Puzzle Corner

Allan J. Gottlieb

The weather here has been very dull lately—no sunshine. Just cloudy, bleak days. With such dull surroundings, I don't feel very creative. My first attempt at an introduction for this column, an extensive account of the miserable weather, now sits in the garbage pail where it belongs. My style is so poor today that I will dispense with any further opening remarks in favor of getting right to the problems.

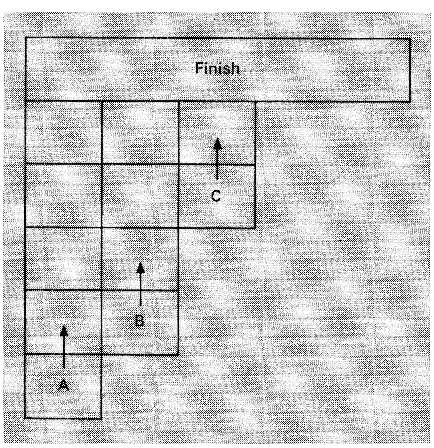
Problems

26 Frank Rubin would like you to find the smallest integers m and n such that $m - n^3$, m, and $m + n^3$ are all perfect squares.

27 Smith D. Turner writes a long explanation which in the end includes an "off-track betting" scheme:

"For betting purposes, a horse race is sometimes simulated by having a number of woolen horses run a course of several moves, the one to move each time being determined by lot. For example, use six horses, throw a die, and the first whose number is thrown (say) 10 times wins. To make this more interesting, I have set it up where one horse must move only a few times, and others increasingly more, to win—thus creating a 'favorite' and 'long shots' in the betting—say 2, 3, 5, 6, 8, and 10 moves with six horses.

"I have found it extremely difficult to calculate the probabilities for such a set-up.



Even with a much simplified race—e.g., three horses having to move 5, 4, and 2 times (below, left)—the calculation was very laborious. In the case illustrated, I get the probability of the favorite C to win as 15001/19683.

"Could anybody check this and—more importantly—come up with a method, computer or otherwise, of handling a more complicated race, say the 2, 3, 5, 6, 8, 10 above?"

The next two problems come from two of my colleagues from M.I.T. student days in Baker House—Andrew Egendorf and George Starkshall. They write:

"In the most recent issue of the *Chinese Journal of Ichthyological Nutrition* we came across the following problem: Give the next number in the series 77, 49, 36, 18, ____. Of course, the answer is trivial. Each number in the series is obtained by multiplying the digits of the preceding number. Thus the next term is 8. This series is unique in that among such series in which the initial term is a two-digit number, it is the only one which contains five terms (before reducing to one digit on the last term). All other such series have at most four terms. The problems:

28 "It is required to show that this series is, indeed, unique (other than by enumerating every series).

29 "Furthermore, given an n-digit number, consider the series formed by multiplying all the digits of that number together, continuing in the same manner as the above series. What is the maximum number of terms in such a series and how many different series will there be with exactly m terms?"

D. Thomas Terwilliger has submitted the following particularly cool problem:

30 "A mathematician moonlighting as a census-taker stops at his friend's house. In this census he is required to obtain the names and ages of all the occupants of the house. After writing down several names and ages the census-taker asks, 'Are there any more people who live here?' His friend replies, 'Yes, there are three more people that live here.' When asked for their ages, the friend reports that the product of the ages is 1296 and the sum is the street number of his house. The census taker makes a few calculations and then says, 'Just tell me one more thing: How many of the three are older than you are?' As soon as his friend replies, the census taker smiles, writes down the ages and leaves. What is the house number?"

Here is a very unusual chess problem from Douglas J. Hoylman, who attributes it to the University of Oklahoma Mathematics Department through his friend Joe Kelley:

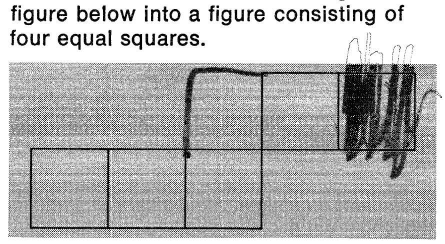
31 The diagram below shows the final position in a chess game in which White has checkmated Black. What was White's last move? (And while you're at it, what was his next-to-last move?)

