The range of a rotor walk and recurrence of directed lattices

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Joint work with Lionel Levine (Cornell University) and Yuval Peres (Microsoft Research)

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- Attach arrows (rotors) at each site pointing in any direction. At each step, move the particle in that direction and then rotate the arrow counter-clockwise by 90 degrees.
- In the square grid $\mathbb{Z}^2$, successive exits could repeatedly cycle through the sequence North, East, South, West.

Motivation

Applications:

1. Theoretical Computer Science

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    1. load balancing in distributed computing (Friedrich et al)
    2. design principles for navigation problems and optimal transport in networks (Li et al)
    3. broadcasting information in networks (Doerr et all)
2. Physics: model of self-organized criticality, connections to abelian sandpile model (Holroyd et al), (Priezzhev et al)

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Work on trees

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- On the other hand if *all* rotors start pointing toward the root, then the rotor walk is recurrent.
- On the regular b-ary tree, the i.i.d. uniformly random initial rotor ρ has escape rate $1/b$ for $b \geq 3$ but is recurrent for $b = 2$.

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3. Markov chains and rotor walks by Holroyd and Propp

Iid rotors

How about an initial configuration of iid rotors on $\mathbb{Z}^2$?

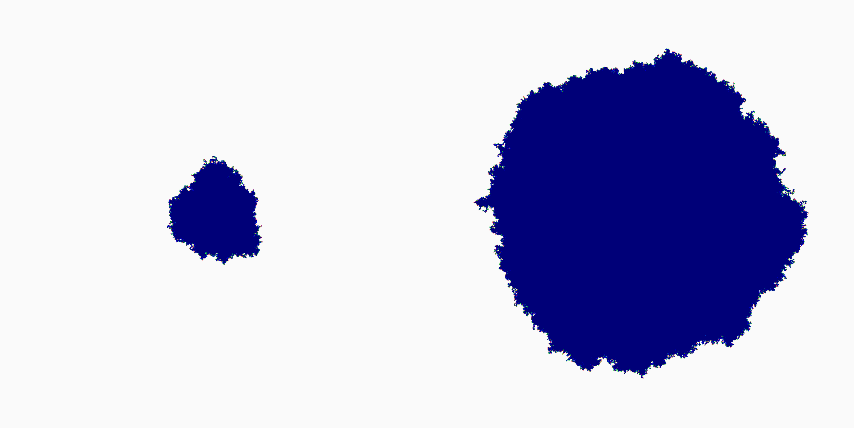

Figure: The set of sites visited after the 18th excursion and the 54th, respectively. Excursion = 4 consecutive visits to o.

i.i.d rotors

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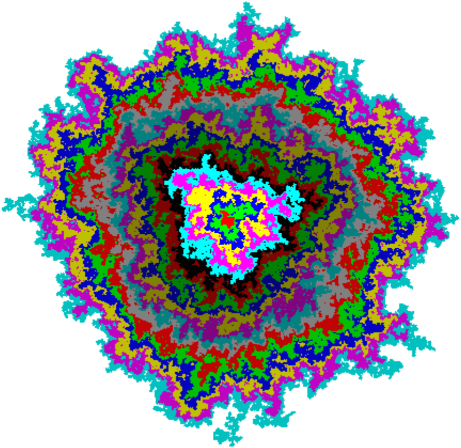
- We know RW visits $t/\log t$ sites in t steps.

iid rotors

Theorem

For all configurations, the number of sites visited by iid rotor walk on $\mathbb{Z}^2$ in t steps is $\Omega(t^{2/3})$.



