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Maximum of 3D Ising interfaces

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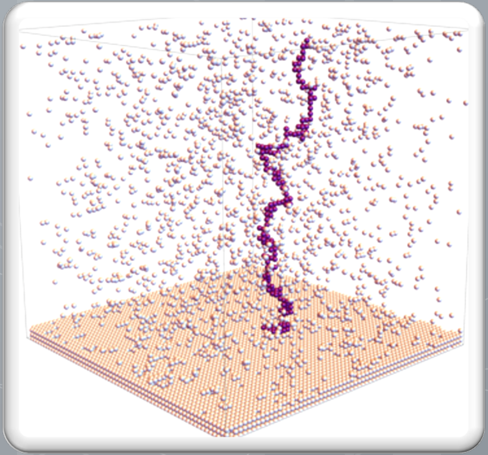
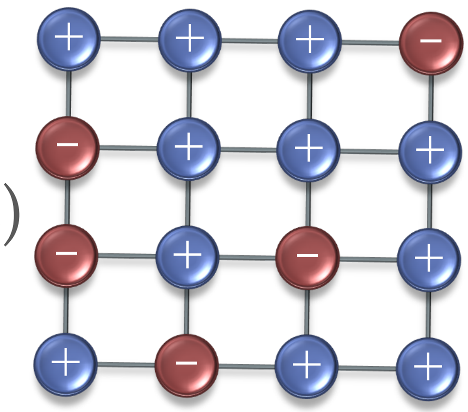


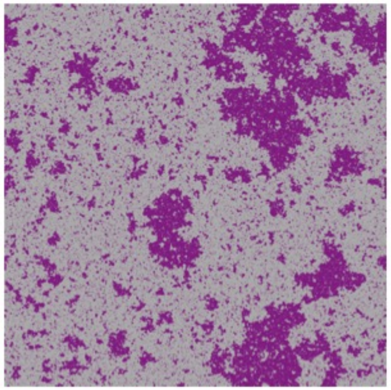
based on joint works with
Reza Gheissari (UC Berkeley)

Definition: the classical Ising model

- Underlying geometry: finite $\Lambda \subset \mathbb{Z}^d$.
- Set of possible configurations:
$$\Omega = \{\pm 1\}^\Lambda$$
(each *site* receives a plus/minus *spin*)
- Probability of a configuration $\sigma \in \Omega$ given by the *Gibbs distribution*:



$$\mu_\Lambda(\sigma) = \frac{1}{Z(\beta)} \exp\left(-\beta \sum_{x\sim y} \mathbf{1}_{\{\sigma_x \neq \sigma_y\}}\right)$$