Black

						WK	
					WP	BK	
						WP	BP
							WP
WP							
	WB						

White

SD12 Russell A. Nahigian wants you to move two matchsticks and change the figure below into a figure consisting of four equal squares.



SD13 Mr. Nahigian also contributes the second speed problem. He writes:

"Three missionaries and three cannibals come to a river. They find a two-man boat, but the missionaries only have time to teach one cannibal how to paddle before the monsoon flood reaches them. All three missionaries can paddle. The missionaries know the cannibals will attack and overcome the missionaries any time when there are more cannibals than missionaries present. Being sagacious, they solve the problem of getting across the river without ever allowing outnumbered missionaries on either shore. How?"

Solutions

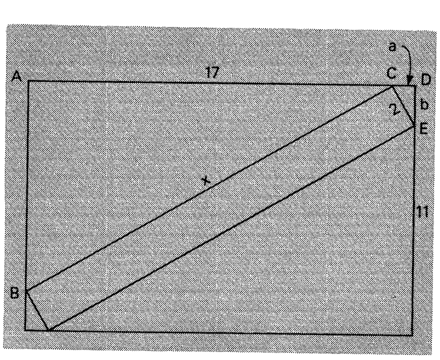
11 Find the range of positive values of x such that, given x, the only positive value of y which satisfies $y^x = x^y$ is the trivial solution $y = x$. For those values of x for which nontrivial solutions for y exist (e.g., $x = 2$, $y = 4$), how many solutions are there for a given x? If a value of x is selected at random from the open interval (o, e), what is the probability that a nontrivial solution for y lies in the same interval?

The following is from Mr. Rubin:
We take the (xy)-th root of both sides to get $x^{1/x} = y^{1/y}$. Consider $f(x) = x^{1/x} = e^{(\log x)/x}$. This is a function which increases from 0 at $x = 0$ and 1 at $x = 1$ to a maximum at $x = e$ and decreases monotonically to 1 as $x \to \infty$ (since $(\log x)/x \to 0$). Thus only the trivial solution exists for $0 < x \leq 1$, exactly one nontrivial solution exists for $x > 1$, and whenever $1 < x < e$ we have $y > e$, so no nontrivial solutions exist with both x and y in (o, e).

Also solved by Mr. Turner, Mr. Hoylman, and the proposer, John E. Prussing.

12 Given a two-foot-wide carpet laid so that all four corners touch the four walls of a room 17 by 11 feet, find the length of the carpet.

Burton Rothleder submits the following:



Triangles ABC and CDE are clearly similar. Labeling the unknown hypotenuse x and unknown sides of CDE a and b, similar triangles yield

$(11 - b)/a = (17 - a)/b = x/2.$
Also, $a^2 + b^2 = 4$ from triangle CDE.
Then
$11b - b^2 = 17a - a^2$
$b^2 = 4 - a^2$
$b = (4 - a^2)^{1/2}$.

Substituting for b and b^2,
$11 (4 - a^2)^{1/2} = - 2a^2 + 17a + 4$
This is a fourth-degree equation which is time-consuming to solve. The drawing was made to scale and shows
$a \sim 1$, $b \sim 1.75$. Using a Wang desk calculator to solve for a and b by trial and error, it only took a few minutes to determine that $a \simeq 1.0027$ and $b \simeq 1.7305$. Thus, $x \simeq 18.4891$.

Also solved by Donald C. Dickson, Mark Yu, William R. Osgood, Richard Joos, and Messrs. Turner, Prussing, and Rubin.

13 Set up all 32 chessmen on the board so that each one has at least six legal moves—except pawns, of course. Bishops belonging to each player must be on different colors. One king may be in check but not both. Pawns may be placed in the first rank but not in the eighth rank.