Comb lattice

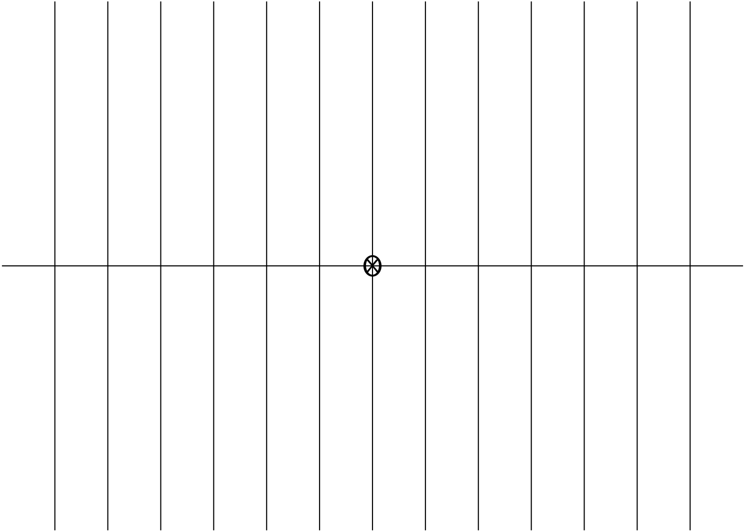


Figure: Comb graph

Comb lattice

Theorem

The number of sites visited by iid rotor walk on the comb lattice in t steps is $\Theta(t^{2/3})$.

It is of note that this result contrasts with random walk on C_2 which expects to visit $(\frac{1}{2\sqrt{2\pi}} + o(1))\sqrt{t}\log t$ as shown in Pach, Tardos.

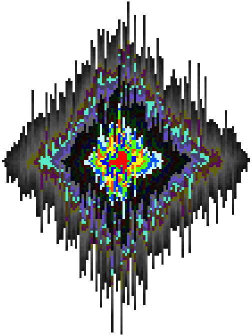


Figure: Set of sites visited by rotor walk on the comb lattice.

Comb lattice

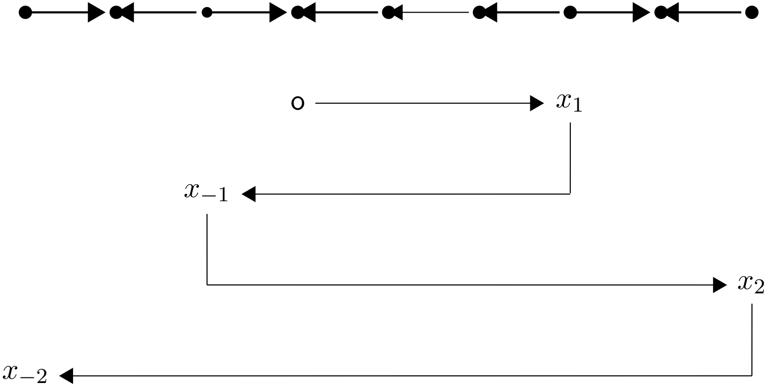
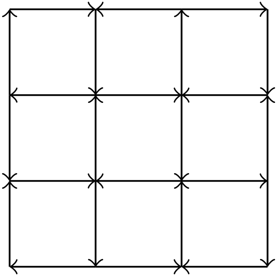
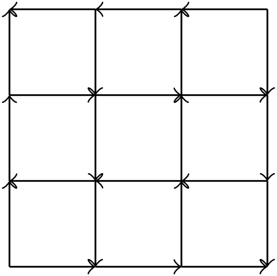


Figure: An initial rotor configuration on $\mathbb{Z}$ (top) and the corresponding rotor walk.

Directed lattices



(a) F-Lattice



(b) Manhattan lattice

Manhattan lattice

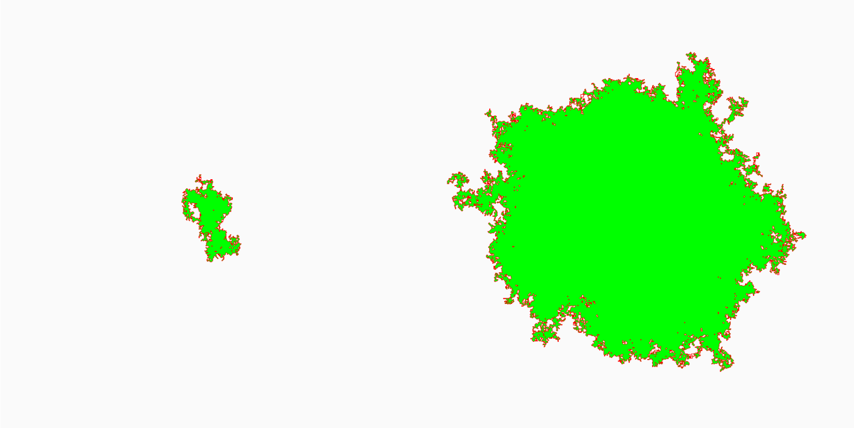


Figure: Set of sites visited by rotor walk on the Manhattan lattice after the 2nd and 11th excursion respectively.