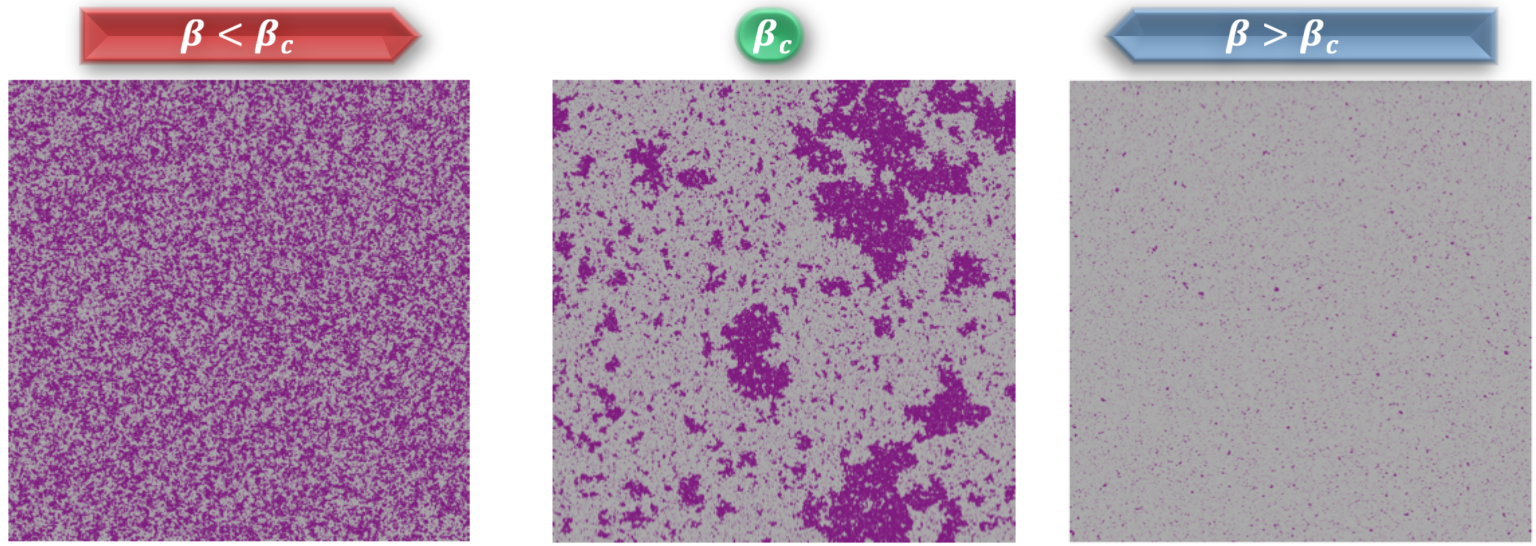


- $\beta \geq 0$: the inverse temperature
- $Z(\beta)$: the partition function

The Ising model phase transition

- Underlying geometry: Λ = finite 2D grid.
- Set of possible configurations: $\Omega = \{\pm 1\}^\Lambda$
- Probability of a configuration: $\mu_\Lambda(\sigma) = \frac{1}{Z(\beta)} \exp\left(-\beta \sum_{x\sim y} \mathbf{1}_{\{\sigma_x \neq \sigma_y\}}\right)$

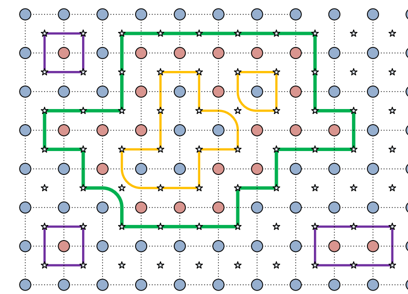
Local (nearest-neighbor) interactions have macroscopic effects:



$\beta = 0.75$ $\beta = 0.88$ $\beta = 1$

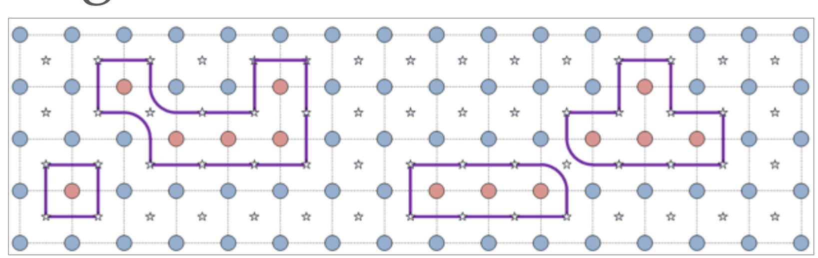
Low temperature representation in 2D

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with -plus boundary.
  - Draw a dual-edge $(u,v)^*$ if $\sigma_x \neq \sigma_y$
  - Bijection between a dual-loop collection and the Ising configuration σ.
  - Induced distribution on the dual-loops:



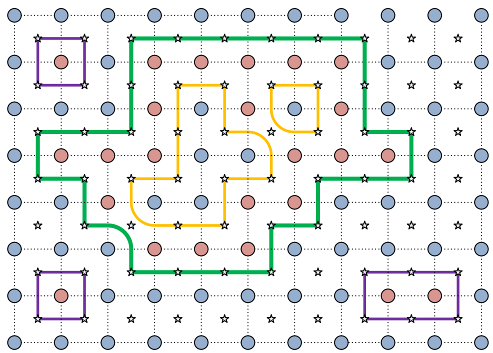
$$\mu_\Lambda^+(\{\gamma_1, \gamma_2, \dots\}) = \frac{1}{Z(\beta)} e^{-\beta \sum |\gamma_i|}$$

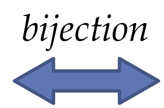
- [Peierls '36]: proof of a phase transition for $\beta > \beta_0$: 1st moment argument on # of sites inside a -"island"

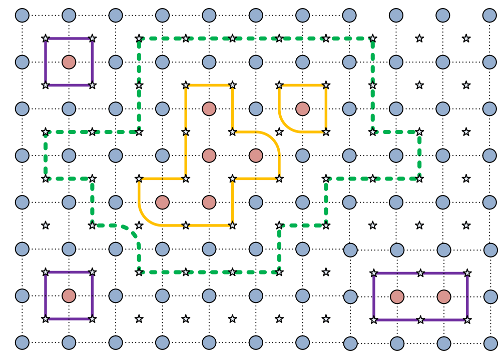


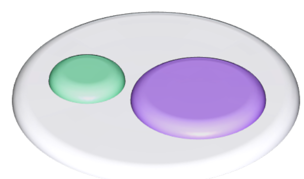
Peierls' phase transition argument

- Setting: $\Lambda \subset \mathbb{Z}^2$ is an $n \times n$ box with -plus boundary.
- For any σ containing γ flip *all spins in the interior of* γ :









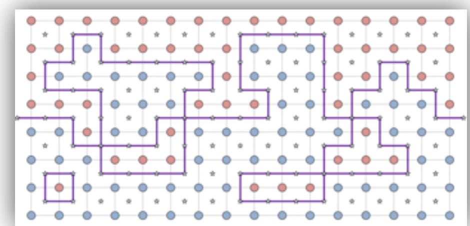
$\mu_\Lambda^+(\sigma) = Z(\beta)^{-1} e^B$ $\qquad$ $\mu_\Lambda^+(\sigma') = Z(\beta)^{-1} e^{B+\beta|\gamma|}$

$\Rightarrow \mu_\Lambda^+(\gamma$ belongs to loop collection$) \leq e^{-\beta|\gamma|}$.

