Five Mathematical Ideas, their Growth, Ramifications and Interconnections Unedited

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Contents

1 Lecture 1: Seeds of Mathematics in the 20th Century Soil: from Pythagorus to Hilbert and Beyond

Much of mathematics springs from a few miraculous seeds-theorems-ideas sown in the soil of the 20 century concepts : spaces, maps, symmetries, where the consistency of these concepts is as improbable as the the properties of the seeds planted in it.

(This is reminiscent of how the propagation of life on the surface of the Earth depends on the fertility of the soil, where this fertility would be impossible if not for the fixation of the free nitrogen from the air and which is possible due to the inequality $2237 < 2346$ in the energy balance of the synthesis of ammonia:

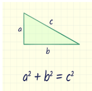
$$3[\mathrm{H}\overset{432}{-}\mathrm{H}] + [\mathrm{N}\overset{941}{\equiv}\mathrm{N}] \mapsto 2[\mathrm{N}\overset{3\times391}{\equiv}3\mathrm{H}]$$

$$2237 = 3\times432+941 < 2\times3\times391 = 2346 \approx 1.05\times2237.$$

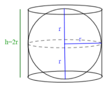
If the energy of the $N-H$ bonds were slightly weaker, say 371 J/mol instead of 391, there would hardly be a chance for Hominidae emerging on Earth: $6\times371 = 2226 < 2237$.)

Here are 5 seeds-theorems-ideas most of mathematics grows from.[1]

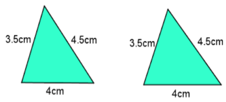
I. Pythagorus (500s BC)[2] : $a^2 + b^2 = c^2$.



II. Archimedes (200s BC): $area(S^2(r)) = 4\pi r^2$.



III. Euclid (300s BC) *Isometric* maps from a subset in the plane, $\mathbb{R}^2 \subset Y \xrightarrow{f_Y} \mathbb{R}^2$, e.g. the vertex set of a triangle, extend to *isometric* maps from the plane, $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$.



IV. Hilbert(1909) generalized Hurwitz' 1908 identity[3]:

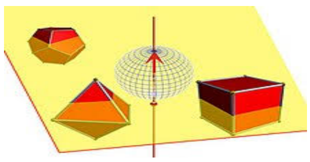
$$5040(a_1^2 + a_2^2 + a_3^2 + a_4^2)^4 =$$

$$= 6\sum_{\pm}(a_1 \pm a_2 \pm a_3 \pm a_4)^8 + \sum_{i>j>k}(2a_i \pm a_j \pm a_k)^8 + 60\sum_{i>j}(a_i \pm a_j) + 6\sum_i (2a_i)^8.$$

V. Borsuk-Ulamn(Lyusternik-Shnirel'man?)(circa 1930): Let $f : S^n \to S^m$ be a continuous map between spheres.

If f is $\pm$-symmetric, i.e $f(-s) = -f(s)$, then $m \geq n$;

if $m = n$ and f is *piecewise smooth* (*C^1-differentiable*), then there exists a point s in the receiving sphere, such that the pullback $f^{-1}(s)$ (in the source sphere S^n) is finite and the number $card(f^{-1}(s))$ is **odd**. Thus *continuous $\pm$-symmetric maps $S^n \to S^n$ are* **onto**.[4]



[1] In the following lectures, where we shall give definitions, precise statements and proofs of everything, which is only briefly indicated in sections 1 and 1.1-1.6 below.

[2] Each minute you inhale millions $N \equiv N$ molecules exhaled by Pythagorus but half of the nitrogen which stays in your body (3% of your body mass), arrived by a chemical route nonexistent in the Pythagorus' times.

[3] `https://www.jstor.org/stable/2370754?seq=10`

[4] Less known to mathematicians is Teller-Ulam implosion design of the thermonuclear bomb.

1.1 Pythagorean World; Hilbert, von Neumann, Atiyah

Pythagorean theorem reconstructs a fragment of Platonic reality by its shadows on walls, where this "fragment" is a rod, a stick, a pencil R floating in the air in the room. This stick casts three shadows S_1, S_2 S_3 on the two walls W_1, W_2 and on the floor W_3 by three sources of lights L_1, L_2, L_3 normal to W_1, W_2, W_3.

If R rotates in the air, the shadows, their lengths and the sum of the lengths change, but, this is supposed to amaze you, the sum of the squares

$$length(S_1)^2 + length(S_2)^2 + length(S_3)^2$$

doesn't change:

by Pythagorean theorem this sum is equal to $2 \cdot length(R)^2$.

