Non-backtracking random walks on expanders

Eyal Lubetzky
Microsoft Research

Based on joint work with
Noga Alon, Itai Benjamini and Sasha Sodin

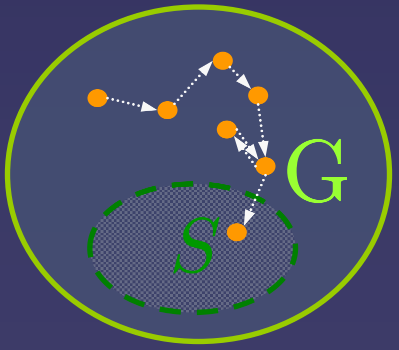
Random Walks on graphs

satisfying some natural properties

- Random walk on G:
  - Simple to analyze:
  - Mixes quickly to stationary distribution.
- $\Longrightarrow$ efficient sampling of the vertices.
- Numerous applications, e.g.:
  - Volume computation and enumeration
  - Space efficient algorithms for STCONN.
  - De-randomization and conservation of random bits.

De-randomization via random walks

- Randomized algorithm $\mathcal{A}$:
  - Requires an n-bit seed.
  - One sided error with fixed probability $0 < p_{\mathrm{e}} < 1$.
- Naïve amplification of p_{e} to $\exp(-\Omega(k))$ requires $\boldsymbol{k\,n}$ random bits.
- Random walks on expanders:
  - W = random walk of length k .
  - S = set of vertices of fixed proportion.
  - $\Pr[W \text{ misses } S] = \exp(-\Omega(k))$
- $p_{\mathrm{e}} \rightsquigarrow \exp(-\Omega(k))$ using only $\boldsymbol{n + \Theta(k)}$ bits!



Non-backtracking random walks

- In many cases (cf. above application) there is "no sense" in backtracking.

Q: Can we benefit from forbidding the random walk to backtrack?

Q: What can be said about the distribution of a set of vertices sampled this way?

some fixed integer

Expanders and random walks

- G = d-regular graph on n vertices.
- RW on G mixes to the stat. dist. π $\iff$ G is connected and non-bipartite.
- Let G have eigenvalues $d = \lambda_1 \geq \ldots \geq \lambda_n$:
  - $\lambda_2 < d$ iff G is connected.
  - $\lambda_n > -d$ iff G is non-bipartite.
- $\Longrightarrow \lambda < d$, where $\lambda = \max\{\lambda_2, \lambda_n\}$.
- How fast does the RW mix in this case?

Mixing rate of RWs

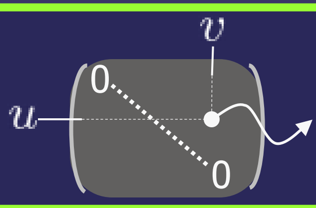
- $P^{(k)}_{uv} = \Pr[$ RW of length k from u ends in v $]$.
- The mixing rate of G is defined as:

$$\rho(G) = \limsup_{k\to\infty} \max_{u,v\in V(G)} \left| P^{(k)}_{uv} - \pi(v) \right|^{1/k}$$

$\log_\rho(\delta)$ steps for the L_∞ distance from π to be $\le \delta$.

- If G is an (n, d, λ)-graph, $\rho(G) = \lambda/d$:

$$\frac{A_G}{d} =$$



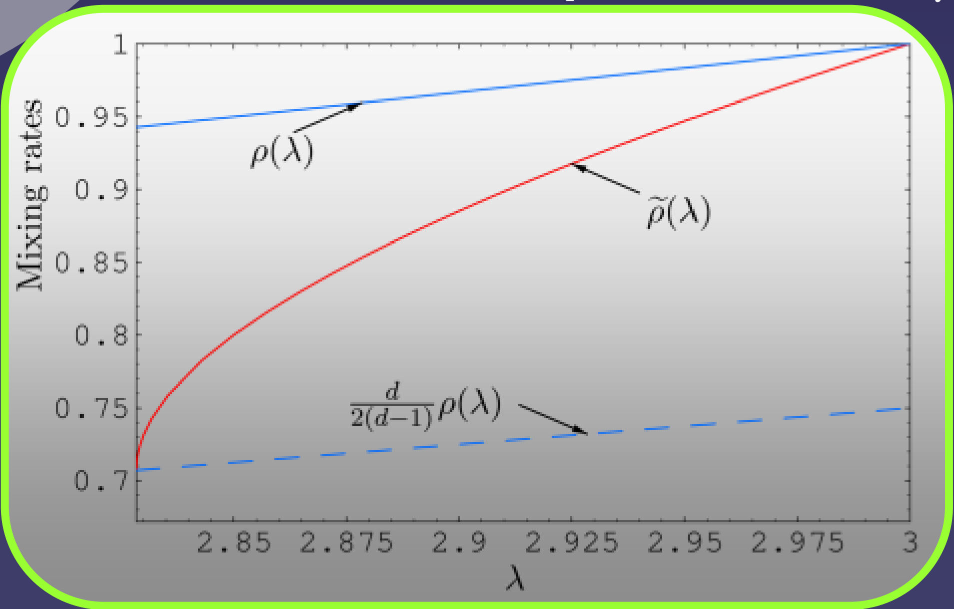
$$\begin{cases} \frac{1}{d} & \text{if } uv \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

$$P^{(k)} = \left(\frac{A_G}{d}\right)^k, \quad \pi = \frac{1}{n} \cdot \underline{1}$$

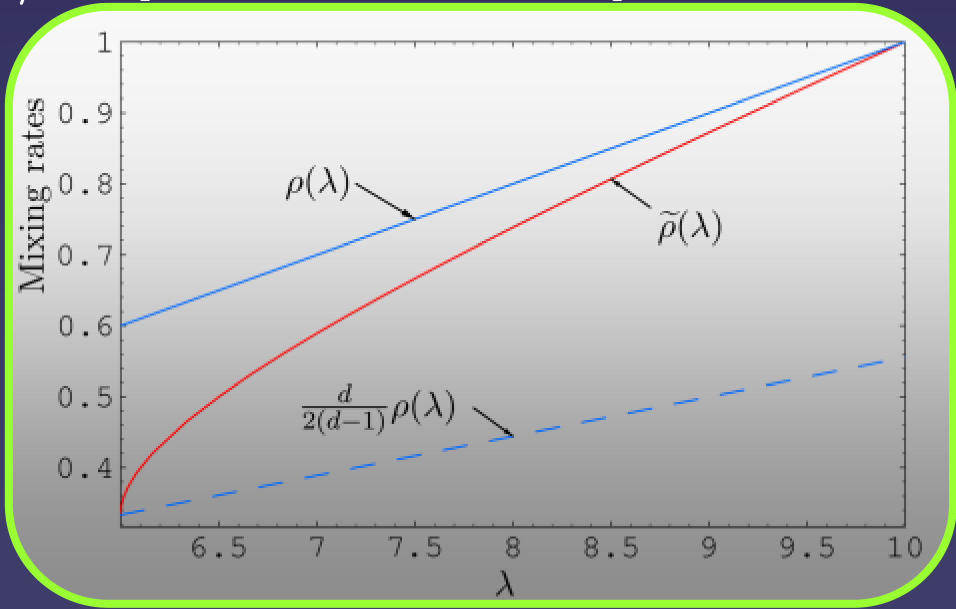
Largest eigenvalue of $P^{(k)} - \frac{1}{n}J$ in absolute value is $(\lambda/d)^k$.

Non-backtracking RWs mix faster

- Define $\tilde{\rho}(G)$ analogously for NBRWs.
- $\tilde{\rho}$ is a function of λ,d , is always $\leq \rho$, and may reach ~$\rho/2$ (twice faster)!



