the graph $H_y \subset L \times \mathbb{R}$ of the function $y(x) + f(x_0) - y(x_0)$ is tangent to the graph $\Gamma_f \subset L \times \mathbb{R}$ at the point $(x_0, f(x_0))$. Hence $\Gamma_f$ lies above the hyperplane $H_y$ for all $y \in D_f(L)$ in the case where $f$ is convex.

Next we observe the subset $L_f^+ \subset L$ is convex, as the inequalities

$$f - y_1 > -\infty \text{ and } f - y_2 > -\infty$$

obviously imply the same inequality for convex combinations of $y_1$ and $y_2$.

$$f - (ty_1 + (1-t)y_2) > -\infty,$$

To conclude the proof we must show that $L_f^+$ is contained in the closure of $D_f(\mathbb{R}^n)$. This is equivalent to

$$\inf_{x \in L} \|d_x g\| = 0$$

for the functions $g = f - y$ satisfying the above (boundness away from $-\infty$) condition (1).

In fact, if (2) is violated and

$\|d_x g\| > \varepsilon > 0$ for all $x \in L$, then $\inf g = -\infty$ as the following trivial lemma shows.

1.2.A. Let $X$ be a complete metric space and $g : X \rightarrow \mathbb{R}$ a continuous function, such that for every $x \in X$ there exists $x' \in x$ different from $x$, such that

$$g(x) - g(x') \geq \varepsilon \text{ dist}(x, x'),$$
where $\varepsilon$ is a fixed positive number. Then

$$\inf_{X} g(x) = -\infty.$$ 

1.2. B. Remark. The above discussion presents a tiny piece of the convex duality theory going back to Legendre whose name is attached to the transform of $f$ from $L$ to $D_{f}(L) \subseteq \mathbb{L}'$ by the map $D_{f}$,

$$f' (y) = f(D_{f}^{-1} y).$$

The Legendre transform $f'$ of $f$ is correctly defined for strictly convex functions $f$ as $D_{f}$ is one-to-one. In this case $f'$ also is strictly convex and satisfies Legendre duality relation $D_{f} = D_{f}^{-1}$ under an appropriate (reflexivity) condition on $L$ (which is obviously satisfied for $L = \mathbb{R}^{n}$).

1.3. Minkowski additivity of $L_{f}'$ and $D_{f}(L)$. If $y_{1} \in L_{f_{1}}'$ and $y_{2} \in L_{f_{2}}'$ (see (1) above), then, obviously) $\inf (y_{1}+y_{2}, y_{2}) > -\infty$, that is $y_{1}+y_{2}$ is contained in the Minkowski sum of $L_{f_{1}}$ and $L_{f_{2}}$.

In other words

$$L_{f_{1}}' + L_{f_{2}}' \subseteq L_{f_{1}+f_{2}}'.$$

(2)

It is equally obvious that

$$D_{f_{1}}(L) + D_{f_{2}}(L) \supseteq D_{f_{1}+f_{2}}(L),$$

(3)

as $d(f_{1}+f_{2}) = df_{1} + df_{2}$.

Thus we obtain the following

1.3.A. Additivity. If $f_{1}$ and $f_{2}$ are strictly convex functions on $\mathbb{R}^{n}$, then

$$D_{f_{1}+f_{2}}(\mathbb{R}^{n}) = D_{f_{1}}(\mathbb{R}^{n}) + D_{f_{2}}(\mathbb{R}^{n}),$$

(4)
1.4. Brunn-Minkowski theorem for convex functions $f$. Let $[D^2f]^n$ denote the determinant of the Hessian $D^2f$ of $f$,

$$ [D^2f]^n = \text{Det} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) $$

and note that $[D^2f]^n$ equals the Jacobian of the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$.

Therefore

$$ \text{Vol} \ D_f (\mathbb{R}^n) = \int_{\mathbb{R}^n} [D^2f]^n $$

(5)

for all strictly convex $C^2$-functions $f$ on $\mathbb{R}^n$.

1.4.A. Remark. For an arbitrary (non-smooth) convex function $f$ one can define $[D^2f]^n$ as a measure on $\mathbb{R}^n$ and show that

$$ \text{Vol} \ L' = \int_{\mathbb{R}^n} [D^2f]^n. $$

Now we apply (Brunn-Minkowski) inequality (*) in 0.1.B. to $D f_i (\mathbb{R}^n)$ $i = 1, 2$ and obtain the following

1.4.B. Theorem. Every two strictly convex $C^2$-functions $f_1$ and $f_2$ on $\mathbb{R}^n$ satisfy

$$ \left( \int_{\mathbb{R}^n} \left[ D^2(f_1 + f_2) \right]^n \right)^{1/n} \geq \left( \int_{\mathbb{R}^n} \left[ D^2f_1 \right]^n \right)^{1/n} + \left( \int_{\mathbb{R}^n} \left[ D^2f_2 \right]^n \right)^{1/n}. $$

(***)

1.4.B1. Remark. This inequality remains valid for all convex functions on $\mathbb{R}^n$. This can be derived from (***)) by a simple approximation argument or proved more directly using (*) and 1.4.A.

1.4.C. Implication (***) $\Rightarrow$ (*) for convex sets in $\mathbb{R}^n$. Let $Y$ be a convex bounded open subset in $L' = \mathbb{R}^n$ and define $f(x)$ on $L = \mathbb{R}^n$ by

$$ f(x) = \log \int_{\mathbb{R}^n} \exp <x,y> dy. $$
One checks by a straightforward computation that $f$ is real analytic and strictly convex, and that the map $D_f : \mathbb{R}^n \to \mathbb{R}^n$ sends each $x$ to the center of gravity of the measure $\exp\left< x, y \right> dy$ on $Y$. It follows that

$$D_f(\mathbb{R}^n) \subseteq Y.$$  

To show that $D_f(\mathbb{R}^n) \supseteq Y$ take a point $x_0$ such that the linear function $\left< x_0, y \right>$ on $L' \ni Y$ has only one maximum point, say $y_0$, in the closure $C\overline{Y} \subseteq L'$ of $Y$. Note that these points $y_0$ are exactly the extremal points of $C\overline{Y}$. Now we look at the measures $\exp(\lambda x_0, y) dy$ on $Y$ and see that these concentrate at $y_0$ as $\lambda \to \infty$. Hence, the closure of the image of $D_f$ contains all extremal points $y_0$ of $C\overline{Y}$. Since the image $D_f(\mathbb{R}^n) \subset Y$ is convex, it necessarily equals $Y$.

1.4.C1. Conclusion. Every convex bounded open subset $Y$ in $\mathbb{R}^n$ admits a surjective diffeomorphism $D_f : \mathbb{R}^n \to Y$ for some strictly convex $C^2$-function $f$ on $\mathbb{R}^n$. Thus (***) $\Rightarrow$ (*) for convex bounded open subsets. This trivially implies that (***) $\Rightarrow$ (*) for all convex subsets in $\mathbb{R}^n$.

1.4.C2. Remark. There are many convex functions $f$ with $D_f(\mathbb{R}^n) = Y$. For example, instead of the Lebesgue's measure $dy$ on $Y$ one can take an arbitrary measure $d\mu$ on $Y$, such that the convex hull of the support of $\mu$ equals the closure of $Y$. Then one sees as earlier that $D_f(\mathbb{R}^n) = Y$ for

$$f(x) = \log \int_{\mathbb{R}^n} \exp\left< x, y \right> d\mu.$$  

However, for every compact convex subset $Y$, there is a distinguished convex function $f_0$ (which is non-smooth and not strictly convex), called the support function of $Y$, such that $L_{f_0} = Y$. This function is characterized by the homogeneity, $f_0(\alpha x) = \alpha f_0(x)$ for all $\alpha \geq 0$ (as well as by convexity and the relation $L_{f_0} = Y$). It is easy to see that $f_0$ equals the infimum of the convex functions $f$, such that $L_f \ni Y$. Usually, one defines $f_0$ as the infimum of the linear functions $y(x)$ on $L$ over all $y \in L \setminus Y$.  

Our choice of $f = \log \int \exp$ is motivated by the Kähler geometry in $\mathbb{C}P^n$ (see 2.4.).
1.5. Kähler formulation of $(\ast \ast)$. Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way.

$$\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^n \oplus \sqrt{-1} \mathbb{R}^n = \mathbb{C}^n,$$

and let us denote by $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the operator corresponding to the multiplication by $\sqrt{-1}$ in $\mathbb{C}^n$. We also denote by $J$ the induced operator on vector fields and differential forms on $\mathbb{R}^{2n}$. We call an exterior 2-form $\omega$ on $(\mathbb{R}^{2n}, J)$ positive if $\omega(\partial, J\partial) \geq 0$ for all vector fields $\partial$ on $\mathbb{R}^{2n}$ and call it strictly positive if $\omega(\partial, J\partial) > 0$ for all non-vanishing fields $\partial$. Say that $\omega$ is a $(1,1)$-form if $J\omega = \omega$, that is $\omega(J\partial_1, J\partial_2) = \omega(\partial_1, \partial_2)$. Since $J^2 = -1$ and $\omega$ is antisymmetric, this is equivalent to the symmetry of the form $h$ defined by $h(\partial_1, \partial_2) = \omega(\partial_1, J\partial_2)$. Note that such a $\omega$ is positive if and only if the (quadratic) form $h$ is positive semidefinite.

Recall that the second differential $H = D^2f$ of a function $f$ is the quadratic form defined by the formula.

$$H(\partial_1, \partial_2) = \partial_1(\partial_2 f)$$

for all translation invariant (parallel) vector fields $\partial_1$ and $\partial_2$, where $\partial f$ stands for the (Lie) derivative of $f$ along $\partial$.

Another useful second order differential operator, now from functions to exterior 2-forms, is

$$f \to \omega = dJdf,$$

where $d$ is the exterior differential first applied to $f$ and then to the 1-form $Jdf$.