Euclidean and Riemannian. A couple of thousand years later (Oresme 1300s, Decartes 1600s, Schläfli 1850s,), the Pythagorean $dist(x,y) = \sqrt{\sum_{i \in I}(x_i - y_i)^2}$ was taken for a defining axiom of the Euclidean space $\mathbb{R}^I$.

Contemporary with Schläfli, Riemann (1854) introduced spaces with "variable Euclidean/Pythagorean" – what we now call *Riemannian*, geometries, where their curvatures – "measures of non-Euclideaness" abided the Pythagorean rule along with all homogeneous quadratic polynomial P on the n-dimensional Euclidean space X:

the sum of values of such a P over n orthonormal vectors in X doesn't depend on which set of such vectors you take.

This generalises the invariance of sums $\sum_{i=1}^n p_i$ under permutations of the group Π_n of the set $\{1, 2, ..., n\}$.

Energy and Probability. Prior to its full geometric interpretation, Pythagorean additivity of squares was seen in every corner of physics as additivity of kinetic energy, e.g. in the *Maxwell distribution* and as the *square root mean square displacement rule* in sums of random variables, where this "rule"

yields the *Law of Large numbers* (Cardano 1500s, Bernoulli1600s)

leads to von Neumann's L_2-proof of the *ergodic theorem.*

configures the random walk theory in science (physics, chemistry, biology, e.g. heat propagation and *Brownian motion*)
as well as in mathematics.

$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{-x^2-y^2} dxdy = \int_0^\infty 2\pi r e^{-r^2} dr = -\pi \int_0^\infty \frac{d}{dr} e^{-r^2} dr = \pi$

Infinite Dimensions and Hilbert Spaces Pythagorean in analyses is demonstrated by the L_2-geometry in function spaces X which, in particular, supports the spectral theory of differential and integral operators (which allows natural realizations of L_2-spaces by l_2-spaces of sequences) and the group representation theory.

Also L_2-estimates for functions, such as the Poincaré's inequalities, provide an essential information on the domains of these functions, such as *measure concentration properties.*

Von Neumann Dimension and Atiyah L_2-index theorem.

Let G be a countable group and X be the l_2-space of functions $\Gamma \to \mathbb{R}^n$.

Then, this depends on the the Pythagorean theorem, there is a well defined dimension function on G-invariant linear subspaces $Y \subset X$, where Y_1 admits an isometric embedding to Y_2 if and only if $dim_{Neum}(Y_1) \leq dim_{Neum}(Y_2)$.

This dimension is also defined on kernels of Γ-invariant elliptic (e.g. Dirac) operators $\mathcal{D}$ on L_2-vector functions on non-compact manifolds acted by Γ, where an evaluation of this dimension via *Atiyah L_2-index theorem* implies the existence of *non-zero L_2-solutions* f of $\mathcal{D}(f) = 0$.

Von Neumann Quantum and von Neumann entropy

Let $P = P(x)$ be a homogeneous quadratic polynomial on the Euclidean space $X = \mathbb{R}^I$ and let us think of P as family of functions or measures $\mu_o = P|E_o$ on the sets (of cardinalities $n = dim(X) = card(I)$) of orthonormal frames $E_o \subset X$, $o \in O^\circ(n)$, in X.

The orthogonal group $O(n)$ naturally simply transitively acts on $O^\circ(n)$, and we think of this action as a "Pythagorization" (quantisation?) of the action of the permutation group Π_n on the set I.

The Pythagorean theorem says that the total *mass* of the measure μ_o, denoted $mass(\mu_o) = trace_{E_o}(P)$, that is a it function on $O^\circ(n)$, is *invariant* under the action of $O(n)$, that is constant in o.

If $P > 0$ (and μ_o are positive measures) and $trace(P) = 1$, then, following physicists, the selfadjoint operator P^*, corresponding to P, i.e. such that $\langle P^*(x), x\rangle = P(x)$, is called a *density state*.

Von Neumann's Definition of Quantum Entropy.(1932) Recall that by the Boltzmann-Planck (circa 1900) and Shannon (1948) formula (sometimes taken for the definition[5] the entropy of a probability measure μ on a countable set I satisfies:

$$ent_{Shan}(\mu) = -\sum_{i\in I} \mu(i) \log \mu(i).$$

Then Von Neumann defined his entropy of P as the ordinary entropy of the spectral measure μ^* of the operator P^*, i. e. where $\mu^*(i)$ is equal the i-th eigenvalue of P^*.