3-regular graphs



10-regular graphs

The mixing rate of NBRWs

- [Alon, Benjamini, L, Sodin '07]:

$\forall$ NBRW on an (n,d,λ)-graph with $d \geq 3$ and $\lambda < d$ converges to the uniform distribution with

$$\tilde{\rho} = \psi\left(\frac{\lambda}{2\sqrt{d-1}}\right) / \sqrt{d-1} \ ,$$

$$\text{where } \psi(x) = \begin{cases} x + \sqrt{x^2 - 1} & \text{If } x \geq 1, \\ 1 & \text{If } x \leq 1. \end{cases}$$

- Corollary: $\lambda \geq 2\sqrt{d-1} \ \Rightarrow \ \frac{d}{2(d-1)} \leq \frac{\tilde{\rho}}{\rho} \leq 1 \ ,$

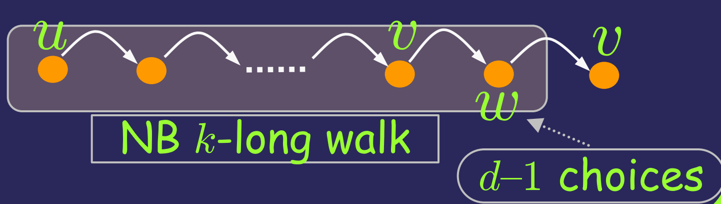
Ramanujan graphs

$$\lambda < 2\sqrt{d-1} \ , \ d = n^{o(1)} \ \Rightarrow \ \frac{\tilde{\rho}}{\rho} = \frac{d}{2(d-1)} + o(1) \ .$$

Computing the mixing rate of NBRWs

- $A^{(k)}_{uv}$:= # k-long NB walks from u to v.
- Goal: determine the spectrum of $A^{(k)}$.
- Claim: $$\begin{cases} A^{(1)} = A\ , \\ A^{(2)} = A^2 - dI\ , \\ A^{(k+1)} = \underbrace{AA^{(k)}}_{\text{all extensions of the walks by 1 edge.}} - \underbrace{(d-1)A^{(k-1)}}_{\text{\# of BT walks we counted:}}\ . \end{cases}$$

Adjacency matrix of G



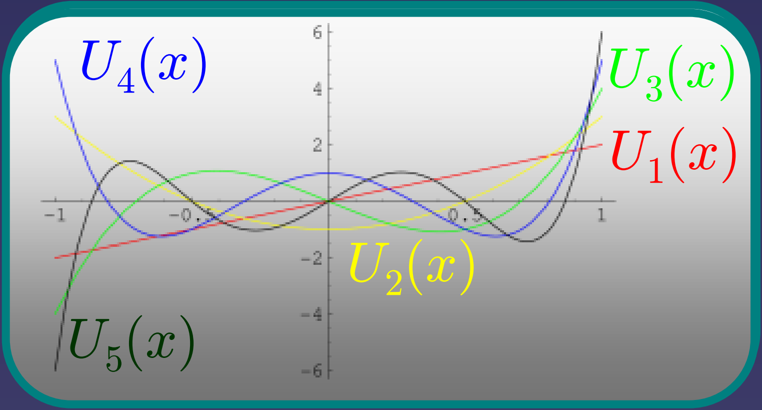
- $A^{(k)}$ is a polynomial of A, yet might be complicated to analyze: $$P_{k+1}(x) = xP_k(x) - (d-1)P_{k-1}(x).$$

