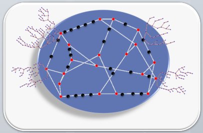
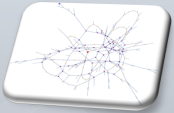
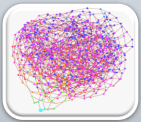




Asymptotics in bond percolation on expanders

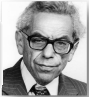
Eyal Lubetzky

March 2018

Courant Institute, New York University

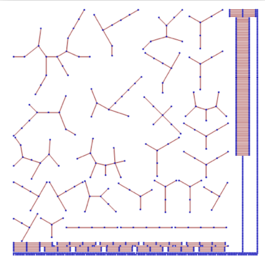
The Erdős–Rényi random graph

"This double 'jump' of the size of the largest component… is one of the most striking facts concerning random graphs." (E–R 1960)



$\mathcal{G}(n,p)$: indicators of the $\binom{n}{2}$ edges are IID $\mathrm{Bernoulli}(p)$.

$n = 1000$
$p = 0.75/n$



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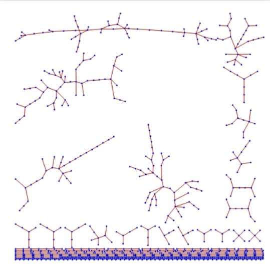

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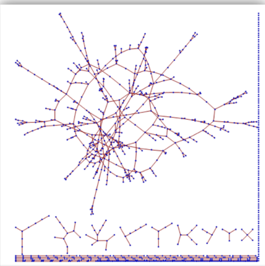

Bond percolation on the complete graph

Definition (*bond percolation* $\mathcal{G}_p$)

keep (open) edges of $\mathcal{G}$ via IID $\mathrm{Bernoulli}(p)$ variables.

$\mathcal{G}(n,p)$: the special case where $\mathcal{G} = K_n$ (complete graph).

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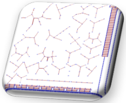

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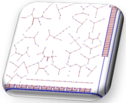

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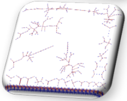

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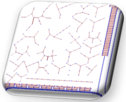

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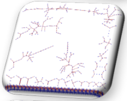

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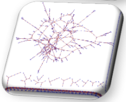

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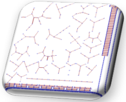
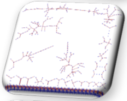
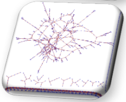

- Emerging from the critical window:
  - $\sim 2\varepsilon n$ when $p = \frac{1+\varepsilon}{n}$ for $n^{-1/3} \ll \varepsilon \ll 1$.

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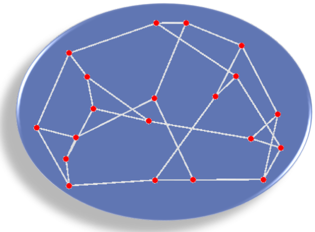
[$c_\lambda \to 0$ as $\lambda \to \infty$, and $c_\lambda \approx 1 - \varepsilon$ when $\lambda = 1 + \varepsilon$ for $\varepsilon = o(1)$.]



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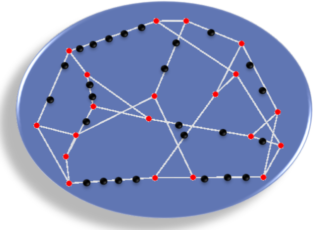

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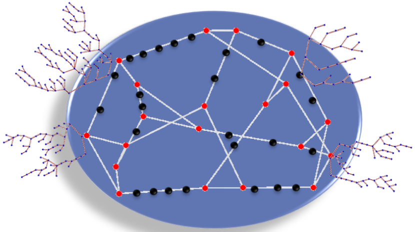

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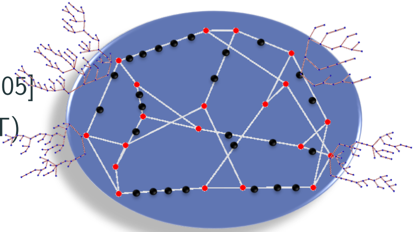