A straightforward computation expresses $dJd$ in terms of $D^2$ by

$$h = H + JH$$

(6)

where $H = D^2f$ and $h$ is defined along with $\omega = dJdf$ by $h(\partial_1, \partial_2) = \omega(\partial_1, J\partial_2)$. Note that the definition of $h$ on the left hand side of identity (6) uses only $J$ while the definition of $H$ via $D^2$ needs the affine structure of $\mathbb{R}^{2n}$.

Since the form $H$ is symmetric, the form $h$ also is symmetric as well as $J$-invariant. Hence, the form $\omega = dJdf$ is $(1,1)$ for all functions $f$ on $\mathbb{R}^{2n}$. Let us divide $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \sqrt{-1} \mathbb{R}^n$ by the $n$-dimensional lattice $\sqrt{-1} \mathbb{Z}^n$ and denote by $V = \mathbb{R}^n \times T^n$ the resulting manifold for the torus.
\[ T^n = \sqrt{-1} \mathbb{R}^{2n} / \sqrt{-1} \mathbb{Z}^n. \]

The notions of a (1,1)-form and of positivity descend from \( \mathbb{R}^{2n} \) to \( V \) along with the operator \( J \) which acts on the tangent bundle of \( V \). Now we can formulate the following

1.5.A. Brunn-Minkowski inequality for \( T^n \)-invariant forms on \( V \).

Let \( \omega_1 \) and \( \omega_2 \) be exact positive (1,1)-forms on \( V \) which are invariant under the natural \( T^n \)-action on \( V \). Then the top-dimensional exterior power \((\omega_1 + \omega_2)^n\) satisfies,

\[
\left( \int_V (\omega_1 + \omega_2)^n \right)^{1/n} \geq \left( \int_V \omega_1^n \right)^{1/n} + \left( \int_V \omega_2^n \right)^{1/n}.
\]

Let us show that (***) is equivalent to (**) in 1.4.B. First we prove the implication (***) \( \rightarrow \) (**) by relating to each function \( f \) on \( \mathbb{R}^n \) the form \( \omega \) on \( V \) which is the differential \( \tilde{f} \) of the pull-back \( f \) of \( f \) to \( V \) for the projection \( V \rightarrow \mathbb{R}^n = V/T^n \). It is clear that

\[
\int_V \omega^n = \int_{\mathbb{R}^n} [D^2f]^n
\]

for all \( f \) on \( \mathbb{R}^n \) and that \( \omega \) is positive if and only if \( f \) is convex. Since \( \omega = df \tilde{f} \) it is an exact (1,1)-form and so (***) \( \rightarrow \) (**).

To prove that (**) \( \rightarrow \) (***) we start with an exact (1,1)-form \( \omega \) on \( V \). The exactness obviously implies that \( \omega \) vanishes on every \( T^n \)-orbit in \( V \). It follows that the associated quadratic form \( h \) admits a unique decomposition

\[
h = \mathcal{H} + J \tilde{h}
\]

where \( \mathcal{H} \) is induced from a quadratic form \( H \) on \( \mathbb{R}^n \). The equality \( d\omega = 0 \) implies (by a straightforward computation) that

\[
\partial_1 H(\partial_2, \partial_3) = \partial_2 H(\partial_1, \partial_3)
\]

for all parallel fields \( \partial_1, \partial_2 \) and \( \partial_3 \) on \( \mathbb{R}^n \). Hence \( H = D^2f \) for some function on \( \mathbb{R}^n \) which is convex as \( \omega \) is positive.

Hence, (***) does follow from (**).
1.6. **Mixed volumes.** Let us use notation \( y^I \) for the monomial \( y_1^{i_1} \ldots y_k^{i_k} \), where \( y = (y_1, \ldots, y_k) \) is a string of variables and \( I = (i_1, \ldots, i_k) \in \mathbb{Z}_+^k \) denotes the multiindex with non-negative integer entries. We denote by \( \text{III} = i_1 + \ldots + i_k \) the degree of \( y^I \) and observe that the monomials \( \{ y^I \}_{\text{III}=n} \) constitute a basis in the space of polynomials in \( y_1, \ldots, y_k \) of degree \( n \). Next, we write \( I y = i_1 y_1 + \ldots + i_k y_k \) and observe that (by a trivial argument) the polynomials \( \{ (I y)^n \}_{\text{III}=n} \) also constitute a basis in the space of polynomials. In other words every monomial is a linear combination of some \( I y \) with universal coefficients. For example,

\[
y_1 y_2 = \frac{1}{2}((y_1 + y_2)^2 - y_1^2 - y_2^2).
\]

Similarly, for strings of differential 2-forms, \( \Omega = (\omega_1, \ldots, \omega_k) \) we write

\[
\Omega^I = \omega_1^{i_1} \wedge \ldots \wedge \omega_k^{i_k}
\]

and we are interested in the integrals \( \int_V \Omega^I \), where \( \dim V = 2n = 2\text{III} \).

We note that every such integral is a linear combination of the integrals

\[
\int_V (I \Omega)^n \text{ for } I \Omega = i_1 \omega_1 + \ldots + i_k \omega_k,
\]

with the above universal coefficients.

Now, let \( V = \mathbb{R}^n \times T^n \) and \( \omega_1, \ldots, \omega_k \) correspond to convex bodies \( Y_1, \ldots, Y_k \). Namely \( \omega_j = d\text{sd} f_j \) for \( j = 1, \ldots, k \) where \( f \) is a smooth \( T^n \)-invariant function on \( V \), such that the corresponding function on \( \mathbb{R}^n \) is convex and \( D_{f_j}(\mathbb{R}^n) = Y_j \) for \( j = 1, \ldots, k \).

1.6.A. **Proposition-Definition.** The integral \( \int_V \Omega^I \), where \( \text{III}=n \) only depends on \( Y = (Y_1, \ldots, Y_k) \) but not on a choice of the functions \( f_j \). This integral is called the \( I \)th mixed volume of \( Y_1, \ldots, Y_k \), and denoted \( [y^I] = \left[ \begin{array}{ccc} Y_1 \ldots Y_k \end{array} \right] \).

**Proof.** By the previous discussion each mixed volume is a universal linear combination of the volumes of the Minkowski sums \( IY = i_1 Y_1 + \ldots + i_k Y_k \) where \( iX \) denotes \( X+X+\ldots+X \).

1.6.A. **Remark.** As it is clear from this definition, the volume \( \text{Vol}(IY) \) expands in the
usual way into the sum of mixed volumes. For example,

\[ \text{Vol}(Y_1+Y_2) = [(Y_1+Y_2)^n] = \sum_{i=0}^{n} b_i \left[ Y_1^n Y_2^{n-i} \right], \]

where \( b_i = \frac{n!}{i!(n-i)!} \).

In order to state the Alexandrov-Fenchel inequality concerning mixed volumes we need the following notion of convexity for real functions on the discrete simplex

\[ \Delta_n^k - 1 = \{ \mathbf{l} \in \mathbb{Z}_+^k | \|\mathbf{l}\| = n \} \subseteq \mathbb{R}^k. \]

In other words, \( \Delta_n^k - 1 \) is the set of multiindices \( (i_1, \ldots, i_k) \) with \( i_1 + \ldots + i_k = n \). We say that a function \( l(\mathbf{l}) \) is \( l \)-concave on \( \Delta_n^k - 1 \) if the restriction of \( l \) to every line parallel to one of the edges of \( \Delta_n^k - 1 \) is concave. For example, if \( k = 1 \), then this is the usual concavity,

\[ l(\sum_\mathbf{v} a_\mathbf{v} I_\mathbf{v}) \geq \sum_\mathbf{v} a_\mathbf{v} l(I_\mathbf{v}) \]

for \( I_\mathbf{v} = (\mathbf{v}, n-\mathbf{v}) \in \Delta_n^1 \) and all those convex combinations, where \( \sum_\mathbf{v} a_\mathbf{v} I_\mathbf{v} \) lies in \( \Delta_n^1 \) (i.e., is integral).

In general, \( \Delta_n^k - 1 \) has \( \frac{k(k-1)}{2} \) edges. A line parallel to an edge is given by fixing \( k-2 \) (out of \( k \)) coordinates \( (i_1, \ldots, i_k) \). For example, a line parallel to the "first" edge is given by fixing the last \( (k-2) \) coordinates \( i_3, i_4, \ldots, i_k \). If \( i_3 + i_4 + \ldots + i_k = m \leq n \), then this line is \( \{ \mathbf{v}, n-m-\mathbf{v}, i_3, \ldots, i_k \} \) for \( \mathbf{v} = 0, 1, \ldots, n-m \), and the 1-concavity condition on this line amounts to the above (7).

1.6.B. **Alexandrov-Fenchel theorem.** Let \( Y = (Y_1, \ldots, Y_k) \) be a sequence of convex bounded open subsets in \( \mathbb{R}^n \). Then the mixed volumes \( [Y^1] \) for \( \|\mathbf{ll}\| = n \) are positive and the function

\[ l_Y(\mathbf{l}) = \log [Y^\mathbf{l}] \]
is 1-concave.

Remark and corollaries. 1.6. B1. The mixed volume $[Y^i]$ for $I = (i_1, \ldots, i_k)$ is bounded from below by the following weighted product of the volumes of $Y_1, \ldots, Y_k$.

$$[Y^i] \geq \frac{i_1}{n} (\text{Vol} Y_1)^{\frac{i_1}{n}}, \ldots, (\text{Vol} Y_k)^{\frac{i_k}{n}}. \quad (8)$$

In fact, every 1-concave function $l(I)$ (obviously) satisfies $l(I) \geq \frac{i_1}{n} l(n, 0, \ldots, 0) + \frac{i_2}{n} l(0, n, 0, \ldots, 0) + \frac{i_k}{n} l(0, \ldots, 0, n)$.