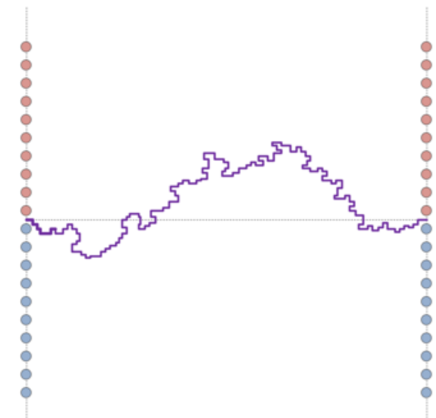
- For a site x: at most $e^{c\ell}$ contours γ of length ℓ around x, and each such γ costs $e^{-\beta\ell}$; overall:

$$\mu_\Lambda^+(\sigma_x = -1) \leq e^{-4(\beta - C)}.$$

2D Ising interfaces



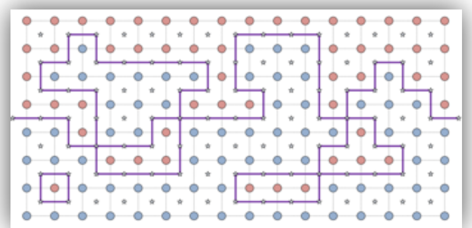
- What does the interface between the + and − phases look like at $\beta > \beta_c$?
- $\mu_\Lambda^{\mp}$: Ising model on
  - 2D cylinder $\Lambda = [\![-n, n]\!] \times (\mathbb{Z} + \frac{1}{2})$
  - Boundary conditions: $\begin{cases} - & \textit{upper half}\text{–plane} \\ + & \textit{lower half}\text{–plane} \end{cases}$
  - Draw a dual-**edge** $(u, v)^*$ if $\sigma_x \neq \sigma_y$.



- **Interface**: (max) connected set $\mathcal{I}$ of dual-edges separating the infinite + and − components of the boundary.

2D Ising interfaces: roughness

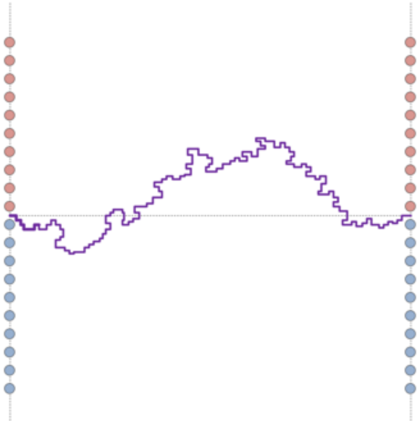
- What does the interface between the + and − phases look like at $\beta > \beta_c$?



- 2D Ising model w. *Dobrushin's boundary conditions* $\mu^{\mp}_{\Lambda_n}$:
  - Interface has a scaling limit:

$$\frac{\mathcal{I}(x/n)}{\sqrt{c_\beta n}} \to \text{Brownian bridge}$$

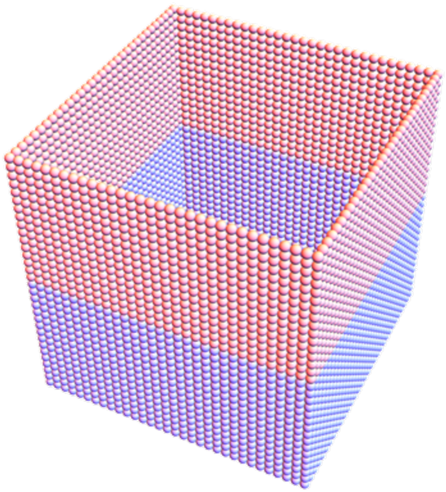
  - Interface is ***rough***: fluctuations of $\sqrt{n}$
  - Maximum M_n is $O_P(\sqrt{n})$, and $M_n - \mathbb{E}[M_n]$ is also $O_P(\sqrt{n})$.