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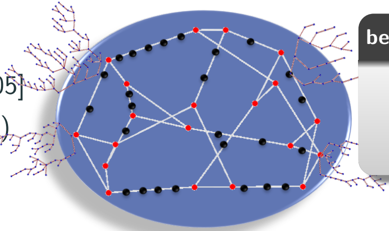
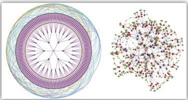


behavior at $p = \frac{1+\varepsilon}{n}$

giant	$\approx 2\varepsilon n$
2-core	$\approx 2\varepsilon^2 n$
excess	$\approx \frac{2}{3}\varepsilon^3 n$

Percolation on expanders



Definition (*edge* (b,d)*-expander*)

sequence of graphs with maximum degree $\leq d$ and conductance $\Phi \geq b$ (for $d \geq 3, b > 0$ fixed), where

$$\Phi(G) = \min_{S:\pi(S)\leq \frac{1}{2}} \frac{|E(S, V\setminus S)|}{\pi(S)} \quad \text{for} \quad \pi(S) = \frac{\sum_{v\in S}\deg(v)}{2|E(G)|}$$

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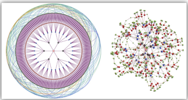

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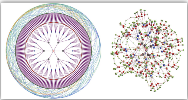

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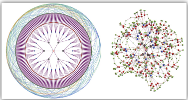

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- [Benjamini, Peres, Nachmias '09] extended high girth regular expanders to sparse graphs with a Benjamini–Schramm limit.

Percolation on expanders (ctd.)

Theorem (Alon, Benjamini, Stacey '04) *uniqueness of giant*

If $\mathcal{G}$ is a (b,d)-expander on n vertices, then $\exists \omega = \omega(b,d) < 1$:

$$\forall p = p_n \,, \qquad \mathbb{P}(|\mathcal{C}_2(\mathcal{G}_p)| > n^{\omega}) = o(1)\,.$$

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Theorem (Alon, Benjamini, Stacey '04) *existence of giant*

Let $\mathcal{G}$ be a regular (b,d)-expander on n vertices with girth $\to \infty$. If $p > \frac{1}{d-1}$ then $\exists c > 0$:

$$\mathbb{P}(|\mathcal{C}_1(\mathcal{G}_p)| > cn) = 1 - o(1)\,,$$

whereas if $p < \frac{1}{d-1}$ then $\forall c > 0$:

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Percolation on random regular graphs

Recall: $|\mathcal{C}_1|$ in the Erdős–Rényi graph $\mathcal{G}(n, p = \frac{\lambda}{n})$ for fixed λ is w.h.p.

$\lambda < 1$	$\lambda = 1$	$\lambda > 1$
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Theorem ([Pittel '08], [Nachmias, Peres '10])

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Well-known: $\mathcal{G} \sim \mathcal{G}(n, d)$ is w.h.p. an expander; what about an *arbitrary* expander? Does $\mathcal{G}_p$ in that case also mirror $\mathcal{G}(n, p)$?

Percolation on K_n vs. $\mathcal{G}(n,d)$ vs. high girth expanders

Comparing $\mathcal{G}_p$ on a d-regular graph $\mathcal{G}$ at $p = \frac{\lambda}{d}$ for $\lambda > 1$:

	$\mathcal{G} = K_n$ ($\mathcal{G}(n,p)$)	$\mathcal{G} \sim \mathcal{G}(n,d)$	$\mathcal{G}$ =high girth expander
$\lvert\mathcal{C}_1\rvert$	$\sim \zeta n$	$\sim \theta_1 n$	$\geq c(b,d,\lambda)n$
$\lvert\mathcal{C}_2\rvert$	$\Theta(\log n)$	$\Theta(\log n)$	$\leq n^{1-\omega(b,d)}$

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Additional geometric features — degree profile? 2-core? excess?

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Additional geometric features — degree profile? 2-core? excess?