1.6.B2. Inequality (8) for $k=2$ reads

$$\left[ Y_1^i, Y_2^{n-i} \right] \geq (\text{Vol} Y_1)^{\frac{i}{n}} (\text{Vol} Y_2)^{\frac{n-i}{n}}$$

which implies the Brunn-Minkowski inequality as

$$\text{Vol} (Y_1 + Y_2) = \sum_i \left[ Y_1^i Y_2^{n-i} \right].$$

1.6.B3. I do not know if $l(Y(I))$ is a concave function in $I$.

1.6.B4. We shall prove the 1-concavity of $\log(|K^I|)$ along with the following

1.6.C. Alexandrov-Fenchel inequality on compact manifolds. Recall that the mixed volume $[Y^i]$ is defined as the integral $\int V^i \Omega^i$, where the string of 2-forms,

$$\Omega = (\omega_1, \ldots, \omega_k)$$

on $V = \mathbb{R}^n \times T^n$, corresponds to convex sets $Y_1, \ldots, Y_k$ in $\mathbb{R}^n$.

Thus the Alexandrov-Fenchel theorem amounts to 1-concavity of $\log |\Omega^I|$ for exact positive $(1,1)$-forms $\omega_1, \ldots, \omega_k$ on $V$. This is proven in §2 where we start with the following result concerning compact manifolds $V$.

1.6.C1. Theorem. Let $V$ be a compact complex manifold. Then for every sequence of strictly positive closed $(1,1)$-forms $\Omega = (\omega_1, \ldots, \omega_k)$ on $V$ the function
\[
\log \int_V \Omega^I, \quad \text{for } |I| = \dim_{\mathbb{C}} V
\]

is 1-concave.

Remarks 1.6.C2. This theorem does not directly imply Alexandrov-Fenchel inequality since the manifold \(\mathbb{R}^n \times \mathbb{T}^a\) is non-compact. Yet this manifold can be approximated in a certain way by compact manifolds (this idea is due to Teissier and Hovanski) and then Alexandrov-Fenchel inequality reduces to the compact case.

1.6.C3. If \(V\) is an algebraic manifold then 1.6.C1 is equivalent to the following property of the index of the intersection of divisors on \(V\), denoted \(D_1^{i_1} \cap \ldots \cap D_k^{i_k}\) for \(i_1 + \ldots + i_k = \dim_{\mathbb{C}} V\), where \(D_i^{i_i}\) stands for \(D_1 \cap \ldots \cap D_i\).

Teissier-Hovanski theorem. If the divisors \(D_1, \ldots, D_k\) are ample then the function

\[
\log \left( D_1^{i_1} \cap \ldots \cap D_k^{i_k} \right)
\]

is 1-concave in \(I = (i_1, \ldots, i_k)\).

Note that this result was proven by Teissier and Hovanski over an arbitrary ground field.

§2. The proof of Alexandrov-Fenchel inequality.

2.1. Algebraic inequality of Alexandrov. Every 2n-linear antisymmetric form \(\Lambda\) on \(\mathbb{C}^n\) is proportional to the standard oriented volume form \(\Lambda_0\) on \(\mathbb{C}^n\), say \(\Lambda = \lambda \Lambda_0\) and we set \(|\Lambda| = \lambda\).

2.1.A. Alexandrov's Lemma. Let \(\omega_0\) be a positive (1,1)-form, let \(\Omega\) be the exterior product of (n-2) positive (1,1)-forms and let a (1,1)-form \(\omega\) satisfy

\[
\Omega \wedge \omega_0 \wedge \omega = 0.
\]

Then
\[ [\Omega \wedge \omega \wedge \omega] \leq 0. \]

**Proof.** First, let \( n = 2 \) and use the fact that \( \omega_0 \) and \( \omega \) can be simultaneously diagonalized (as the corresponding quadratic forms \( \omega_0(x, Jz) \) and \( \omega(x, Jz) \) can be diagonalized) by a complex linear transformation of \( \mathbb{C}^n \). Such a transformation makes

\[ \omega_0 = a_0 \, dx_1 \wedge dy_1 + b_0 \, dx_2 \wedge dy_2 \]

and

\[ \omega = a \, dx_1 \wedge dy_1 + b \, dx_2 \wedge dy_2. \]

Then the relation

\[ \omega_0 \wedge \omega = a_0 \, b + a \, b_0 = 0 \]

implies that

\[ [\omega_0 \wedge \omega_0] = ab \leq 0 \]

as \( a_0 \geq 0 \) and \( b_0 \geq 0 \) (because of the positivity of \( \omega_0 \)).

Next, assume the lemma is true for some \( n \geq 2 \) and pass to \( n+1 \) in two steps.

**Step 1.** Let \( \Omega = \omega' \wedge \Omega \), where \( \omega' \) is a **monomial** \((1,1)\)-form, that is \( \omega' \) induced from a \((1,1)\)-form on \( \mathbb{C}^1 \) by a \( \mathbb{C} \)-linear map \( l : \mathbb{C}^{n+1} \to \mathbb{C}^1 \). Denote by \( L \subset \mathbb{C}^{n+1} \) the kernel of \( l \) and assume without loss of generality that \( \dim_{\mathbb{C}} L = n \) (i.e. \( \omega' \neq 0 \)). Then for every \( J \)-invariant \((2n-2)\)-form, say, \( \Omega' \), the exterior product \( \omega' \wedge \Omega' \) has the same sign as the restriction of \( \Omega' \) to \( L \). (This becomes more obvious if \( \omega' \) is transformed by a \( \mathbb{C} \)-linear automorphism of \( \mathbb{C}^{n+1} \) to the form \( dx_{n+1} \wedge dy_{n+1} \) where \( L = \mathbb{C}^n \subset \mathbb{C}^{n+1} \)). Hence,

\[ \Omega \wedge \omega_0 \wedge \omega = 0 \iff \Omega' \wedge \omega_0 \wedge \omega \mid L = 0 \]

and

\[ [\Omega \wedge \omega \wedge \omega] \leq 0 \iff [\Omega' \wedge \omega \wedge \omega \mid L] \leq 0. \]
Thus the lemma for $\Omega = \Omega' \land \omega'$ follows from the inductive assumption applied to the forms $\Omega'$, $\omega$, and $\omega_0$ restricted to $L = \mathbb{C}^n$.

Step 2. We need the following

2.1. A1. **Trivial Lemma.** Let $A$ be a linear function on the space of antisymmetric two forms on $\mathbb{C}^{n+1}$. Then every positive (1,1)-form $\omega'$ can be decomposed into a sum of positive monomial forms,

$$\omega' = \sum_{i=1}^{k} \omega'_i$$

such that

$$A(\omega'_i) = A(\omega') \text{ for } i = 1, \ldots, k.$$

**Proof.** Since $\omega'$ is $I$-invariant, the kernel $K$ of $\omega'$ is a $\mathbb{C}$-linear subspace in $\mathbb{C}^{n+1}$ of dimension $\dim K = 2(n+1)-\text{rank } \omega'$. Then for every $\mathbb{C}$- hyperplane $L \ni K$ there exists a unique positive monomial form $\omega'_L$ such that $\text{Ker } \omega'_L = L$ and $\omega'' = \omega' - \omega'_L$ is positive of rank by one less than that of $\omega'$. (The reader who feels more comfortable with quadratic forms will see this by looking at $\omega' (z, I z)$ which is a positive semidefinite quadratic form). Then by induction on $k = 1/2 \text{rank } \omega'$, one sees that $\omega' = \sum_{i=1}^{k} \omega'_L$ i for some hyperplanes in $L_i \ni K$. Since

$$A\omega' = \sum_{i=1}^{k} A(\omega'_L)$$

the function $A(\omega') - A(\omega'_L)$ necessarily changes sign as $L$ runs over all hyperplanes containing $K$ and so that function vanishes for some $L$. This gives

$$\omega' = \omega'' + \omega'_L$$

where $A(\omega'') = A(\omega'_L) = A(\omega')$ and the proof is concluded by induction on $k = 1/2 \text{rank } \omega'$. (One can also derive the lemma from Kakutani's theorem which says that every continuous function on $S^n$ is constant on some orthonormal $n$-frame).
Now we can complete the inductive step \((n) \rightarrow (n+1)\) by writing \(\Omega = \omega' \wedge \Omega'\) and by observing that
\[
\omega' \rightarrow [\Omega \wedge \omega_0 \wedge \omega]
\]
is a linear function on the space of forms. Then the trivial lemma allows a monomial decomposition \(\omega' = \sum_{i=1}^{k} \omega'_i\) such that
\[
\omega'_i \wedge \Omega' \wedge \omega_0 \wedge \omega = 0
\]
for all \(i = 1, \ldots, k\) and by applying Step 1 to all \(\omega'_i\) we obtain
\[
[\Omega \wedge \omega_0 \wedge \omega] = \sum_{i=1}^{k} [\omega'_{i} \wedge \Omega' \wedge \omega_0 \wedge \omega] \leq 0.
\]
Q.E.D.

2.1. A2. Remark. Observe that
\[
\Omega(\omega, \omega) = [\Omega \wedge \omega_0 \wedge \omega]_{\text{def}}
\]
is a quadratic form on the space of \((1,1)\)-forms \(\omega\). Then the condition (1) of the lemma claims \(\Omega\)-orthogonality between \(\omega_0\) and \(\omega\). Note also that \(\Omega\) is (non-strictly) positive on positive forms as is seen by taking monomial decomposition of all forms in question. In particular,
\[
\Omega \wedge \omega_0 \wedge \omega_0 \geq 0
\]
and the Lemma states that \(\Omega\) is negative on the orthogonal complement of \(\omega_0\). This implies by elementary theory of quadratic form, that
\[
[\Omega \wedge \omega_0 \wedge \omega_1]^2 \geq [\Omega \wedge \omega_0 \wedge \omega_0] [\Omega \wedge \omega_1 \wedge \omega_1] \quad (**)
\]
for all \((1, 1)\)-forms \(\omega_1\).
2.1.B. Corollary. Let \( \omega_1, \ldots, \omega_k \) be positive \((1,1)\)-forms on \( \mathbb{C}^n \) and let \( I = (i_1, \ldots, i_k) \) with \( i_1 + \cdots + i_k = n \). Then the function

\[
I(I) = \log \left( \frac{\omega_1^{i_1} \wedge \cdots \wedge \omega_k^{i_k}}{\Omega} \right)
\]

is 1-concave.