Equivalently, $ent(P) = ent_{Neum}(P)$ can be defined as the *minimum* of $ent_{Shan}(\mu_o)$ over all orthonormal frames o in X.

(The equality of the two, we shall see it later, follows from the Pythagorean theorem.)

$$ent(P_{12}) \le ent(P_1) + ent(P_2),$$

where P_{12} is a density state on the tensor product $X_1 \otimes X_2$ and P_i on X_i, $i = 1, 2$, are natural *reductions* – "quantum shadow" of P_{12} on $X_1$and X_2.

More generally, this is *Lieb-Ruskai's strong subadditivity theorem*

$$ent(P_3) + ent(P_{123}) \le ent(P_{23}) + ent(P_{13})$$

for the reductions of P_{123} on $X_1 \otimes X_2 \otimes X_3$.

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Besides the Pythagorean orthogonal group $O(n)$, that is the group of the linear transformations of the $\mathbb{R}$-linear n-space which preserves the quadratic form $\sum_{i=1}^n x_i^2$, all basic (classical) symmetry groups: *orthogonal, symplectic* and their complex-geometric relatives are defined following Pythagorus dictum via forms of degree 2 on linear spaces.

[5]One doesn't define the area of the disk as the limit of the areas of the regular inscribed n-gons, $n \to \infty$.

1.2 Archimedes Map, Symplectic Geometry and Complex Manifolds

Archimedes map A, that is the normal projection from the unit 2-sphere S^2 onto the segment $[-1,1]$ (positioned on the "vertical' x-axes), is measure preserving for the measure $2\pi dx$. on $[-1,1]$; in fact, the radial map A_+ from the 2-sphere to the cylinder $[-1,1]\times S^2$ has Jacobian $1=\frac{\sin r}{\sin r}$, where $r=r(s)$ is the spherical (angular) distance from $s\in S^2$ to the South pole.

Symplectic interpretation/generalisation of A: forget the metric in S^2, remember only the area form ω and use the fact that ω is invariant under the the circle action, which keeps the poles of the sphere fixed.

This defines A, since the ω-gradient of A equals the vector field of this action[6]

There is a better algebraically described version of the Archimedes map, that is $\mathbb{C}\to\mathbb{R}_+$ for $z\mapsto r=|z|=z\bar{z}$, which is measure preserving for $dz\rightsquigarrow 2\pi dr$.

This, raised to the nth power, $\mathbb{C}^n\to\mathbb{R}^n_+$, is measure preserving for $\prod_i dz_i\rightsquigarrow (2\pi)^n\prod_i dr_i$ and this leads to the measure preserving map A_n from the complex projective space $\mathbb{C}P^n$ to the "probability simplex" $\Delta^n=\{p_i\geq 0\}_{\sum_i p_i=1}\subset\mathbb{R}^{n+1}$, which is *the moment map* for the diagonal action of the n-torus on $\mathbb{C}P^n$ and which is equal to the original Archimedes map for $n=1$.

The real part of A_n, which sends $\mathbb{R}^{n+1}\supset S^n\to\Delta^n$ for

$$S^n\ni(x_0,x_1,...,x_n)\mapsto(p_0=x_0^2,p_1=x_1^2,...,p_n=x_n^2)\in\Delta^n,$$

is also significant. For instance the spherical metric transported by A_n on the simplex Δ^n is equal to the minus Hessian of the entropy function $\sum_i p_i\log p_i$ on Δ^n by the *Fisher entropy formula.* (1922).

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The measure preserving property of the map $\mathbb{C}P^n\to\Delta^n\subset\mathbb{R}^{n+1}$, generalises to all symplectic manifolds (X^{2n},ω) acted by m-tori:

the push forward of the ω^n-measure μ under the moment map $M:X\to\mathbb{R}^m$ is a piecewise polynomial and the Fourier transform of μ can be expressed as a stationary phase approximation at the critical points of M, according to the *Duistermaat-Heckman formula.*

Infinite dimensional moment maps.

Example (Atiyah and Bott 1982): the curvature can be viewed as a moment map for the action of the gauge group on the space of connec- tions of unitary bundles over surfaces.[7]

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Beside its symplectic aspect, the map $\mathbb{C}P^n\to\Delta^n\subset\mathbb{R}^{n+1}$ has a complex analytic one, which allows an uplifting of convexity to pseudo-convexity, as it is illustrated by the following.