For instance:

Behavior at $\lambda = 1+\varepsilon$ for $\varepsilon \ll 1$:

	$\mathcal{G} = K_n$ $(\mathcal{G}(n,p))$	$\mathcal{G}$=high girth expander
$\|\mathcal{C}_1\|$	$\sim 2\varepsilon n$	$\geq c(b,d,\varepsilon)n$
2-core	$\sim 2\varepsilon^2 n$	?
excess	$\sim \frac{2}{3}\varepsilon^3 n$	?

New results: the giant

Let $\mathcal{G}$ be a regular n-vertex (b, d)-expander ($b > 0$, $d \geq 3$ fixed) with girth $\to \infty$, and fix $\frac{1}{d-1} < p < 1$.

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$$\big|V(\mathcal{C}_1)\big| = (\theta_1 + o(1))n\,, \qquad \big|E(\mathcal{C}_1)\big| = (\eta_1 + o(1))n\,,$$

- q is the extinction probability on a $\mathrm{Bin}(d-1, p)$-G–W-tree;
- θ_1 is the probability of p-percolation on a d-reg tree.
- η_1 is the fraction of edges which are open, and the $\mathrm{Bin}(d-1, p)$-G–W-tree from at least one of their endpoints survived.

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$$\left|\frac{1}{n}|V(\mathcal{C}_1)| - \theta_1\right| < \varepsilon\,, \qquad \left|\frac{1}{n}|E(\mathcal{C}_1)| - \eta_1\right| < \varepsilon\,, \tag{1}$$

$$\left|\frac{1}{n}|V(\mathcal{C}_1^{(2)})| - \theta_2\right| < \varepsilon\,, \qquad \left|\frac{1}{n}|E(\mathcal{C}_1^{(2)})| - \eta_2\right| < \varepsilon\,. \tag{2}$$

In particular, w.h.p.,
$\mathrm{excess}(\mathcal{C}_1) \approx (\eta_1 - \theta_1)n$,
and
$\mathrm{excess}(\mathcal{C}_1^{(2)}) \approx (\eta_2 - \theta_2)n$.

$0 < q < 1$ solves $q = (1 - p + pq)^{d-1}$

$\theta_1 := 1 - q(1-p) - pq^2$

$\eta_1 := \frac{1}{2}pd(1-q^2)$

$\theta_2 := 1 - q - (d-1)pq(1-q)$

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Example: asymptotics for large d

[Recall: w.h.p. $\frac{1}{n}|\mathcal{C}_1(\mathcal{G}(n,\frac{\lambda}{n}))| \sim \zeta = \mathbb{P}$(survival of a Po($\lambda$)-G–W-tree).]

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Degree distributions of the giant and the 2-core

Let D_k be the number of degree-k vertices in $\mathcal{C}_1$ and let D_k^* be the number of degree-k vertices in its 2-core $\mathcal{C}_1^{(2)}$.

Theorem (Krivelevich, L., Sudakov)

Fix $d \geq 3$, $1 < \lambda < d-1$, $p = \frac{\lambda}{d-1}$, *and* q *as above; define*

$$\alpha_k = \binom{d}{k} p^k (1-p)^{d-k} (1-q^k) \qquad (k = 1, \ldots, d)\,,$$

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For all $b, \varepsilon > 0$ *there exist some* $c, R > 0$ *so that, if* $\mathcal{G}$ *is a regular* (b,d)*-expander on* n *vertices with girth at least* R*, w.h.p.,*

$$\left|\frac{D_k}{n} - \alpha_k\right| < \varepsilon \;\; \forall 1 \leq k \leq d \quad \text{and} \quad \left|\frac{D_k^*}{n} - \beta_k\right| < \varepsilon \;\; \forall 2 \leq k \leq d\,.$$

