BASIC PROBABILITY : HOMEWORK 4

**Question 1:** Suppose that $X$ has an exponential distribution with parameter 1, and $Y = X^2$. Compute the expectation of $Y$.

**Question 2:** Let $X$ be a continuous random variable with density distribution function $f_X$ and $g(x) = 2x + 3$. Find the density distribution function of $g(X)$.

**Question 3**(Solved): Let $X_1$ and $X_2$ be independent exponentially distributed random variables with the same parameter $\lambda$. Compute the density of $(Y_1, Y_2)$ where

$$Y_1 = X_1 + X_2 \text{ and } Y_2 = X_1/X_2,$$

and deduce that $Y_1$ and $Y_2$ are independent and find their marginals.

**Question 4:** Let $X$ and $Y$ be independent random variables with distribution $\mathcal{N}(a, \sigma_1^2)$ and $\mathcal{N}(b, \sigma_2^2)$. Find the distribution of $X + Y$.

**Question 5:** Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with exponential distribution of parameter $\lambda$. Find the density function of $S = X_1 + \cdots + X_n$. (This distribution function is called gamma distribution with parameters $n$ and $\lambda$.

*Hint: proceed by induction*

**Question 6:** Show that for two independent continuous random variables $X$ and $Y$, we have

$$E[XY] = E[X]E[Y],$$

and deduce that

$$\text{var}(X + Y)\text{var}(X) + \text{var}(Y).$$
Solution to **Question 3**

First notice that

\[
P[Y_1 \leq a, Y_2 \leq b] = P[X_1 + X_2 \leq a, X_1/X_2 \leq b] = \int_{x_1, x_2 \geq 0} 1\{x_1 + x_2 \leq a, x_1/x_2 \leq b\} \lambda^2 e^{-\lambda(x_1+x_2)} \, dx_1 \, dx_2.
\]

Then make the change of variable \( y_1 = x_1 + x_2 \) and \( y_2 = x_1/x_2 \), which has an inverse map given by

\[
x_1 = \frac{y_1 y_2}{1 + y_2} \quad \text{and} \quad x_2 = \frac{y_1}{1 + y_2},
\]

which has a Jacobian equal to

\[
J(y_1, y_2) = -\frac{y_1}{(1 + y_2)^2}.
\]

Thus

\[
P[Y_1 \leq a, Y_2 \leq b] = \int_{y_1, y_2 \geq 0} 1\{y_1 \leq a, y_2 \leq b\} \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1 + y_2)^2} \, dy_1 \, dy_2
\]

\[
= \int_0^b \int_0^a \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1 + y_2)^2} \, dy_1 \, dy_2,
\]

which means that \( f_{(Y_1, Y_2)}(y_1, y_2) = \lambda^2 e^{-\lambda y_1} \frac{y_1}{(1 + y_2)^2} \) for \( y_1, y_2 \geq 0 \). This can be rewritten

\[
f_{(Y_1, Y_2)}(y_1, y_2) = (\lambda^2 y_1 e^{-\lambda y_1})(1 + y_2)^{-2},
\]

so the marginals of \( Y_1 \) and \( Y_2 \) are given by the following

\[
f_{Y_1}(y_1) = \lambda^2 y_1 e^{-\lambda y_1} \quad \text{and} \quad f_{Y_2}(y_2) = (1 + y_2)^{-2}.
\]