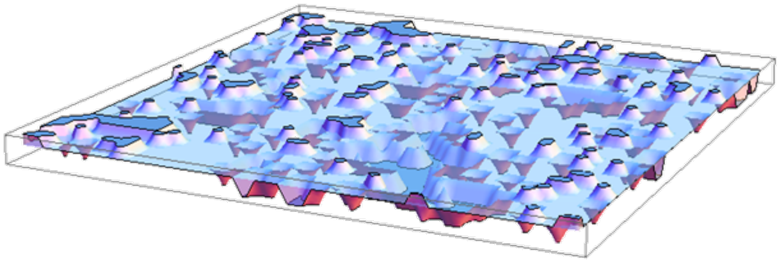
[Higuchi '79], [Dobrushin, Hryniv '97], [Hryniv '98], [Dobrushin, Kotecky, Shlosman '92]

3D Ising Interfaces

- $\mu_\Lambda^{\mp}$: Ising model on
  - 3D cylinder $\Lambda = [\![-n,n]\!]^2 \times (\mathbb{Z} + \frac{1}{2})$
  - Boundary conditions: $\begin{cases} - & \textit{upper half-space} \\ + & \textit{lower half-space} \end{cases}$
  - Draw a dual-**face** $(u,v)^*$ if $\sigma_x \neq \sigma_y$.

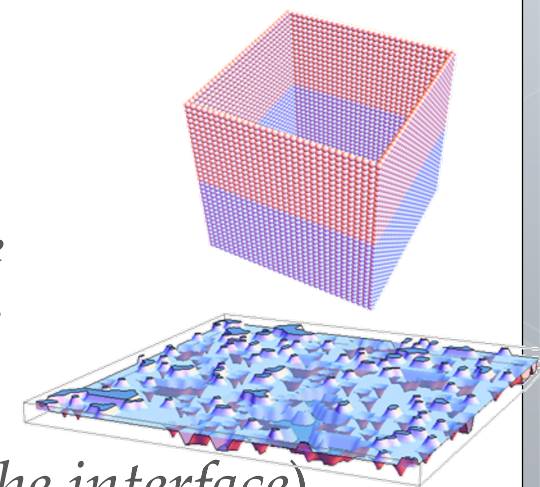


- **Interface**: (max) connected set $\mathcal{I}$ of faces separating the infinite $+$ and $-$ components of the boundary.



3D Ising interfaces: rigidity

- $\mu_\Lambda^\mp$: Ising model on
  - 3D cylinder $\Lambda = [\![-n,n]\!]^2 \times (\mathbb{Z} + \frac{1}{2})$
  - Boundary conditions: $\begin{cases} - & \textit{upper half-space} \\ + & \textit{lower half-space} \end{cases}$



- THEOREM: [Dobrushin '72] (*rigidity of the interface*)

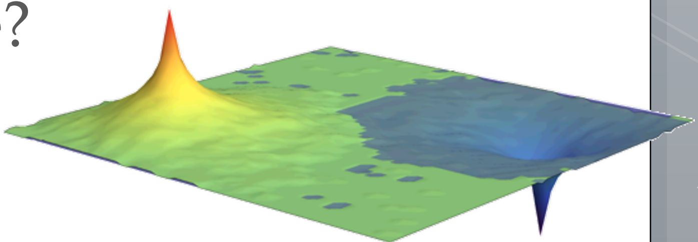
> There exists $\beta_0 > 0$ such that $\forall \beta > \beta_0$ and $\forall x_1, x_2, h$,
> $$\mu_\Lambda^\mp\big(\mathcal{I} \ni (x_1, x_2, h)\big) \leq \exp\left(-\tfrac{1}{3}\,\beta h\right)$$

- COROLLARY: [Dobrushin '72, '73] for $\beta > \beta_0$:

> 1. $\exists$ non-translation invariant $\mathbb{Z}^3$ Gibbs measures
> 2. Maximum height of $\mathcal{I}$ is $O_{\mathrm{P}}(\log n)$.

Plus/minus interface in 3D Ising

- M_n = maximum height of the interface $\mathcal{I}$ in 3D Ising with Dobrushin's boundary conditions.
  - [Dobrushin '72]: $\exists C_\beta$ s.t. $\mu_\Lambda^{\mp}(M_n \le C_\beta \log n) \to 1$.
  - $\Rightarrow$ (via straightforward matching order lower bound) the maximum of the interface has **order** $\log n$.
- Asymptotics of the maximum (LLN)? Tightness?
- Structure of interface conditioned on LDs?
  - conditioned on $(x_1, x_2, 0)$ belonging to a "**pillar**" reaching height h, what can we say about that pillar, e.g., its surface area? its volume? xy-coords of its tip?