Proof. Let

\[
\Omega = \omega_1^{i_1-1} \wedge \omega_2^{i_2-1} \wedge \cdots \wedge \omega_k^{i_k-1}
\]

write

\[
\omega_1^{i_1} \wedge \cdots \wedge \omega_k^{i_k} = \Omega \wedge \omega_1 \wedge \omega_2
\]

apply (**) and take logarithms. Then one sees the 1-concavity along the "first" edge in \( \Delta_{k-1} \) (compare 1.6.A1) and the rest of edges are taken care of in the same way. Q.E.D.

Remarks 2.1.B1. It is probably not hard to figure out whether the above \( I(I) \) is concave.

2.1.B2. One can generalize the proof of 2.1.A. Using more general Hodge bilinear relations as the base of induction. This gives, for example, the following result.

Let \( \omega \) be a \((k,k)\) form on \( \mathbb{C}^n \) (i.e. \( \omega \) is a \( J \)-invariant \(2k\)-form), \( \Omega \) be the exterior product of \( l \) positive \((1,1)\)-forms, such that \( k+1 = n \), and let \( \omega_0 \) be a strictly positive \((1,1)\)-form. Then the equality

\[
\Omega \wedge \omega_0 \wedge \omega = 0
\]

implies that

\[
(-1)^k \left[ \Omega \wedge \omega \wedge \omega \right] \geq 0.
\]

2.1.B3. Let \( \omega_1, \omega_2, \ldots, \omega_n \) be diagonal \((1,1)\)-forms,
\[ \omega_i = \sum_{j=1}^{n} a_{ij} \, dx_i \wedge dy_j. \]

Then \( [\omega_1 \wedge \ldots \wedge \omega_n] \) equals the permanent of the matrix \((a_{ij})\) that is the sum of the same products of \(a_{ij}\) which go into the determinant, but now all appearing with the positive sign. Alexandrov's lemma for diagonal forms was recently used to prove the following:

**Van der Waerden Conjecture.** If \(a_{ij} \geq 0\) then the permanent \(\text{Per}(a_{ij})\) is bounded from below in terms of \(p_i = \sum_{j=1}^{n} a_{ij}\) and \(q_j = \sum_{i=1}^{n} a_{ij}\)

as follows

\[ \text{Per}(a_{ij}) \geq n^{-n} n! \prod_{i=1}^{n} p_i q_i. \]

2.2. **Laplace equation.** Let \((V,J)\) be an almost complex manifold, that is \(J\) is an automorphism of the tangent bundle of \(V\), such that \(J^2 = -\text{Id}\). We extend the notions from §1.5. to \((V,J)\) in an obvious way and then for every \((2n-2)\)-form \(\Omega\) on \(V\) for \(2n = \dim V\), we assign the following differential operator of second order from functions on \(V\) to \(2n\)-forms,

\[ f \to \Delta f = \delta(\Omega \wedge Jdf). \]

Note that if \(\Omega\) is closed, then

\[ \Delta f = \Omega \wedge dJdf. \]

2.2.A. Call \(\Omega\) strictly positive if \([\Omega \wedge \omega] > 0\) for all non-vanishing positive \((1,1)\)-forms \(\omega\) on \(V\). For example, an exterior product of strictly positive \((1,1)\)-forms is strictly positive.

2.2.A. **Lemma.** If \(\Omega\) is strictly positive then \(\Delta\) is a positive elliptic operator. In particular, if \(V\) is compact then the kernel of \(\Delta\) consists of constant functions only.

**Proof.** If \(f\) has compact support, then

\[ \int f \Delta f = \int \Omega \wedge Jdf \wedge df > 0 \]
for \( f \neq \text{const} \) as \( \text{Jac} \wedge \text{df} \) is a positive \((1,1)\)-form. This shows \( \Delta \) is positive and hence elliptic.

2.2.A. Corollary. If \( V \) is compact and \( \Omega \) is strictly positive then the image of \( \Delta \) consists of all exact forms on \( V \).

Proof. In fact, the index of \( \Delta \) is zero since it acts on functions and so

\[
\dim \text{Coker} \, \Delta = \dim \text{Ker} \, \Delta = 1.
\]

Q.E.D.

2.3. Alexandrov-Fenchel on compact manifolds \( V \) (see 1.6.C). Let \( (V,J) \) be a compact complex manifold, which signifies local isomorphism between \( (V,J) \) and \((\mathbb{R}^n, J=\sqrt{-1})\). The relevant property is that \( \text{Jac} \) is a \((1,1)\)-form for all functions \( f \) on \( V \).

2.3.A. Let \( \omega_0 \) be a closed strictly positive \((1,1)\)-form on \( V \) and \( \Omega \) the exterior product of \((n-2)\) closed strictly positive \((1,1)\)-forms. Then the equality

\[
\int_V \Omega \wedge \omega_0 = 0
\]

implies that

\[
\int_V \Omega \wedge \omega \leq 0
\]

for all closed \((1,1)\)-forms \( \omega \) on \( V \), provided \( V \) is connected.

Proof. The integral does not change if we replace \( \omega \) by \( \omega + \text{Jac} \) for all functions \( f \) on \( V \). By 2.2.A and connectedness of \( V \), there exists an \( f_0 \), such that

\[
\Omega \wedge \omega_0 \wedge \text{Jac}f_0 = -\Omega \wedge \omega_0 \wedge \omega,
\]

and so we may assume that instead of (1) the pointwise relation

\[
\Omega \wedge \omega_0 \wedge \omega = 0
\]
holds true on \( V \), as we can switch to \( \omega + \text{d}J\text{d}f_\omega \). Then by Alexandrov's lemma 2.1.A.,

\[
\Omega^2 \wedge \omega \wedge \omega \leq 0
\]
everywhere on \( V \) and (*) follows by integration.

2.3.B. Now 1.6.C1 follows from 2.3.A. by the (trivial) argument used in 2.1.B.

2.3.C. Remark. The above 2.3.A. remains true for non-strictly positive forms if \( V \) is Kähler, which means the existence of at least one strictly positive closed (1,1)-form on \( V \) called a Kähler form. In such a case every positive form \( \omega \) can be approximated by a strictly positive form, that is

\[
\omega_\varepsilon = \omega + \varepsilon \text{ (Kähler form)}.
\]

2.3.D. Singular varieties. Let \( V \) be a compact complex space with singularities. Then the above result extends to \( V \) with the following convention. A differential form on \( V \) means a form \( \omega \) on the non-singular locus \( V_{\text{reg}} \subseteq V \), such that the following regularity condition is satisfied. For every complex manifold \( U \) and every holomorphic map \( \alpha : U \to V \), the form \( \alpha^* (\omega) \) on \( \alpha^{-1} (V_{\text{reg}}) \) extends to a smooth form on all of \( U \).

There are two approaches to singular spaces \( V \). First, one can use Hironaka theorem, which claims the existence of a compact complex manifold \( U \) of dimension \( n = \dim V \) which admits a holomorphic map onto \( V \). If \( V \) is irreducible (i.e. \( V_{\text{reg}} \) is connected) then \( U \) can be assumed connected and we can apply 2.3.C. if \( U \) is Kähler. Such a \( U \) exists by Hironaka theorem if, for example, \( V \) admits a holomorphic embedding into a Kähler manifold. In particular, this applies to projective algebraic varieties.

Another approach consists in extending 2.2.A2 to the open manifold \( V_{\text{reg}} \) by using appropriate boundary conditions. This eventually leads to slightly more general results and will be discussed somewhere else.

2.3.E. Subvarieties in Kähler manifolds. A Riemannian metric \( h \) on a complex manifold \( (W,J) \) is called Kähler, if the associated 2-form \( \omega \) for

\[
\omega(x, y) = h(x, -Jy)
\]
is a Kähler form, that is \( \omega \) a closed (1,1)-form. If \( V \) is an \( n \)-dimensional complex
subvariety in $W$, then the $2n$-dimensional volume of $V$ equals the integral of $\omega^n$ over the non-singular part of $V$,

$$\text{Vol}_h V = \int_{V_{\text{reg}}} \omega^n.$$ 

In fact, $\omega^n|_{V_{\text{reg}}}$ obviously equals the Riemannian $2n$-volume form at every point of $V_{\text{reg}}$ and the singular part $V_{\text{sing}} = V - V_{\text{reg}}$ has zero $2n$-dimensional volume, as it is a variety of dimension $2n-2$. Now let $V$ be a subvariety in a Cartesian product of Kähler manifolds

$$V \subset W = W_1 \times \ldots \times W_k$$

where $W_i = (W_i, \omega_i)$ and where $W$ is given the product metric. Then the corresponding $\omega$ on $W$ is $\bar{\omega}_1 + \ldots + \bar{\omega}_k$, where $\bar{\omega}_i$ denotes the pull-back of $\omega_i$ for the projection $W \to W_i$. Thus

$$\text{Vol}_V = \int_{V} (\bar{\omega}_1 + \ldots + \bar{\omega}_k)^n$$

and so our inequalities can be interpreted in terms of $\text{Vol}_V$ and

$$\text{Vol}_V = \int_{V} \omega_1 = \int_{V_i} \bar{\omega}_i,$$

where $V_i$ is the projection of $V$ to $W_i$. In particular, we can reduce inequality (+) in 0.2.A. to 1.6.C.1. since the standard metric in the projective space is Kähler (see below).