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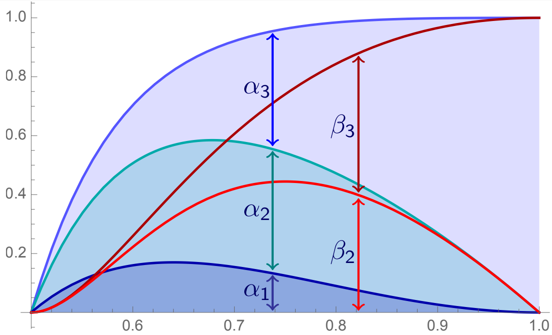
$$\left(\theta_1 = \sum_{k=1}^d \alpha_k\,, \quad \eta_1 = \tfrac{1}{2}\sum_{k=1}^d k\alpha_k\,, \quad \theta_2 = \sum_{k=2}^d \beta_k\,, \quad \eta_2 = \tfrac{1}{2}\sum_{k=2}^d k\beta_k.\right)$$

Example: the giant in percolation on cubic expanders

Asymptotic degree distribution in $\mathcal{G}_p$ for $d = 3$ and $\frac{1}{2} < p < 1$:
w.h.p., the giant has $(\alpha_k + o(1))n$ vertices of degree $k \in \{1, 2, 3\}$;
its 2-core has $(\beta_k + o(1))n$ vertices of degree $k \in \{2, 3\}$.



$\alpha_1 = \frac{3}{p}(1-p)^2(2p-1)$ $\alpha_2 = \frac{3}{p^2}(1-p)(1-4p+6p^2-4p^3)$ $\alpha_3 p^3\left(1-\left(\frac{1-p}{p}\right)^6\right)$

$\beta_2 := \frac{3}{p^3}(1-2p)^2(1-p)$ $\beta_3 := \left(\frac{2p-1}{p}\right)^3$

The second largest component

Recall: w.h.p., $\exists \omega(b,d) > 0 \, : \, |\mathcal{C}_2(\mathcal{G}_p)| < n^{1-\omega}$ ([Alon et al. '04]).

	$\mathcal{G} = K_n$ ($\mathcal{G}(n,p)$)	$\mathcal{G} \sim \mathcal{G}(n,d)$	$\mathcal{G}$ =high girth expander
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Perhaps surprisingly, the $n^{1-\omega}$ from above is essentially tight:

Theorem (Krivelevich, L., Sudakov)

For every $d \geq 3$, $R \geq 1$, $p \in (\frac{1}{d-1}, 1)$ and $\alpha \in (0,1)$ there exist $b > 0$ and a regular (b,d)-expander $\mathcal{G}$ on n vertices with girth at least R where $G \sim \mathcal{G}_p$ has $|V(\mathcal{C}_2)| \gtrsim n^{\alpha}$ w.h.p.

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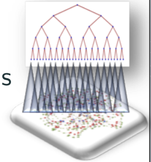
Similarly, for any fixed sequence $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k < 1$ one can construct an expander $\mathcal{G}$ such that w.h.p. $G \sim \mathcal{G}_p$ has components with sizes $\Theta(n^{\alpha_1}), \ldots, \Theta(n^{\alpha_k})$ plus the giant.



A related question of Benjamini: predicting a giant

Question (Benjamini '13)

Let $\mathcal{G}$ be a bounded degree expander. Further assume that there is a fixed vertex $v \in \mathcal{G}$, so that $G \sim \mathcal{G}_{1/2}$ satisfies

$$\mathbb{P}\big(\operatorname{diam}(\mathcal{C}_v(G)) > \tfrac{1}{2}\operatorname{diam}(\mathcal{G})\big) > \tfrac{1}{2}\,.$$

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Is there a giant component w.h.p.?

Variant of our construction for $\mathcal{C}_2$ gives a negative answer to this:

Theorem (Krivelevich, L., Sudakov)

For every $\varepsilon > 0$ and $0 < p < 1$ there exist $b, d, \delta > 0$ and, for infinitely many values of n, a (b, d)-expander $\mathcal{G}$ on n vertices with a prescribed vertex v, such that the graph $G \sim \mathcal{G}_p$ satisfies

$$\mathbb{P}\big(\operatorname{diam}(\mathcal{C}_v(G)) \geq (1-\varepsilon)\operatorname{diam}(\mathcal{G})\big) \geq 1-\varepsilon\,,$$

yet there are no components of size larger than $n^{1-\delta}$ in G w.h.p.