3D Ising Roughening phase transition

- Conj.: *Roughening phase transition in 3D* at $\beta_R \approx 0.83$:

β_c | $\beta_c < \beta < \beta_R$ | β_R | $\beta > \beta_R$

rough (delocalized)

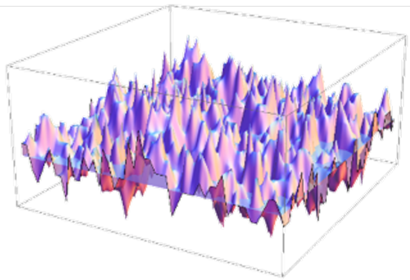
$\mathrm{Var}(\mathrm{ht}_x(\mathcal{I})) \to \infty$

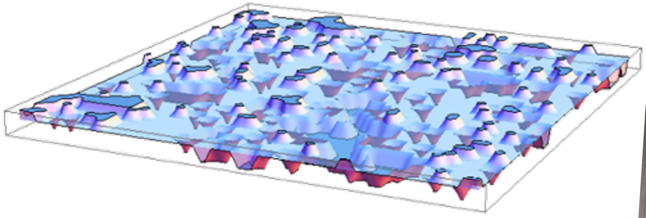
$\mathcal{I} \approx$ DGFF

rigid (localized)

$\mathrm{Var}(\mathrm{ht}_x(\mathcal{I})) = O(1)$

$\max_x \mathrm{ht}_x(\mathcal{I}) \asymp \log n$

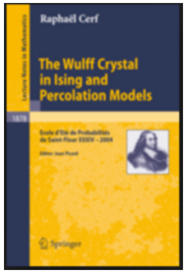




"Evidence that $T_R < T_c(3)$ strictly was obtained by Weeks et al. (1973) … To this day, there still appears to be no proof that $T_R < T_c(3)$." [Abraham '86]

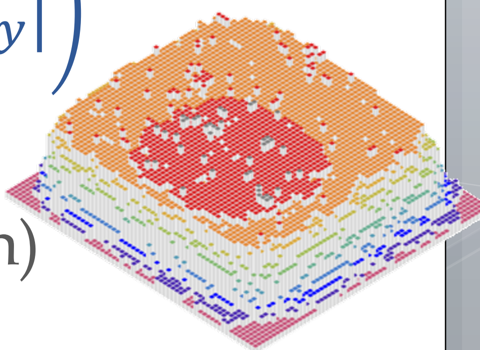
Related work on 3D Ising interfaces

- Alternative simpler argument by [van Beijeren '75] for [Dobrushin '72]'s result on the rigidity of the 3D Ising interface.
- Rigidity argument extended to
  - Widom–Rowlinson model [Bricmont, Lebowitz, Pfister, Olivieri '79a], [Bricmont, Lebowitz, Pfister '79b, '79c]
  - Super-critical percolation / random cluster model conditioned to have interfaces [Gielis, Grimmett '02]
- Tilted interfaces: [Cerf, Kenyon '01] (zero temperature, 111 interface), [Miracle Sole '95] (1-step interface), [Sheffield '03] ($|\nabla\phi|^p$ models), **many** works on the conjectured behavior, related to the (non-)existence of non-translational invariant Gibbs measures
- Wulff shape, large deviations for the magnetization, surface tension [Pisztora '96], [Bodineau '96], [Cerf, Pisztora '00], [Bodineau '05], [Cerf '06]
- Plus/minus phases away from the interface [Zhou '19]



Approximating random surface models

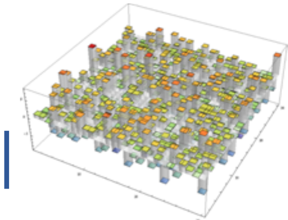
- DEFINITION: *(2+1)-dimensional SOS above a wall* [Temperley '52]
probability measure on *height functions* ϕ on $\Lambda = \{1, \dots, L\}^2$ with $\Lambda \ni x \mapsto \phi_x \in \mathbb{Z}$ and $\phi_x = 0$ for $x \notin \Lambda$ given by
$$\pi_\Lambda(\phi) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(-\beta \sum_{x\sim y} |\phi_x - \phi_y|\right)$$
  - no bubbles (distribution on interfaces)
  - no overhangs (interface = height function)



- $|\nabla\phi|^p$ model: $\pi_\Lambda(\phi) \propto e^{-\beta \sum_{x\sim y} |\phi_x - \phi_y|^p}$ for $p \geq 1$
($p = 1$ is *SOS*; $p = 2$ is the *discrete Gaussian;* $p = \infty$ is *RSOS*)

SOS: roughening transition

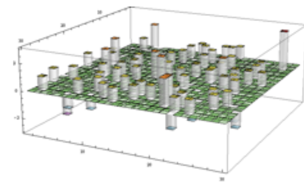
- (2+1)D surface *delocalized* (rough) at $\beta \ll 1$:



$$\mathrm{Var}(\phi_x) \asymp \log n\,, \qquad \mathbb{E}\,\phi_x\phi_y \asymp \log|x-y|$$