2.4. Toral varieties. Let us describe the canonical metric on the complex projective space $\mathbb{P}^N$, restricted to $\mathbb{C}^N = \mathbb{P}^N \setminus \mathbb{P}^{N-1}$. Denote by $\|w\|$ the norm $\left( \sum_{i=1}^{N} \bar{w}_i w_i \right)^{1/2}$ on $\mathbb{C}^N$ and let $h = d\text{Id} \log(1 + \|w\|^2)$. Then one easily checks that $\omega$ smoothly extends to $\mathbb{P}^N \supset \mathbb{C}^N$ and is a $U(N+1)$-invariant Kähler form on $\mathbb{P}^N$, where the associated quadratic form $h$ is called the Fubini-Study metric on $\mathbb{P}^N$. Note that $h$ restricts to the standard metric of constant curvature on the real projective space $\mathbb{P}^N_{\mathbb{R}} \subset \mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$.

For a covector $y \in (\mathbb{R}^n)^*$, that is a linear function $y : \mathbb{R}^n \to \mathbb{R}$, and all $x = x' + \sqrt{-1} \ x'' \in \mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1} \ \mathbb{R}^n$ we set $\langle x, y \rangle = y(x') + \sqrt{-1} \ y(x'')$. Then we take $N$ covectors $y_1, \ldots, y_N$ and consider the $\mathbb{C}$-span of the functions $\varphi_i(x) = \exp \langle x, y_i \rangle$, $i = 1, \ldots, N$. This is
the N-dimensional complex linear space, say \( \mathbb{C}^N \), and \( \mathbb{C}^n \) acts on this \( \mathbb{C}^N \) by translation, \( \varphi(x) \to \varphi(x + x_0) \) for all \( x_0 \in \mathbb{C}^n \). Note that this action is diagonal for the basis \( \exp<\cdot, \cdot> \) in \( \mathbb{C}^N \).

Next we pull-back the above \( \omega \) and \( h \) on \( \mathbb{C}^N \) to the \( \mathbb{C}^n \)-orbit of \( \varphi(x) = \sum_{i=1}^{N} a_i \exp<\cdot, \cdot> \) and observe the following.

**Obvious property.** If the coefficients \( a_i \) are real then the pulled back forms \( \omega \) and \( h \) are \( (\sqrt{-1} \otimes \mathbb{R}^n) \)-invariant on \( \mathbb{C}^N = \mathbb{R}^n \oplus \sqrt{-1} \otimes \mathbb{R}^n \) and the induced Riemannian metric \( h \) on \( \mathbb{R}^n \) equals the Hessian \( D^2f \) for the function

\[
\text{f(x) = log(1 + \sum_{i=1}^{N} (a_i \exp<\cdot, \cdot>)^2) = log(1 + \sum_{i=1}^{N} a_i^2 \exp 2 <\cdot, \cdot>).}
\]

Furthermore, \( f(x) \) is a convex function (see 1.4.C2) and if none of \( a_i \) is zero, then the closure of the image \( Df(\mathbb{R}^n) \subset (\mathbb{R}^n)^* \) equals the convex hull of the covectors

\[
0, y_1, y_2, \ldots y_N \text{ in } (\mathbb{R}^n)^*.
\]

In what follows we assume for simplicity's sake that the covectors \( y_i \) linearly span \( (\mathbb{R}^n)^* \) and we call the system \( \{y_1, \ldots, y_N\} \) rational if the group \( A' \) generated by \( y_i \), that is

\[
A' = \left\{ \sum_{i=1}^{N} m_i y_i \mid m_i \in \mathbb{Z} \right\}
\]

is discrete in \( (\mathbb{R}^n)^* \). Then we denote by \( A \subset \sqrt{-1} \otimes \mathbb{R}^n \subset \mathbb{C}^n = \mathbb{R}^n \oplus \sqrt{-1} \otimes \mathbb{R}^n \) the 2π-dual group,

\[
x \in A \iff <x, a> \in 2\pi \sqrt{-1} \mathbb{Z},
\]

which is in the rational case is an n-dimensional lattice (isomorphic to \( \mathbb{Z}^n \)) \( \im \sqrt{-1} \otimes \mathbb{R}^n \), whose action on \( \mathbb{C}^N \) is trivial. Thus our action of \( \mathbb{C}^n \) factors through that of the group

\[
(\mathbb{C}^n)^n = \mathbb{R}^n \oplus \mathbb{C}^n / A.
\]

Then we use the following elementary fact (see 3.3.).

*For every linear diagonal action of \( (\mathbb{C}^n)^n \) on \( \mathbb{C}^N \subset \mathbb{P}^N \) the closure in \( \mathbb{P}^N \) of every*
orbit is an irreducible complex algebraic variety in $\mathbb{P}^N$.

It is also easy to show in our case that for a generic orbit this closure is a non-singular variety, provided the convex hull $Q$ of the set $\{o, y_1, \ldots, y_N\} \subseteq \mathbb{R}^n$ is a simple polyhedron which means at every vertex $p$ of $Q$ exactly $n+1$ faces of codimension one come together. For example, if $N = n$ then $Q$ is a simplex (which is a simple polyhedron) and our variety is isomorphic (by a trivial argument) to $\mathbb{P}^n$.

Now we combine the above observations with the earlier discussion and arrive at the following.

**Conclusion.** The Alexandrov-Fenchel inequality holds true for rational polyhedra $Q_1, \ldots, Q_k$ where $Q$ is called rational if the system of vertices of $Q$ is rational. Then this inequality extends to all convex sets $Y$ as every $Y$ can be (obviously) approximated by rational polyhedra.

**Remarks 2.4.A.** Instead of the approximation one can prove Alexandrov-Fenchel inequality more directly by extending to $\mathcal{F}^n/V$ the argument we used for compact manifolds $V$. In fact, this is exactly what Alexandrov does in [Al]$_2$ where he works with supporting functions viewed as functions on $S^{n-1}$ to which he applies the elliptic theory. On the other hand, Alexandrov’s proof in [Al]$_1$ first applies to simple polyhedra and then the general case follows by approximation.

2.4.B. The algebraic inequality indicated in 2.1.B$_2$ leads by a usual Hodge-theoretic argument to the following.

**Theorem.** Let $\omega_0$ be a Kähler form on a compact variety $V$ and $\Omega$ be the exterior product of $k$ Kähler forms. Let $\omega$ be a closed $(k,k)$-form, where $k+l = \dim \mathbb{C} V$, such that the form $\Omega \wedge \omega_0 \wedge \omega$ is exact. Then

$$(-1)^k \int_V \Omega \wedge \omega_0 \wedge \omega \geq 0.$$ 

I do not know if this result has any significance for convex sets.

2.4.C. **Open questions.** Let $V$ be a complex manifold with an action of a complex Lie group $G$, such that $V/G$ is compact. Then one asks if the Alexandrov-Fenchel inequality holds true for $K$-invariant closed positive $(1,1)$-forms on $V$, where $K$ is the maximal compact subgroup in $G$. If $G = (\mathbb{C}^*)^p$, then we have already proven this in two extreme
cases, where $G = V$ or on the contrary, $n = 0$. The general case here looks easy. Next interesting case is where $G$ is semisimple and $V = G$.

Looking into another direction we forget about complex structures and take a $C^\infty$-manifold $V$ with a symmetric affine connection. Then for each (convex) function $f$ on $V$ one has (positive definite) Hessian $h = D^2f$ and one may speak of the total volume

$$ [h^n] \overset{\text{def}}{=} \int_V \text{Det}(h) $$

which is just the total volume of the (Riemannian) manifold $(V, h)$. Similarly one defines the mixed volumes $[h_{1, \ldots, k}^{n_1, \ldots, n_k}]$ and asks about possible inequalities between these. For example, one asks these questions for the hyperbolic space $V = \mathbb{H}^n$ with the Levi-Civita connection.

§3. Miscellany.

3.1. Brunn-Minkowski for non-convex sets. To avoid irrelevant complications we consider the case where $X$ and $Y$ are bounded open sub sets in $\mathbb{R}^n$ with smooth boundaries.

3.1.A. Main lemma. There exists a bijective map $f : X \rightarrow Y$ with the following two properties

(i) $f$ is almost everywhere differentiable and the Jacobian of $f$ is a.e. constant.

(ii) The Jacobian matrix of $f$ (that is the differential $Df$ in the standard basis of $\mathbb{R}^n$) is triangular with positive eigenvalue almost everywhere on $X$.

Proof: The claim is obvious for $n=1$, where condition (ii) just requires $f$ to be monotone increasing. In fact, for any two absolutely continuous measures $\mu$ and $\nu$ on $\mathbb{R}^1$ there exists a (essentially) unique monotone increasing map $f_1$ sending $\mu \rightarrow \nu$.

Now, for $n \geq 2$, we denote by $p : \mathbb{R}^n \rightarrow \mathbb{R}^1$ the projection on the first coordinate line and we apply the above remark to the measures $\mu$ and $\nu$ which are the push-forwards of the measures of $X$ and $Y$. (Namely, $\mu(I) = \text{Vol}(p^{-1}(I) \cap X)$ and $\nu(I) = \text{Vol}(p^{-1}(I) \cap Y)$ for all $I$ in $\mathbb{R}^1$). Thus we obtain a map $f_1$ on $\mathbb{R}^1$ and then, by induction on $n$, we construct maps with properties (i) and (ii) between the intersections of $X$ and $Y$ with the hyperplanes $p^{-1}(t)$.
\[ p^{-1}(t) \cap X \rightarrow p^{-1}(f_1(t)) \cap Y \]

for all \( t \in p(X) \subset \mathbb{R}^1 \). The collection of these is the required map \( f : X \rightarrow Y \).