Brunn-Minkowski inequality for complex algebraic manifolds. Let $X\subset\mathbb{C}P^{N_1}\times\mathbb{C}P^{N_2}$ be a connected algebraic submanifold of complex dimension n and let $X_i\subset\mathbb{C}P^{N_i}$, $i=1,2$, be the images of X under the coordinate projections $\mathbb{C}P^{N_1}\times\mathbb{C}P^{N_2}\to\mathbb{C}P^{N_1},\mathbb{C}P^{N_2}$.

[6]Miraculously, the fundamental physical forces – gravitational and electric and -magnetic are symplectic ω-gradients for the Louiville form ω on the cotangent bundles of configuration spaces.

[7]*Moment maps in differential geometry(2003) by S.K. Donaldson*

Then

$$vol(X)^{1/n} \geq vol(X_1)^{1/n} + vol(X_2)^{1/n}.$$

1.3 Extension of Isometries and Lipschitz Maps in Euclidean and non-Euclidean Spaces

An essential feature of of the plane $\mathbb{R}^2$, regarded as a metric space, which is shared with many (but not all) spaces, is

1. **Intermediate distance property**. Given points $x_1, x_2 \in \mathbb{R}^2$ and a real number $0 \geq r \leq 1$, there exists a point $x \in \mathbb{R}^2$ such that $dist(x_1, x) = r \cdot dist(x_1, x_2$ and $dist(x, x_2) = (1-r) \cdot dist(x_1, x_2)$.

Exercise: Every metric space X is isometric to a subspace. in a space with the Intermediate distance property.

But the following is highly restrictive and nearly characterises the plane;

2. **Full homogeneity**. Every isometry map from a subset $\mathbb{R}^2 \supset U \to \mathbb{R}^2$ extends to an isometry $\mathbb{R}^2 \supset U \to \mathbb{R}^2$.[8]

There little to add to 1&2 to uniquely characterise the plane.

3. $\mathbf{dim(\mathbb{R}^2) = 2}$. The maximal number of points in the plane with mutually equal distances is 3.

4. **Self-similarity (scaling symmetry) "Fifth postulate"**. There exists a map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$dist(\phi(x_1), \phi(x_2)) = \lambda \cdot dist(x_1, x_2), x_1, x_2 \in \mathbb{R}^2,$$

for some $\lambda \neq 0, 1$.

There are many interesting spaces, where only parts of these properties are satisfied, historically starting with hyperbolic and spherical spaces, many homogeneous spaces and/or self-similar spaces , p-adic spaces, etc.

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Kirszbraun's theorem. Partially defined λ-Lipschitz maps

$$\mathbb{R}^n \supset U \xrightarrow{f} \mathbb{R}^m$$

extend to everywhere defined λ-Lipschitz maps

$$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m.$$

The Kneser-Poulsen monotonicity conjecture for volumes of unions and intersections of balls. If $B_i, B'_i \subset \mathbb{R}^n, i \in I$, are two finite sets of pairwise equal balls with centres $c_i, c'_i \in \mathbb{R}^n$, such that

$$dist(c_i, c_j) \leq dist(c'_i, c'_j) \text{ for all } i, j \in I,$$

Then

$$vol(\bigcup_i B_i) \leq vol(\bigcup_i B'_i) \text{ and } vol(\bigcap_i B_i) \geq vol(\bigcap_i B'_i)$$

[8]Full homogeneity of the physical space allows an unrestrictive freedom of movements of our bodies – one can hardly imagine living in a different kind of space.

Example of Lang-Schröder's Theorem. Let the surface $X_1 \subset \mathbb{R}^{n+1}$ be defined by the equation $x_3 = x_1^2 + x_2^2$ and the surface X_2 by $x_3 = x_1^2 - x_2^2$ and let these surfaces be endowed by the shortest paths metrics on them.

Then partially defined λ-Lipschitz maps $X_1 \supset Y \overset{f}{\to} X_2$ extend to everywhere defined λ-Lipschitz maps $X_1 \overset{F}{\to} X_2$.

Stoker's conjecture. If the corresponding dihedral angles of two combinatorially equivalent convex polyhedra $X, X' \subset \mathbb{R}^n$ satisfy $\angle' \leq \angle$ then all corresponding face angles are mutually equal.