[Fröhlich, Spencer ('81), ('83)]

- (2+1)D surface *localized* (rigid) at $\beta \gg 1$:



$$\mathrm{Var}(\phi_x) \asymp 1\,, \qquad \mathbb{E}\phi_x\phi_y \simeq e^{-c|x-y|}$$

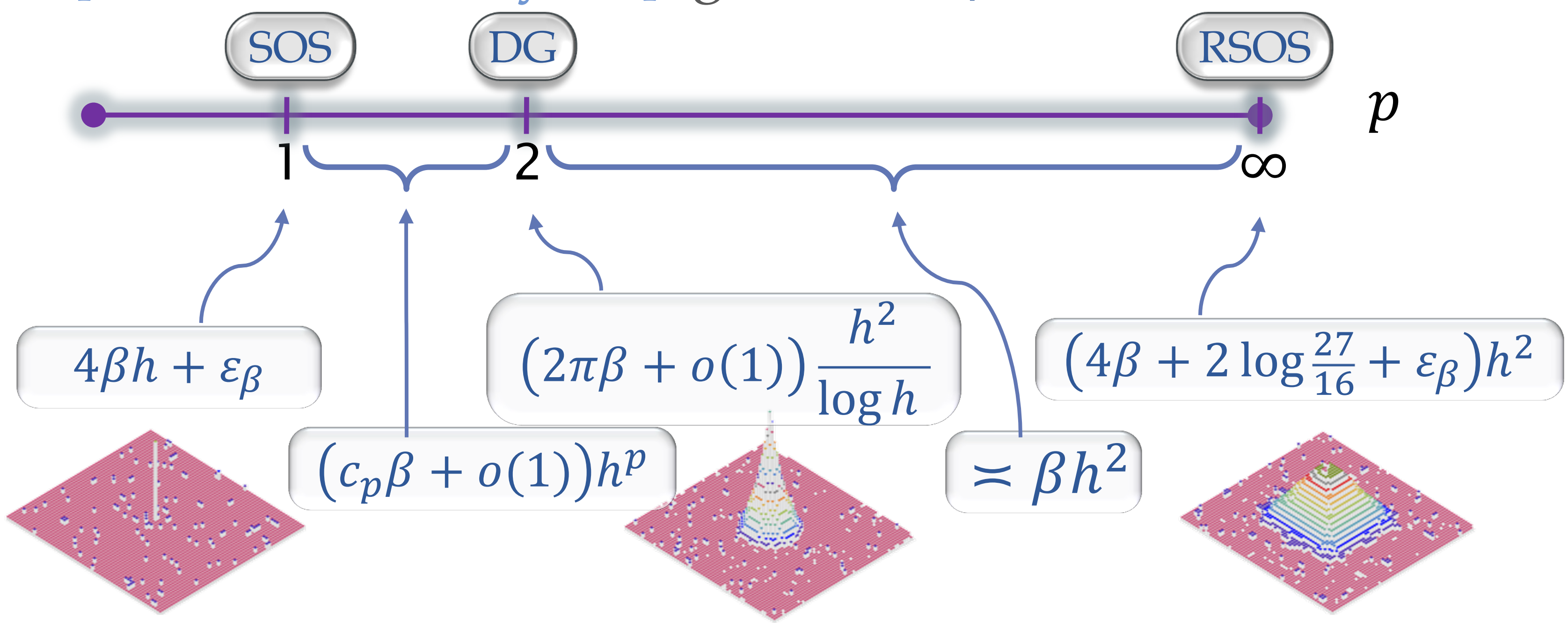
[Gallavotti, Martin-Löf, Miracle-Solé ('73)], [Brandenberger, Wayne ('82)]

- Maximum M_n of the rigid (2+1)D surface at $\beta \gg 1$:
  - $\mathbb{E}[M_n] \asymp \beta^{-1}\log n$ [Bricmont, El-Mellouki, Fröhlich '86]
  - $M_n = \frac{1}{2\beta}\log n + O(1)$ (+*shape theorem, with and w/o a floor*)

[Caputo, L., Martinelli, Sly, Toninelli '12, '14, '16]

Maximum dominated by LD at origin

- Maximum governed by ∞-volume large deviation rate
$$\lim_{h\to\infty} -\frac{1}{h^a}\log \pi_{\mathbb{Z}^2}(\phi_x \geq h)$$
which is tied to the shape of ***tall pillars***:
- [L., Martinelli, Sly '16]: general $|\nabla\phi|^p$ surface models:



LLN for the maximum

- Recall: M_n = maximum of the interface $\mathcal{I}$ in 3D Ising with Dobrushin's b.c.; [Dobrushin '72]: $M_n = O_{\mathrm{P}}(\log n)$.
- THEOREM: ([Gheissari, L. '19a])

There exists β_0 such that for all $\beta > \beta_0$,

$$\lim_{n\to\infty} \frac{M_n}{\log n} = \frac{2}{\alpha}\,, \qquad \textit{in probability,}$$

LLN

where

$$\alpha(\beta) = \lim_{h\to\infty} -\frac{1}{h}\log \mu^{\mp}_{\mathbb{Z}^3}\left((0,0,0) \stackrel{+}{\longleftrightarrow} (\mathbb{R}^2 \times \{h\})\right)$$

and satisfies $\alpha(\beta)/\beta \to 4$ as $\beta \to \infty$.