3.1.A. Remark. The map \( f \) is itself "triangular" in the obvious sense with respect to the (sequence of) partitions of \( \mathbb{R}^n \) into subspaces of dimensions \((n-1), (n-2), \ldots, 1\), parallel to the standard flag of linear subspaces in \( \mathbb{R}^n \). It is also clear that the map \( f \) satisfying (i) and (ii) is a.e. unique.

3.1.B. The proof of the inequality

\[
(Vol (X + Y))^{1/n} \geq (Vol X)^{1/n} + (Vol Y)^{1/n}.
\]

Observe (using induction on \( n \) as earlier) that the map \( g = \text{Id} + f : X \rightarrow \mathbb{R}^n \) for the above \( f \) is a.e. injective and the image, say \( Z \subseteq \mathbb{R}^n \), of this map (obviously) is contained in \( X + Y \).

The volume of \( Z \) equals the integral of the Jacobian of \( g \),

\[ Vol Z = \int_X \text{Det} \,(1+Df) \]

and because of (ii) this Jacobian satisfies

\[ (\text{Det} \,(1+Df))^{1/n} \geq 1 + (\text{Det} \,Df)^{1/n}. \]

Next we see with (i) that

\[ \text{Det} \,Df = C = \text{Vol} \,Y/Vol \,X, \]

and so

\[
(Vol Z)^{1/n} \geq \left[ \frac{1}{X} \left( 1 + C^{1/n} \right) \right]^{1/n} = (Vol X)^{1/n} + (Vol Y)^{1/n},
\]

Q.E.D.

3.1.C. Remarks. The above argument is standard (compare [MS]) and it applies to all
solvable Lie groups $G$ in place of $\mathbb{R}^n$. This yields the inequality

$$(\text{Vol } (X+Y))^{1/k} \geq (\text{Vol } X)^{1/k} + (\text{Vol } Y)^{1/k}$$

(*)

for the codimension $k$ of the maximal compact subgroup $K \subset G$.

Note that (*) is sharp only for Abelian groups $G$ and the best bound

$$\text{Vol } (X_1 + X_2) \geq B_G(V_1, V_2),$$

for $V_i = \text{Vol } X_i$, $i = 1, 2$, is unknown for non-Abelian (solvable and unsolvable) groups $G$. Some information for $G = 0(n+1)$ and $0(n,1)$ is provided by the classical isoperimetric inequality for $S^n$ and $H^n$ which concerns $0(n)$-invariant subsets in $0(n+1)$ and $0(n,1)$.

3.1.D. Here is a Kähler version of 3.1.A.

Let $\omega$ be a Kähler form on a compact manifold $V$ and $\Omega$ be a positive $2n$-form for $n = \dim G$, such that $\Omega - \omega^n$ is exact. Then there exists a function $f$ on $V$, such that

$$(\omega + df)^n = \Omega.$$ 

This is the celebrated Calabi conjecture solved by Yau. Using this result one obtains an alternative proof of the Brunn-Minkowski inequality on Kähler manifolds.

3.2. Gradient actions and the moment map. Consider a manifold $V$ with a bilinear form $g$ which is viewed as a homomorphism between tangent and cotangent bundles of $V$,

$$g : T(V) \to T^*(V).$$

A vector field $\partial$ on $V$ is called $g$-gradient (or Hamiltonian) if

$$g(\partial) = df$$

for some function $f$ on $V$, called the potential (or Hamiltonian) of $f$.

Next, an action of a Lie algebra $L$ on $V$ is called gradient (Hamiltonian) if all vector fields $\partial$ on $V$ constituting the action are gradient. Then the potentials of these fields form a
map of \( V \) into the linear space of linear functions on \( L \).

\[
f : V \to L',
\]
such that

\[
d(f(\vartheta)) = g(\vartheta) \tag{*}
\]

for all fields \( \vartheta \) in \( L \).

**Remarks on terminology.** The words "gradient" and "potential" are customary used if \( g \) is a symmetric form while "Hamiltonian" refers to an antisymmetric form. In the first case the above \( f \) is called the gradient map and in the second the moment map.

3.2A. Example. Consider the standard action of the Abelian algebra \( L = \mathbb{R}^n \) on \( \mathbb{R}^n \) and let this action be gradient for some symmetric form \( g \) on \( \mathbb{R}^n \). Then (*) implies that \( dg(\vartheta) = 0 \), which is equivalent to the symmetry of the (full) differential \( Dg \) defined by

\[
Dg(\vartheta_1, \vartheta_2, \vartheta_3) = \vartheta_1 g(\vartheta_2, \vartheta_3),
\]

for all parallel fields \( \vartheta_i = 1, 2, 3 \) on \( \mathbb{R}^n \). Now we observe that the symmetry of \( Dg \) that is

\[
\vartheta_1 g(\vartheta_2, \vartheta_3) = \vartheta_2 g(\vartheta_1, \vartheta_2),
\]

implies that \( g \) equals the Hessian \( D^2P \) of some function \( P \) on \( V \),

\[
g(\vartheta_1, \vartheta_2) = \vartheta_1(\vartheta_2 P).
\]

In this case, the gradient map of the action equals the gradient (or Legendre) map of \( P \),

\[
v \to d_v P \in L', \text{ for all } v \in \mathbb{R}^n.
\]

In particular, we see with Legendre theorem (see 1.2.) that if \( g \) is positive definite, then the image of the gradient map is convex.

A similar convexity property is satisfied by a gradient action of an Abelian algebra \( L \) on every compact connected Riemannian manifold. In this case, the action integrates to that of the Abelian group, that is \( \mathbb{R}^n \), and the gradient map \( f : V \to L' \) obviously satisfies the following two properties.
(i) Monotonicity of on the orbits (compare §1). Let \( l \in \Lambda \setminus \{0\} \) and let \( l_t(v) \in V \) for \( t \in \mathbb{R} \) be the orbit of some \( v \in V \) under the one parameter group \( l_t \) corresponding to \( l \). Then the function \( \alpha_t(l,v) = \langle l_t(v), l \rangle \) is monotone increasing in \( t \), where \( \langle l, l \rangle \) stands for \( \ell(l) \). In particular, \( \alpha_t(l,v) \) converges for \( t \to +\infty \).

(ii) The limit \( \alpha_t(l,v) = \lim_{t \to +\infty} \alpha_t(l,v) \) is constant on every \( \mathbb{R}^n \)-orbit in \( V \) for all \( l \in \Lambda \).

These two properties easily yield the convexity of the image \( f(V) \) in \( L' = \mathbb{R}^n \) (see [At1] and [At2]).

**Example.** The standard (diagonal) action of \( (\mathbb{R}^+)^n = \mathbb{R}^n \setminus \{0\} \) on the real projective space \( \mathbb{P}^n_{\mathbb{R}} \) (as well as on \( \mathbb{P}^n_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{R}} \)) is gradient for the standard metric in \( \mathbb{P}^n_{\mathbb{R}} \) and \( f \) sends \( \mathbb{P}^n_{\mathbb{R}} \) onto a simplex in \( L' = \mathbb{R}^n \) (compare 2.4.).

3.2.B. **Remark.** The above action can be described as follows. Take \( n+1 \) points \( y_0, \ldots, y_n \) in \( L' \) and let \( P^n_{\mathbb{R}} \) be identified with the projectivisation of the space \( \mathbb{R}^{n+1} \) of the maps \( \varphi: \{y_0, \ldots, y_n\} \to \mathbb{R} \). Then \( \mathbb{R}^n \) acts on this \( \mathbb{R}^{n+1} \) and hence on \( P^n_{\mathbb{R}} \) by

\[ \varphi(y) \to \varphi(y) \exp\langle x, y \rangle \]

for all \( x \in \mathbb{R}^n \) and \( f(x) \in L' = (\mathbb{R}^n)' \) equals the center of gravity of the points \( \{y_i\}_{i=0}^{n} \) with masses \( \varphi^2(y_i) \exp\langle x, y_i \rangle \) attached to them. This agrees with the discussion in 1.4.C2 which applies to an arbitrary probability measure \( \mu \) with compact support in \( L' = \mathbb{R}^n \), and where \( \mathbb{R}^n \) acts on \( \mu \) by

\[ \mu \to x \cdot (\mu) = \mu' |x| (L') \]

for \( \mu' = \mu \exp\langle x, \cdot \rangle \) for all \( x \in \mathbb{R}^n \). Then \( x \) is sent to \( f(x) \in L' \) which is the center of gravity of \( x(\mu) \) (which is the same as the center of \( \mu' \)). We have seen in §1 that the map \( x \to f(x) \) sends \( \mathbb{R}^n \) onto the interior of the convex hull of the support of \( \mu \), provided this support spans \( L' \). It is also not hard to see that \( f \) continuously extends to the closure of the \( \mathbb{R}^n \)-orbit \( \{x(\mu)\}_{x \in \mathbb{R}^n} \) in the space of probability measures on \( L' \) with the weak topology. One can show (we leave this to the reader) that \( f \) homeomorphically maps this closure onto the convex hull of the support of \( \mu \).
To grasp the geometry of the map $x \to f(x)$ it is useful to replace $\exp$ by the step function. Now, to every half-space $H \subset L'$ which intersects the support of $\mu$ we assign the measure

$$H(\mu) = \chi(H) \mu(\mu(H)),$$

where $\chi$ is the characteristic function of $H$. Here one sees more clearly the structure of the weak closure of the measures $H(\mu)$ as well as of the map $H \to \text{(center of gravity of } H(\mu)).$ As for the above $\exp$-case, here the picture is especially simple if the convex hull of the support of $\mu$ is strictly convex. (This never happens for finite measures, but these are easy anyway).

3.2.C. Convex maps. The convexity of the image is often accompanied by the following stronger property.