Proof ideas: the giant

Sprinkling argument of [Alon et. al '04] can be used to characterize nearly all edges in the giant: *most components that are suitably large should join the giant once we sprinkle some extra edges*.

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Sprinkling argument of [Alon et. al '04] can be used to characterize nearly all edges in the giant: *most components that are suitably large should join the giant once we sprinkle some extra edges*.

Definition (local predictor for the giant)

$$E_1(H) := \left\{ xy \in E(H) \ : \ \begin{array}{l} \text{the component of either } x \text{ or } y \\ \text{in } H \setminus \{xy\} \text{ has size at least } R \end{array} \right\}$$

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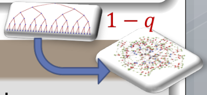

Proposition

$\forall b, \varepsilon > 0\ \exists R, c > 0$ s.t., if $\mathcal{G}$ is a regular (b,d)-expander on n vertices with girth greater than $2R$, and $G \sim \mathcal{G}_p$, then w.h.p.

$$\left| E_1(G) \triangle E(\mathcal{C}_1(G)) \right| \le \varepsilon n \quad \text{and} \quad \left| V_1(G) \triangle V(\mathcal{C}_1(G)) \right| \le \varepsilon n\,.$$

Proof of giant edge and vertex characterization

Upper bound on $V(\mathcal{C}_1) \triangle V_1$ and $E(\mathcal{C}_1) \triangle E_1$ is trivial:

$$\bigcup \{E(\mathcal{C}) : \mathcal{C} \text{ is a conn. component of } H \text{ with } |\mathcal{C}| \geq 2R\} \subseteq E_1(H)$$
$$\bigcup \{V(\mathcal{C}) : \mathcal{C} \text{ is a conn. component of } H \text{ with } |\mathcal{C}| > dR\} \subseteq V_1(H)$$

First step in lower bound: via Hoeffding–Azuma,

$$\mathbb{P}\left(\Big| |E_1(H)| - \mathbb{E}[|E_1(H)|] \Big| \geq a\right) \leq e^{-a^2/(4dn(d-1)^{2R})}$$

and similarly for $\big| |V_1(H)| - \mathbb{E}[|V_1(H)|] \big|$.

Together, these imply that if $p' = p - \varepsilon$ then $G' \sim \mathcal{G}_{p'}$ w.h.p. has

$$|E_1(G')| \geq (\tfrac{1}{2} p' d (1 - q'^2) - \varepsilon) n \,,$$
$$|V_1(G')| \geq (1 - q'(1 - p' + p'q') - \varepsilon) n \,.$$

Proof of giant edge and vertex characterization (2)

Claim

For every $\varepsilon, b, d > 0$ there exist $c, R > 0$ such that, if

- $\mathcal{G}$ is a regular (b, d)-expander with n vertices,
- $(\mathcal{S}_i)$ are disjoint vertex subsets of $\mathcal{G}$ with $|\mathcal{S}_i| \geq R \ \forall i$,

and $H \sim \mathcal{G}_\varepsilon$, then w.h.p. there are no disjoint sets $\mathcal{A} = \bigcup_{i \in I} \mathcal{S}_i$ and $\mathcal{B} = \bigcup_{j \in J} \mathcal{S}_j$ with $|\mathcal{A}|, \mathcal{B}| \geq \varepsilon n$ and no path between them in H.

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Proof.

By Menger's Theorem: $\exists \geq \lceil \frac{b\varepsilon}{2} n \rceil$ edge-disjoint paths of length $\leq \lfloor \frac{d}{b\varepsilon} \rfloor$ between such $\mathcal{A}, \mathcal{B}$ in $\mathcal{G}$. The probability that none survive in H is at most

$$\left(1 - \varepsilon^{d/(b\varepsilon)}\right)^{\frac{1}{2} b\varepsilon n} \leq \exp\left[-\tfrac{1}{2} b\varepsilon^{1+d/b\varepsilon} n\right] .$$

A union bound over at most $2^{2n/R}$ subsets of the $\mathcal{S}_i$'s:

$$\exp\left[\left(R^{-1} 2 \log 2 - \tfrac{1}{2} b\varepsilon^{1+d/b\varepsilon}\right) n\right] . \qquad \square$$

Proof of giant edge and vertex characterization (3)

Corollary

For every $\varepsilon, b, d > 0$ there exist $c, R > 0$ s.t., if $\mathcal{G}$ is a regular (b,d)-expander on n vertices with girth greater than $2R$, then w.h.p. there $\exists$ a connected component $\mathcal{C}$ of $G' \cup \mathcal{G}_\varepsilon$ containing all but at most $2\varepsilon n$ of the vertices $V_1(G')$.