- existence of the limit α nontrivial: relies on new results on the interface shape conditioned on LD.

Tightness and tails for the maximum

- THEOREM: ([Gheissari, L. '19b])

1. There exists β_0 such that for all $\beta > \beta_0$,
$$M_n - \mathbb{E}M_n = O_{\mathrm{P}}(1).$$
2. There exist $C, \bar{\alpha}, \underline{\alpha}$ such that $\forall\, r \geq 1$,
$$\begin{cases} e^{-(\bar{\alpha}r+C)} \leq \mu_n^{\mp}(M_n \geq \mathbb{E}[M_n] + r) \leq e^{-(\underline{\alpha}r-C)} \\ e^{-e^{\bar{\alpha}r+C}} \leq \mu_n^{\mp}(M_n \leq \mathbb{E}[M_n] - r) \leq e^{-e^{\underline{\alpha}r-C}} \end{cases}$$
where $\bar{\alpha}/\underline{\alpha} \to 1$ as $\beta \to \infty$.

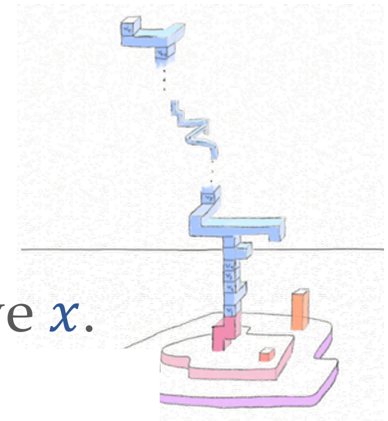
Tightness

Gumbel tails

- PROPOSITION: ([Gheissari, L. '19b])

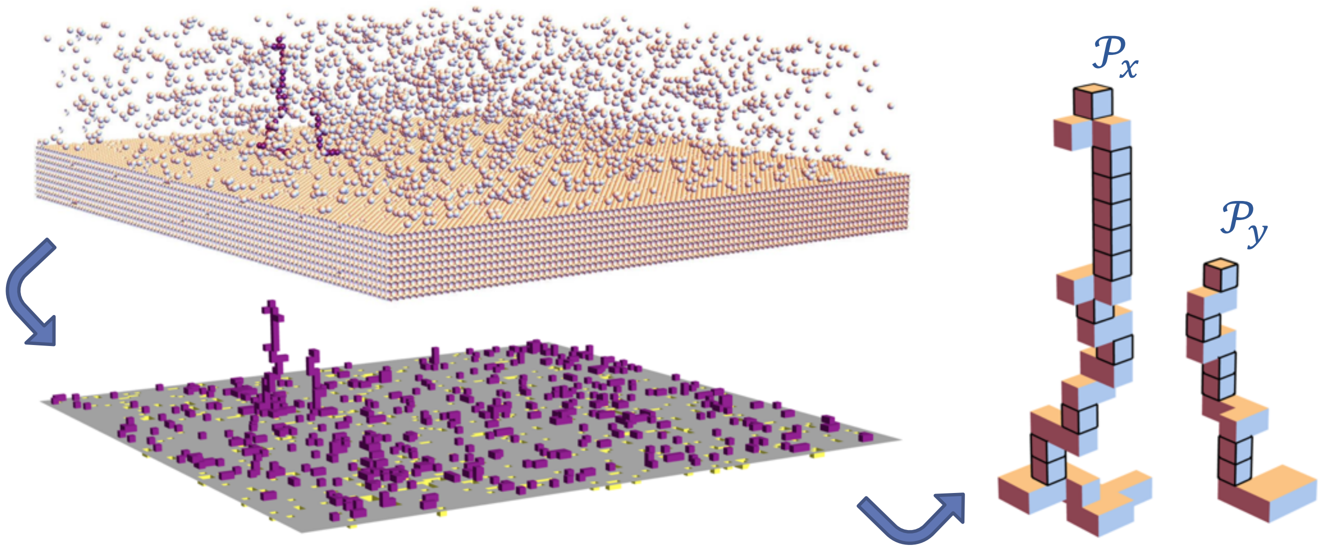
There *does not* exist a deterministic sequence (m_n) s.t. $(M_n - m_n)$ converges weakly to a nondegenerate law.