**Definition.** A continuous map $f: V \to \mathbb{R}^n$ is called convex if it satisfies the following three conditions:

(i) the image $f(V) \subset \mathbb{R}^n$ is convex;
(ii) the pull-back $f^{-1}(x) \subset V$ is connected or empty;
(iii) for every open subset $U \subset V$ the image $f(U)$ is an open subset of $f(V)$ with the topology induced from $\mathbb{R}^n \supset f(V)$.

3.2.C1. Remarks. (a) The conditions (i) and (ii) are equivalent to the following

(i') For every convex subset $X \subset \mathbb{R}^n$ the pull-back $f^{-1}(x) \subset V$ is connected or empty.

(b) Property (iii) is satisfied by every embedding $V \to \mathbb{R}^n$. In this case (i) and (ii) just say that the image is convex.

Next, call $f$ locally convex if every point in $V$ admits a neighborhood $U \subset V$ such that $f$ restricts to a convex map $U \to \mathbb{R}^n$.

Now we invoke the following well known

3.2.C2. Lemma. If $V$ is a compact connected space then every locally convex map $V \to \mathbb{R}^n$ is convex.

One sees the idea by looking at the case where $V \hookrightarrow \mathbb{R}^n$ is an embedding and the convexity of $V \subset \mathbb{R}^n$ is obtained by showing that the shortest curve in $V$ between any two points necessarily is a straight segment since the local convexity does not allow such a
curve to touch the boundary of \( V \) from inside.

3.2.C.3 Remark. The Lemma remains valid for maps into spaces more general than \( \mathbb{R}^n \) which possess a good notion of convexity. For example, one can replace \( \mathbb{R}^n \) by a complete simply connected Riemannian manifold with non-positive curvature (This is an exercise to the reader).

3.2.D. Convexity of the moment maps. Let \( \omega \) be a nonsingular antisymmetric 2-form on \( V \) and let the Abelian Lie algebra \( L = \mathbb{R}^m \) act on \( V \), such that the following four conditions are satisfied

(i) The action preserves \( \omega \).
(ii) The action is Hamiltonian (i.e. \( \omega \)-gradient).
(iii) If a vector field \( \partial \in L \) on \( V \) vanishes at some point \( v \in V \), then the differential of \( \partial \) on the tangent space \( T_v(V) \), say \( D_\partial : T_v \to T_v \) is an operator diagonalizable over \( \mathbb{C} \) with purely imaginary eigenvalues.

(iii)' The zero set of \( \partial \) is a smooth submanifold in \( V \) whose dimension at \( v \) equals that of the zero eigenspace of \( D_\partial \).

If \( V \) is compact and connected, then the moment map (i.e. the \( \omega \)-gradient map) \( f : V \to L^* \) is convex.

This is proven by checking the local convexity of \( f \), where the local geometry of the action is seen by looking at linear Hamiltonian actions on \( V = \mathbb{R}^{2m} \) with \( \omega = \sum_{i=1}^{m} dx_i \wedge dy_i \). Details of the proof are left to the reader.

3.2.D.1. Remarks (a). The most (if not the only) interesting case of the above convexity (due to Atiyah [At] and Guillemin-Sternberg [G-S] is where \( \omega \) is a closed (and hence symplectic) form and where the action of \( L \) integrates to an action of a compact torus \( T^n \) on \( V \) (or to a non-compact subgroup in such a torus). Note that for closed forms \( \omega \) condition (i), that is \( \partial \omega = 0 \) for \( \partial \in L \), is equivalent to \( d(\omega(\partial)) = 0 \), and so (ii) \( \Rightarrow \) (i). The opposite implication (i) \( \Rightarrow \) (ii) also holds true in many cases, for example if \( V \) is simply connected or at least \( H^1(V, \mathbb{R}) = 0 \). Also note that (iii) and (iii)' are obviously satisfied for actions coming from (compact !) tori \( T^n \).

(b) If \( \omega \) is a Kähler form on a complex manifold \((V,J)\) and the action of \( L \) preserves \( J \), then \( JL \) also acts on \( V \) and preserves \( J \). It is also clear that the action of \( L \) is \( \omega \)-gradient if and only if the action of \( JL \) is \( h \)-gradient for the quadratic form \( h(\partial_1, \partial_2) = \omega \).
(\partial_1, \partial_2).

(c) If \((V, \omega)\) is a symplectic manifold with a Hamiltonian \(T^n\)-action then the moment map \(f : V \to \mathbb{R}^n\) pushes forward the measure (associated to) \(\omega^m\), for \(2m = \dim V\), to a measure on the image \(f(V)\) with \textit{piecewise} polynomial density, provided the moment map is \textit{proper} (see [D-H] and [At1]). In the special case of \(n = m\) this polynomial is constant and so the volume of \((V, \omega^m)\) equals, up to a universal constant, to that of the image of the moment map. For the standard action of \(T^1\) on the sphere \(S^2\) this result goes back to Archimedes (this remark I owe to Michael Atiyah) who proved that the orthogonal projection \(S^2 \to \mathbb{R}^n\) sends the spherical measure to \(2\pi\) (Lebesgue measure) on the segment \([-1, 1] \subset \mathbb{R}\).

3.3. \textit{Recollection on algebraic sets}. Here we give basic definitions and state some elementary properties of algebraic varieties.

3.3.A. A subset \(V \subset \mathbb{C}^N\) is called (complex) \textit{algebraic} if there exists a complex polynomial map \(p : \mathbb{C}^N \to \mathbb{C}^m\) for some \(m\), such that

\[ V = p^{-1}(0). \]

A point \(v_0 \in V\) is called \textit{regular} if there exists a polynomial map \(p_0 : \mathbb{C}^N \to \mathbb{C}^{m_0}\) of rank \(m_0\) at \(v_0\), such that \(p_0^{-1}(0)\) equals \(V\) in a small neighborhood in \(\mathbb{C}^N\) around \(v_0\). The set of regular points \(V_{\text{reg}} \subset V\) obviously is a smooth manifold of (real) dimension \(2n\) at \(v_0\), for \(n = N - m_0\), where \(n\) is called the \textit{complex dimension} of \(V\). Then an easy argument shows that the singular part

\[ V_{\text{sing}} = V \setminus V_{\text{reg}} \]

is an algebraic set and

\[ \dim_{\mathbb{C}} V_{\text{sing}} < \dim_{\mathbb{C}} V_{\text{reg}} \quad \text{dim}_{\mathbb{C}} V_{\text{reg}}. \]

One says that \(V\) is \textit{reducible} if it is a union \(V = V_1 \cup V_2\), where \(V_1\) and \(V_2\) are algebraic subsets both different from \(V\). Otherwise \(V\) is called \textit{irreducible}. It is easy to see that \(V\) is irreducible if and only if \(V_{\text{reg}}\) is connected.

3.3.B. Let us observe an obvious correspondence between \textit{cones} in \(\mathbb{C}^{N+1}\) and subsets in \(\mathbb{P}^N\), where a subset in \(\mathbb{C}^n\) is called a cone if it is a union of complex lines through the
origin. Call a subset $V \subset \mathbb{P}^N$ algebraic if the corresponding cone $CV \subset \mathbb{C}^{N+1}$ is algebraic. Then one extends the above definitions from $\mathbb{C}^{N+1}$ to $\mathbb{P}^N$ in an obvious way (Warning: $V$ in $\mathbb{P}^N$ is non-singular if the cone $CV$ is non-singular away from the origin. In fact if some cone is non-singular at the origin, then it is a linear subspace).

Similarly one defines algebraic subsets in products of projective spaces $\mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_k}$ using $k$-cones in $\mathbb{C}^{N_1+1} \times \ldots \times \mathbb{C}^{N_k+1}$ which are unions of Cartesian products of lines in $\mathbb{C}^{N_i+1}$. It is not hard to show that the projection of an (irreducible) algebraic subset from $\mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_k}$ to $\mathbb{P}^{N_i}$ an (irreducible) algebraic subset $\mathbb{P}^{N_i}$.

3.3.C. For an $n$-dimensional subset $V \subset \mathbb{P}^N$, define $\deg V$ as the maximal (possible) number of points in the intersections of $V$ with $(N-n)$-dimensional projective subspaces in $\mathbb{P}^N$. If $V$ is algebraic, then the above number of intersection points is constant and equals the degree for almost all $(N-n)$-dimensional subspaces. Namely, the subset of exceptional subspaces (where the number of intersection points less than the degree) is algebraic in the Grassmann $\text{Gr}_{N,n} \mathbb{P}^N$ of all $(N-n)$-subspaces, where one naturally define the notion of an algebraic subset.

If $V$ is irreducible, one knows that the top-dimensional homology group $H^{2n}(V)$ is free cyclic and the index of the image of $H^{2n}(V)$ in $H^{2n}(\mathbb{P}^N)$ (which is also free cyclic) equals $\deg V$. Then in the reducible case one sees that there are no more than $\deg V$ of top-dimensional irreducible components in $V$ as their degrees add up to $\deg V$.

3.3.D. The space $\mathbb{P}^N$ carries a natural complex structure and one constructs many Kähler forms on $\mathbb{P}^N$ as follows. Let $\mu$ be a smooth positive measure on the (dual) projective space of hyperplanes in $\mathbb{P}^N$, that is $\text{Gr}_{N-1} \mathbb{P}^N$. Then (by an easy argument) there exists a unique smooth 2-form $\omega = \omega_\mu$ on $\mathbb{P}^N$ such that for all oriented surfaces $S \subset \mathbb{P}^N$ the integral $\int_S \omega$ equals the $\mu$-average of the algebraic (i.e. counted with the sign defined by the orientations) intersection number of $S$ with the hyperplanes $H \in \text{Gr} = \text{Gr}_{N-1} \mathbb{P}^N$,

$$\int_{\text{Gr}} (S \cap H) d\mu.$$

It is not hard to show that such an $\omega$ is Kähler and if $\mu$ is normalized by $\mu(\text{Gr}) = 1$, then

$$\int_V \omega^n \overset{\text{def}}{=} \int_{\text{reg}} \omega^n = \deg V,$$
for all algebraic subsets \( V \subset \mathbb{P}^N \) which agrees with the previous homological definition of \( \deg V \). Another easy fact (valid for all Kähler forms) is

\[
\int_V \omega^n = \text{Vol} V
\]

where "Vol" stands for the 2n-dimensional volume for the Riemannian metric \( h \) associated to \( \omega \), that is \( h(\partial_1, \partial_2) = \omega(\partial_1, \partial_2) \).