Proof.

Let $\mathcal{S}_i$ be the connected components in G' of all $y \in V_1 = V_1(G')$, and form U by collecting connected components in G of (arbitrary) $\mathcal{S}_i$'s until

$$|U \cap V_1| \geq \varepsilon n\,,$$

so $\varepsilon n \leq |U \cap V_1| < \varepsilon n + |\mathcal{C} \cap V_1|$ for some connected component $\mathcal{C}$ in G. If $|\mathcal{C} \cap V_1| \leq |V_1| - 2\varepsilon n$, the cut $(U \cap V_1, V_1 \setminus U)$ violates the claim. $\square$

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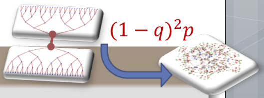
Goal: mimic the analysis of the giant to show:



Proposition

$\forall b, \varepsilon > 0\ \exists R, c > 0$ s.t., if $\mathcal{G}$ is a regular (b,d)-expander on n vertices with girth greater than $2R$, and $G \sim \mathcal{G}_p$, then w.h.p.

$$\left|E_2(G) \triangle E(\mathcal{C}_1^{(2)}(G))\right| \leq \varepsilon n \quad \text{and} \quad \left|V_2(G) \triangle V(\mathcal{C}_1^{(2)}(G))\right| \leq \varepsilon n\,.$$

Proof ideas: the 2-core

`Intuition`: if both endpoints of an edge are suitably large, then sprinkling should form a cycle through it…

Definition (local predictor for the 2-core)

$$E_2(H) := \left\{ xy \in E(H) \;:\; \begin{array}{l}\text{the component of both } x \text{ and } y \\ \text{in } H \setminus \{xy\} \text{ has size at least } R\end{array} \right\}$$

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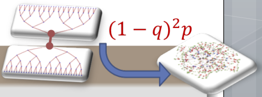

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Problem: sprinkling may reuse the edge xy and not create a cycle!

Proof ideas: the 2-core (ctd.)

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- Partition the edge set of $\mathcal{G}$ randomly (independently) into blue and red, where the probability of an edge to be blue is ε.
- Modify the definition of $E_2(\mathcal{G})$ to include blue edges xy where the red clusters of x and y in $\mathcal{G} \setminus \{xy\}$ are suitably large.

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This should imply there are (roughly) $\geq \varepsilon p(1-q)^2|E(\mathcal{G})|$ blue edges in the 2-core, thus $\geq p(1-q)^2|E(\mathcal{G})|$ that are blue or red.

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Problem: red graph is no longer an expander—e.g., it typically has linearly many isolated vertices—sprinkling argument fails...

Proof ideas: the 2-core (ctd.)

Recall: the edges pf $\mathcal{G}$ are randomly partitioned into blue and red, where the probability of an edge to be blue is ε (independently of other edges).

Definition (k-thick subsets)

A subset $S \subset V(H)$ is k-thick if there exists disjoint connected subsets of H, $\{S_i\}$, each of size at least k, such that $S = \bigcup S_i$.

Key: although the red graph is not an expander, w.h.p., sets that are k-thick do maintain edge expansion in it:

Claim

There exists $k(\varepsilon, b, d)$ such that, with probability $1 - O(2^{-\varepsilon n})$,

$$\#\{\text{red } (x,y) \in E(\mathcal{G}) \;:\; x \in S\,,\, y \in S^c\} \geq \tfrac{1}{2} b|S|$$

for every k-thick $S \subset V(\mathcal{G})$ with $\varepsilon n \leq |S| \leq n/2$.

Thank you!