This is proven to be true[9] for many (all?) cases where X' is a curvelinear polyhedron X' such that the codimension 1 faces have $mean.curv \geq 0$.

For instance, if X is a regular n-simplex and X' is a convex curved linear simplex, such that $\angle' \leq \angle$, the Wang-Xie-Yu-Li-Brendle theorem implies that X' is an ordinary (hence regular) simplex, but no simple proof (not using the index theorem for Dirac operators, at least for $N \geq 8$) of this is available.

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Uryson spaces $\mathcal{U}$. These are universal objects in the category of metric spaces. and isometric maps:

isometric embeddings from subspaces

$$X \supset Y \to \mathcal{U}$$

extend to isometric embeddings $X \to \mathcal{U}$ for compact metric spaces X.

In fact, properly understood random metric spaces are Uryson.

1.4 Hilbert's Spherical Designs, Cubature Formulas, Mean Value Theorems, and Curvatures of Immersions

Hilbert's Spherical Design Lemma *allows an imbedding of the n-torus to the Euclidean unit ball, $f_o : \mathbb{T}^n \to B^N(1) \subset \mathbb{R}^N$ for large $N = N(n)$ with curvature $curv(f) \leq \sqrt{3\frac{n}{n+2}}$.*

Petrunin's theorem. *There is no C^2-immersion $f : \mathbb{T}^n \to B^N(1) \subset \mathbb{R}^N$ for all N with curvature everywhere $< \sqrt{3\frac{n}{n+2}}$.*

The relies on nonexistence of Riemannian metrics on tori with $scal.cirv > 0$ and also applies to certain manifolds homeomorphic to spheres, where the proofs are based on the Atiyah-Singer index theorem.

(The real projective space $\mathbb{R}P^n$ imbeds to $B^N(1)$, $N \approx n^2$, with curvature $\sqrt{2\frac{n}{n+1}}$.)

Seymour -Zaslavsky Mean value theorem. *Let $\{f_i(x)\}$, $i \in I$, be a finite set of continous functions on the unit interval $X = [0,1]$, Then there exists a finite subset $\{x_j\} \subset [0,1]$, $j \in J$, such that*

$$\int_X f_i(x)dx = \frac{1}{card(J)} \sum_{j \in J} f_i(x_j) \text{ for all } i \in I.$$

1.5 Combinatorial Applications and Fredholm Generalisations of the Topological Intersection Theorems

Intersections and (self) Linking of submanifolds.

[9]Wang, Xie, Yu, Li, Brendle, whete some proves are long and difficult

Kneser conjecture: *if the n-subsets of a (2n+k)-set are divided into k+1 classes, then two disjoint subsets are contained in the same class.*

Bisection of Necklaces by Alon-West

Every interval n-coloring has a bisection of size at most n.

That is, given continuous functions $f_1(t), ..., f_j(t), , ..f_n(t)$*,* $t \in [0, 1]$*, there exist* n *points* $t_1 < ... < t_i < .. < t_n \in [0, 1]$ *and a partition of the set of the* $n + 1$ *segments* $S_0 = [0, t_1], S_1 = [t_1, t_2], ..., S_n = [t_n, 1]$ *into two subsets , say* I_+ *and* I_-*, such that the integrals of the functions* f_j *over the unions of these intervals, called*

$$S_+ = \bigcup_{i \in I_+} S_i \text{ and } S_- = \bigcup_{i \in I_-} S_i$$

satisfy

$$\int_{S_+} f_j(t)dt = \int_{S_-} f_j(t)dt, j = 1, ..., n.$$

KUKUTANI-YAMABE-YUJOBO'S THEOREM: Let $f : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function.*

Then there exists an orthonormal frame $\{x_i\} \subset \mathbb{R}^n$*,* $i = 1, ..., n$*, such that* $f(x_1) = f(x_2) = ... = f(x_n)$*.*

DYSON'S THEOREM if $n = 2$*, then there exists a pair of unit ortogonal vectors.* x_1, x_2 *such that* $f(x_1) = f(x_2) = f(-x_1) = f(-x_2)$

Nash equilibrium theorem

Dvoretzky's theorem. and counterexamples to the Knaster conjecture

1.6 Spaces, Transformations, Perturbations, Deformations and Approximations Genericity versus Symmetry, Maps and Categories, Linearisation and Unitarization...

Powers and Cartesian products

dimensions finite and infinite

Genericity and Symmetry in Physics

Indistinguishability of different random sequences