F-lattice

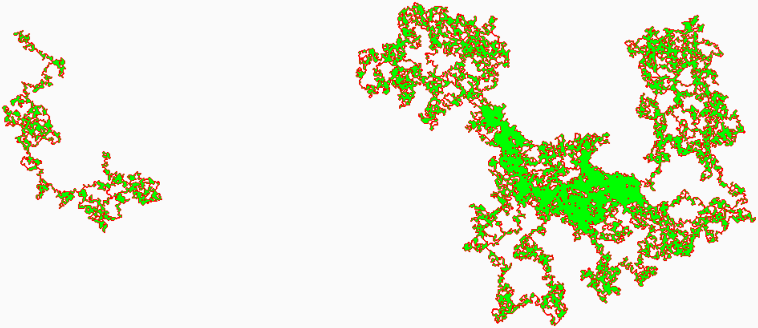


Figure: Set of sites visited by rotor walk on the F-lattice at the first and second excursion after 100000 steps respectively.

Recurrence

Theorem

The F- and Manhattan lattices are recurrent, through connection to critical percolation.

The Stochastic pin-ball model: place at each vertex x a *mirror* with $\mathbb{P} = 1/2$ (either NW or NE).

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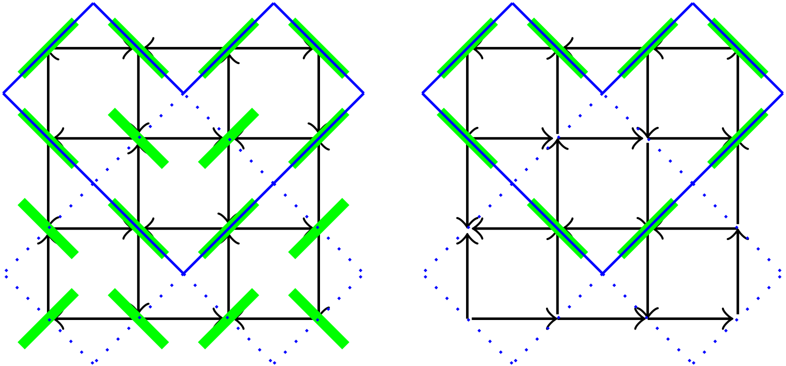
Mirrors



(a) F-Lattice (b) Manhattan lattice

Figure: Percolation on $\mathbb{L}$: dotted blue edges are open, solid blue edges are closed. Shown in green are the corresponding mirrors on the F-lattice (left) and Manhattan lattice.

Manhattan lattice

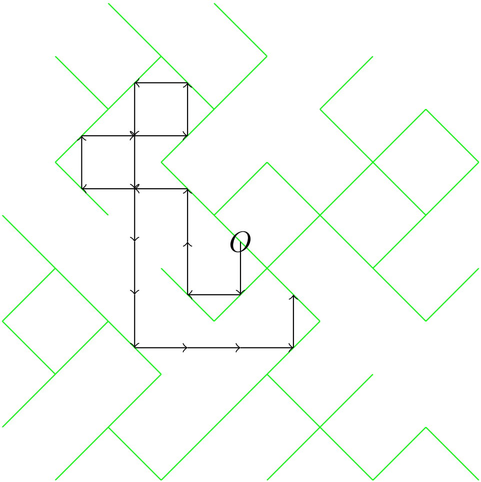


Figure: Mirrors and rotor walk on the Manhattan lattice.

Recurrence of directed lattices

Definition

A *contour* of mirrors is a set of rotors creating a directed cycle.

Lemma

Before the rotor particle exits a contour of mirrors, it visits the sites inside the contour d times, thus performing an excursion.

Critical percolation: $\exists$ infinite number of contours.

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- shape of range in 2d is ball?

Thank you!