Everyone found solutions satisfying additional conditions as well. The nicest was by the proposer, Steven Scott, who increases the requirement to seven legal moves from six:

Black

				WB	BQ	WR	BP
WP		WK		WB			WP
BP	BP			WKt			BR
WP			WP	WKt	WP		
BP			BP	BKt	BP		
WP	WP			BKt			WR
BP		BK		BB			BP
				BB	WQ	BR	WP

White

14 Let $n_1 = x^x$, $n_2 = n_1{}^x, \ldots$
$n_i = X^{n_{(i-1)}} = X^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$
and let $N = \lim n_i$ as i grows large beyond bound. Do real values of x exist for all positive N? If so, what is the relationship? If not, what are the limitations?

Several people noted that the problem was stated incorrectly (corrected above) and even figured out the correct question. But only Messrs. Yu and Rubin could solve it. Here is Mr. Yu's solution:

Since $R\{n_i\}$ is a monotonically increasing sequence for all positive $x \neq 1$, it suffices to prove $N < \infty$ in order to assure than $\lim n_i$ converges in R^1. Surely if $x > 1$, $N = \infty$. Otherwise, suppose $x < 1$ so $x = e^{-p}$ (some $p > 0$). By repeated application of the inequality $e^{-y} < 1/(1 + y)$ we have:
$n_i < f(p, i)/[f(p, i) + p] \leq 1$,
where f is a polynomial whose degree is $(i - 2)$ (where $i > 2$) and $f > 1$ (this says $n_i > 1/(1 + p)$. This means that $N = x^N$, or $\log x = (\log N)/N < 0$, since $x < 1$. Hence $N < 1$. The solution to the problem then is that x can be found so that $n_i \to N$ if $D < N \leq 1$ (for $x = 1$, $N = 1$).

15 This is a logical set of numbers, mathematically derived: 10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, 31, 100, ___, 10,000. What is the missing number?

Elliot D. Friedman submits the following "honest" solution:
Since the sequence of numbers is the base ten number 16 expressed in bases ten through two, then the missing number is 121, expressing base ten 16 in base 3. But Mr. Friedman also writes, "I won't claim to be so smart that I worked that one out. The problem appeared at least five years ago in advertisements of Litton Industries. If you aren't aware of that particular source of problems, Litton publishes usually yearly and distributes at the I.E.E.E. show in March booklets called 'Problematical Recreations.' Copies are available from Litton on request."

An amazing letter came from the family of Juan Maran:
"I have been arguing with my wife over the answer to your problem. We both recognize, of course, that the series represents the first 15 most significant (non-one-digit) integers—i.e., 10 = first composite which is unit backwards; 11 = first prime which is itself backwards; 12 = first composite which is composite backwards; 13 = first prime which is larger prime backwards; 14 = first composite which is prime backwards; 15 = first composite whose digit sum is composite; 16 = first square which is prime backwards; 17 = first prime whose digit difference is composite; 20 = first composite which is lesser prime backwards; 22 = first composite which is itself backwards; 24 = first number which read forward or backward is divisible by the same composite; 31 = first prime which is smaller prime backward; and 100 = first square which is unit backward. (10,000, of course, is fascinating—being the first integer that requires a comma.) The missing number, I say, is 121, that being the first square which is itself backwards. I am reinforced by the

observation that the squares (base ten) of the first three nonunites (base two) are 100, 121, and 10,000. My wife says the missing number is 1729, being the first totally uninteresting number and hence quite distinguished. Our son agrees with me about 121, but as he thinks the next number is 1,111,111,111,111,111, I scoff.

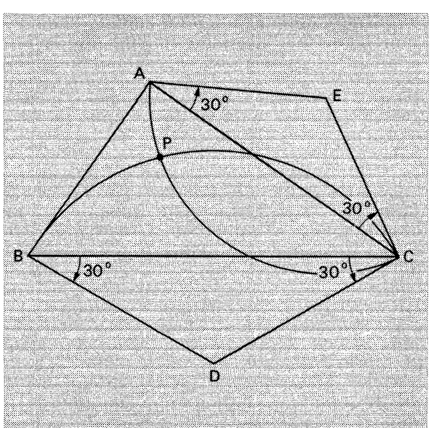
Also solved by Messrs. Hoylman, Yu, Rubin, and Joos and by Thomas Jabine, Charles Russell, F. W. Hawkes, and Thomas Nelson.