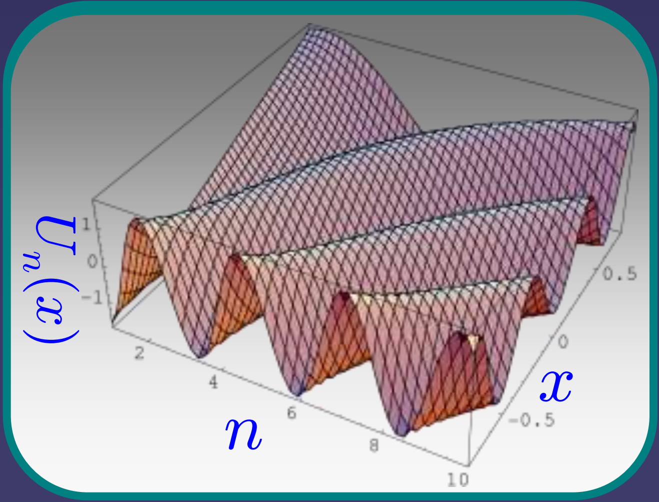
Chebyshev polynomials of the 2nd kind

- The polynomials $U_k(\cos\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}$ satisfy:

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)$$

Reminds the recursion that $A^{(k)}$ satisfies…





Chebyshev polynomials of the 2nd kind

- The polynomials $U_k(\cos\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}$ satisfy:

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)$$

Reminds the recursion that $A^{(k)}$ satisfies…

- Indeed:

$$A^{(k)} = \sqrt{d(d-1)^{k-1}}\,q_k\left(\frac{A}{2\sqrt{d-1}}\right),$$

where:

$$q_k(x) = \sqrt{\tfrac{d-1}{d}}U_k(x) - \tfrac{1}{\sqrt{d(d-1)}}U_{k-2}(x)\,.$$

- Result now follows from an asymptotic analysis of the behavior of $q_k(x)$. ■

Distribution of sampled vertices: RW

- Recall: n-long RW on an expander:
  - Costs $\Theta(n)$ random bits.
  - $\Pr[$ missing a linear set $] = \exp(-\Omega(n))$.

"right" probability

Q: What about frequencies of visits at vertices?

- Random setting:

Classical n balls $\to$ n bins

Poisson visits at a given vertex.

Max # visits $\sim \log n \,/\, \log\log n$.

- RW setting: # of visits reaches $\Omega(\log n)$ … (too much)

Large probability of traversing an edge back & forth $\Omega(\log n)$ times 

Distribution of sampled vertices: NBRW

Backtracking → Too many visits to a vertex ← Short cycles

- What about NBRWs and high girth?
- [Alon, Benjamini, L, Sodin '08]:

> Almost $\forall$ NBRW of length n on a high-girth n-vertex expander has the "right" maximum # of visits to a vertex: $(1+o(1))\ \log n\ /\ \log\log n$.

- Girth requirement: $\Omega(\log\log n)$ (tight).
- Indeed, $\text{maximum} =$ balls & bins setting. What about the entire distribution?

Poisson approximation for NBRW

- Recall: unbounded girth is *necessary* for a Poisson dist. of visits to vertices.
- [Alon, L]: this requirement is *sufficient*:

Almost $\forall$ NBRW of length n on an n-vertex expander of girth $g = \omega(1)$ makes t visits to $(1+o(1))\ n/(\mathrm{e}\ t!)$ vertices.

Brun's Sieve

- Moreover, high-girth $\Longrightarrow$ relative point-wise convergence to the Poisson distribution:

If in addition $g = \Omega(\log \log n)$, the above holds uniformly over all t up to the "right" maximum of the distribution.

Stronger version of Brun's Sieve (error estimate)

Open problems

- Recall:

  Maximum # of visits to a vertex in n-long NBRWs on high-girth n-vertex expanders is w.h.p. $(1+o(1))\,\frac{\log n}{\log\log n}$.

- For which other families of d-regular graphs, $d \geq 3$, is this maximum $\sim\frac{\log n}{\log\log n}$?

- Does a NBRW on any n-vertex d-regular ($d \geq 3$) graph visit some vertex w.h.p. at least $(1+o(1))\frac{\log n}{\log\log n}$ times?

Thank you.