Note that \( h = h_\mu \) is \( \text{U}(N+1) \)-invariant on \( \mathbb{P}^N \) if and only if the measure \( \mu \) on \( \text{Gr}_{N,1} \mathbb{P}^N \) is \( \text{U}(N+1) \)-invariant.

3.3.E. Let \( V \subset \mathbb{P}^N \times \mathbb{P}^N \) be an irreducible subset of \( \dim \, V = 2 \)

and recall Hodge's inequality

\[
\left( \int_V \omega_1 \wedge \omega_2 \right)^2 \geq \int_V \omega_1^2 \int_V \omega_2^2
\]

\((*)\)

where \( \omega_1 \) and \( \omega_2 \) are the pull-backs of the standard Kähler forms on \( \mathbb{P}^N \times \mathbb{P}^N \). This inequality imposes a restriction on the homology class

\[
[V] \in H^2(\mathbb{P}^N \times \mathbb{P}^N) = \mathbb{Z}^3.
\]

which can be equivalently expressed in terms of the intersection numbers of \( V \) with pull-backs to \( \mathbb{P}^N \times \mathbb{P}^N \) of hyperplanes in \( \mathbb{P}^N \times \mathbb{P}^N \),

\[
(V \cap H_1 \cap H_2)^2 \geq (V \cap H_1^2) \times (V \cap H_2^2),
\]

\((**)\)

where \( H_i^2 \) for \( i = 1, 2 \) stands for the intersection of \( H_i \) and the hypersurface \( H_i^1 \) obtained from \( H_i \) by a generic (holomorphic) motion of \( \mathbb{P}^N \times \mathbb{P}^N \).

Next consider \( V \subset \mathbb{P}^N \times \mathbb{P}^N \times Q \) where

\[
Q = \mathbb{P}^{N_3} \times \ldots \times \mathbb{P}^{N_k}
\]

and let \( H \subset \mathbb{P}^N \times \mathbb{P}^N \times Q \) be the intersection of the pull-backs of projective subspaces...
in \( P^3, P^4, \ldots, P^k \), in general positions, such that the codimensions of these subspaces add up to \( n-2 \) (i.e. \( \text{codim}_P H = n-2 \)) for \( n = \dim_C V \). Now Tschirn-Hovanski theorem can be expressed by the following inequality

\[
(V \cap H \cap H_1 \cap H_2)^2 \geq (V \cap H \cap H_1^2) \times (V \cap H \cap H_2^2)
\] (\( \text{***} \))

for all irreducible \( V \).

**Proof.** By Bertini theorem the intersection \( V \cap H \) is an irreducible variety for generic \( H \) and thus (\( \text{***} \)) follows from (**) applied to the projection of \( V \cap H \) to \( P^1 \times P^2 \).

Note that (\( \text{***} \)) applied to Abelian varieties \( V \) yields Alexandrov's lemma (see 2.1.) and that Alexandrov's proof in [Al] is similar in spirit to the above use of Bertini's theorem.

3.3.E. In order to clarify the relation between (\( \text{***} \)) and the corresponding Kahlerian inequality (see 1.6.C.) we recall the following fundamental theorem of Lefschetz-Kodaira.

Let \( \omega \) be a Kahler form on compact complex \( n \)-dimensional manifold \( V \), such that \( \int_S \omega \) is an integer for all closed oriented (real) surfaces \( S \) in \( V \). Then there exists a holomorphic embedding \( i: V \to \mathbb{P}^N \) for \( N = 2n+1 \), such that the image \( i(V) \) is an algebraic subvariety and the homology class of \( [V \cap H] \subset H^{n-2}(V) \) for a generic hyperplane \( H \subset \mathbb{P}^N \) is the Poincaré dual of the class \( [\omega] \in H_2(V) \).

That is

\[
V \cap H \cap S = \int_S \omega
\]

for all closed oriented surfaces \( S \) in \( V \).

3.3.F. Consider a finite subset of multi-indices \( I \subset \mathbb{Z}_+^n \) and let \( x^I = x_1^{i_1} \cdots x_n^{i_n} \) for \( x = (x_1, \ldots, x_n) \) and \( I = (i_1, \ldots, i_n) \) be the monomials for all \( I \in I \), which are viewed as \( \mathbb{C} \)-valued functions on \( \mathbb{C}^n \). All these together define a polynomial map, say \( \alpha: \mathbb{C}^n \to \mathbb{C}^{N+1} \) where \( N+1 \) is the number of elements in \( I \). By multiplying each entry \( x_i \) of \( x^I \) by a non-zero complex number we obtain a natural action of \( (\mathbb{C}^*)^n \) on monomials as well as a
monomorphism

\[ \beta = \beta_I : (\mathbb{C}^x)^n \to (\mathbb{C}^x)^{N+1} \]

such that the map \( \alpha \) is equivariant with respect to \( \beta \) for the standard (diagonal) actions of \((\mathbb{C}^x)^n\) on \(\mathbb{C}^n\) and of \((\mathbb{C}^x)^{N+1}\) on \(\mathbb{C}^{N+1}\).

Next we take away zero from the image \( \alpha(\mathbb{C}^n) \subset \mathbb{C}^{N+1} \) and project \( \alpha(\mathbb{C}^n) \backslash \{0\} \) from \(\mathbb{C}^{N+1} \backslash \{0\}\) to \(\mathbb{P}^N\). One knows (for all polynomial maps \( \alpha \)) that the topological closure \( V \) of the image of this projection is an irreducible subvariety in \(\mathbb{P}^N\) and it is quite easy to see that \( \dim_{\mathbb{R}} V \) equals the dimension of the convex hull \( \bar{I} \subset \mathbb{R}^n \) of \( I \subset Z_+ \subset \mathbb{R}^n \).

The previous equivariance discussion shows that \( V \) is a toral variety and the moment map restricted to \( V_0 \subset V_{reg} \subset V \), where the pertinent action of \( T^n \subset (\mathbb{C}^x)^n \) is free, is a fibration of \( V_0 \) over the interior of the convex hull \( \bar{I} \) assuming \( \dim I = n \). Then by Archimedes theorem \( \text{Vol} V = \int_0^n = n! \text{Vol} \bar{I} \) and so \( \text{deg} V \) also equals \( n!\text{Vol} \bar{I} \).

3.3.F1. Recall that \( \text{deg} V \) equals the number of intersection points of \( V \) with \( n \) hyperplanes in general positions,

\[ V \cap H_1 \cap H_2 \cap \ldots \cap H_n \]  

and observe that every hyperplane \( H_i \) is given by \( l_i = 0 \) for a linear function \( l_i \) on \( \mathbb{C}^{N+1} \) whose zero set is the cone over \( H_i \). Then we observe that the functions \( p_i = l_i \circ \alpha \), \( i = 1, \ldots, n \), are polynomials on \( \mathbb{C}^n \) which are linear combinations of monomials \( x^l \) and that the intersection points in \((+)\) correspond to common zeros of \( p_i \). Thus we arrive at Kushnirenko's theorem equating the number of common zeros of \( p_i \) (which are generic linear combinations of \( x^l \) for \( I \in I \)) to \( n!\text{Vol} I \).

3.3.F. Let \( V_{\mathbb{R}} \) be the real locus of the above \( V \). Now the intersection number in \((+)\) varies as we vary the real hyperplanes \( H_i \), but yet the average number of points in \((+)\) equals the (properly normalized) volume of \( V_{\mathbb{R}} \) by the standard integral geometry. An explicit formula for length of \( V_{\mathbb{R}} \) for \( n = 1 \) and \( I = \{0,1,\ldots,d\} \subset Z_+ \) can be extracted from formulas in Example 2 of Ch.1 in [Ka] and a similar computation can be made for corresponding subsets \( I \subset Z_+^n \). But in general one does not know the behaviour of \( \text{deg} V \) and \( \text{Vol} V \) for \( V_{\mathbb{R}} \) (compare [Ho]). It is worth noticing that the imaginary part of \( V \), (that is the \( T^n \)-orbit instead of the \( (\mathbb{R}^+)^n \)-orbit) has simpler Riemannian geometry as the action of \( T^n \) is isometric on \( \mathbb{P}^N \). In particular, the average number of zeros is easier to compute for
trigonometric polynomials.

3.3. F₁. The above map φ on ℝⁿ has many amusing properties besides sheer size. Take for example the curve

\[(t, t^2, t^3, \ldots, t^{2m}) \subset ℝ^{2m} \text{ for } t \in ℝ\]

(which is accidentally called the moment curve) let F be a finite subset in this curve and \(\bar{F}\) the convex hull of F. Then (by an easy argument) for every subset \(F \subset \bar{F}\) containing m points or less, the convex hull \(\bar{F}'\) lies in the boundary of \(\bar{F}\). For example, if \(m = 2\), then \(\bar{F}\) is 4-dimensional and the segment between any two vertices in \(\bar{F}\) is an edge in \(\bar{F}\). There is no such convex polyhedron in ℝ³ apart from the simplex and with \(\bar{F}\) one arrives at a counter-example for \(n = 4\) (This example has been appearing in literature every other year since the last century).

REFERENCES


[G-S] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping,