Better Late Than Never

Returning to problem 1, published in the October/November, 1968, issue of the *Review,* the following improvement is credited to Russell L. Mallett.

1 Given an arbitrary triangle, find (by geometrical construction) the point such that the sum of the distances to the three vertices is a minimum.

Mr. Mallett writes: "The solution you published for this problem (*see* Technology Review *for February, page 73*) was not a geometric 'construction' at all, since it used limiting processes (impractical ones, at that). My solution was a simple ruler-and-compass construction: Let A, B, and C be the vertices of an arbitrary triangle with largest angle α at A. If $\alpha \geqq 120°$, then the desired point P coincides with A. If $\alpha \geqq 120°$, then P is determined as the intersection of two circular arcs with easily constructed centers D and E as shown.



Although I used a bit of calculus in my original proof, purely geometric arguments can be used to show:
1. P cannot lie outside the triangle.
2. If P lies on the triangle, then $\alpha \geqq 120°$ and P coincides with A
3. If P lies inside the triangle, then AP, BP, and CP make equal angles (120°) with each other and $\alpha < 120°$.
These lead to the given solution.

Mr. Rubin solved **2**, **6**, and **10**. And someone finally solved **4**—Frank Rubin, of course!

4 Consider a sequential gambling system on sequences of Heads and Tails in which the bet at each stage may depend only on the outcomes of the previous events. The gambler has an initial capital w and may never bet more than he currently has. Given capital a after k trials, the gambler may bet any amount b, $0 \leqq b \leqq a$, on either Heads or Tails, and the gambler's fortune at the next stage will then be $a + b$ or $a - b$ accordingly as the $k + 1$st event is correctly guessed or not. There are 2^n possible sequences of Heads and Tails of length n. Let $w_1, w_2, \ldots, w_{2^n}$ be the *corresponding* terminal fortunes (for a given betting scheme). Suppose that a sequence of n bets is to be made. Show that there exists a sequential gambling system on n trials achieving $w_1, w_2, \ldots, w_{2n}$ if and only if $w_i \geqq 0$, all i, and $(1/2)^n \Sigma w_i = w$. Thus, subject to the above rather obvious constraints, every terminal distribution on the gamber's fortune is achievable.

Mr. Rubin writes: "I introduce the following improved (?) notation: 0 = Head, 1 = Tail. $W(i_1, i_2, \ldots, i_n)$ is the wealth desired after the sequence $i_1, i_2, \ldots, i_n$ where each i is 0 or 1. Let $W(i_1, \ldots, i_x)$ be the amount to be had after $i_1, \ldots, i_x$, $0 \leqq k \leqq n$, and $B(i_1, \ldots, i_k)$ the amount bet after $i_1, \ldots, i_k$ on the occurrence of tails. Then we have

$W(i_1, \ldots, i_k, 0) = W(i_1, \ldots, i_k) - B(i_1, \ldots, i_k)$

$W(i_1, \ldots, i_k, 1) = W(i_1, \ldots, i_k) + B(i_1, \ldots, i_k)$

which solves to

$W(i_1, \ldots, i_k) = 1/2\,[W(i_1, \ldots, i_k, 1) + W(i_1, \ldots, i_k, 0)]$ (1)

$B(i_1, \ldots, i_k) = 1/2\,[W(i_1, \ldots, i_k, 1) - W(k_1, \ldots, i_k\, 0)]$ (2)

From the first relation we obtain by induction, working backwards from n,

$$W(i_1, \ldots, i_k) = 1/2^{n-k} \sum_{i_{n^3}=0}^{1} \cdots \sum_{i_{k+1}=0}^{1} W(i, \ldots, i_n) \quad (3)$$

from which the bets at any stage can be determined from (2). I note that for $k = 0$, the relation (3) is just $W = 1/2^n\, \Sigma\, W(i_1, \ldots, i_n)$, the original condition of the problem.

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