Pillars in the 3D Ising interface



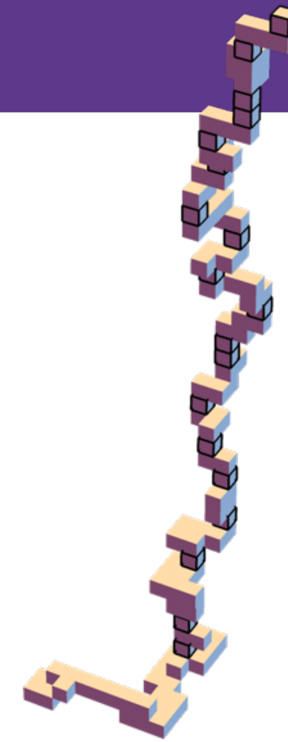
<u>**DEFINITION**</u>: [$\mathcal{P}_x$, the **pillar** at $x \in \mathbb{R}^2 \times \{0\}$]

1. Fill in all the bubbles to obtain the interface $\mathcal{I}$
2. Discard $\mathbb{R}^2 \times (-\infty, 0)$ from the sites below $\mathcal{I}$
3. The pillar $\mathcal{P}_x$ is the remaining component above x.



Goal: second moment argument for $M_n = \max_x \mathrm{ht}(\mathcal{P}_x)$

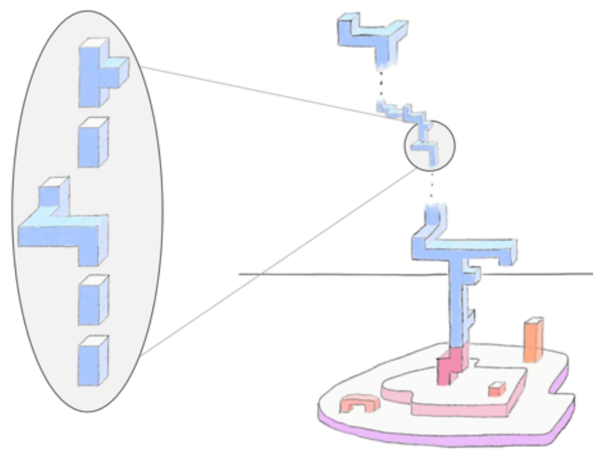
Decomposition of pillars



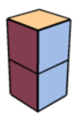
- DEFINITION: [**cutpoint** of the pillar]
  a cell v_i which is the only intersection of the pillar $\mathcal{P}_x$ with a horizontal slab.

- DEFINITION: [pillar **increment**]
  $\mathcal{X}_i$ = segment of $\mathcal{P}_x$ bounded between the cutpoints v_i, v_{i+1} (inclusively).

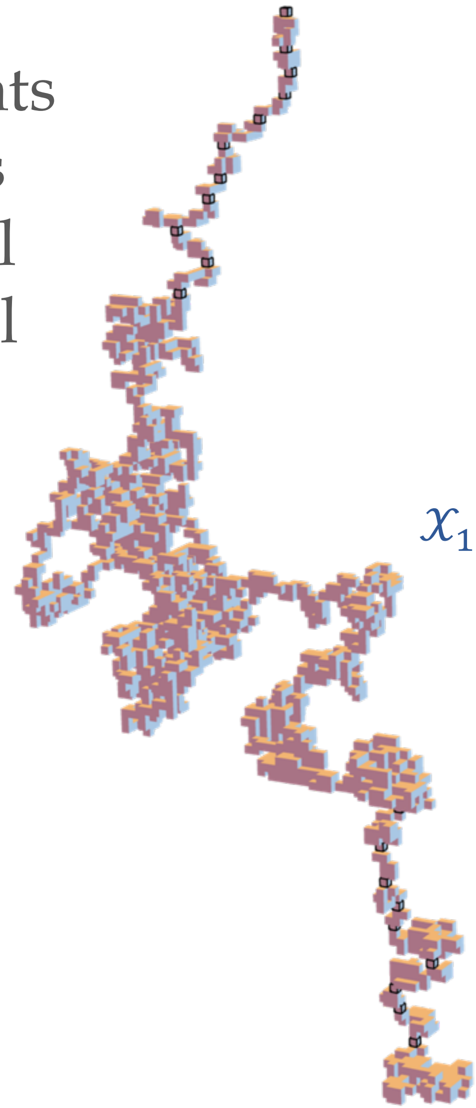
- Decompose $\mathcal{P}_x$ into:
  1. *increments* $(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_T)$
  2. *base* $\mathfrak{B}_x = \mathcal{P}_x \cap (\mathbb{R}^2 \times [0, \mathrm{ht}(v_1)])$

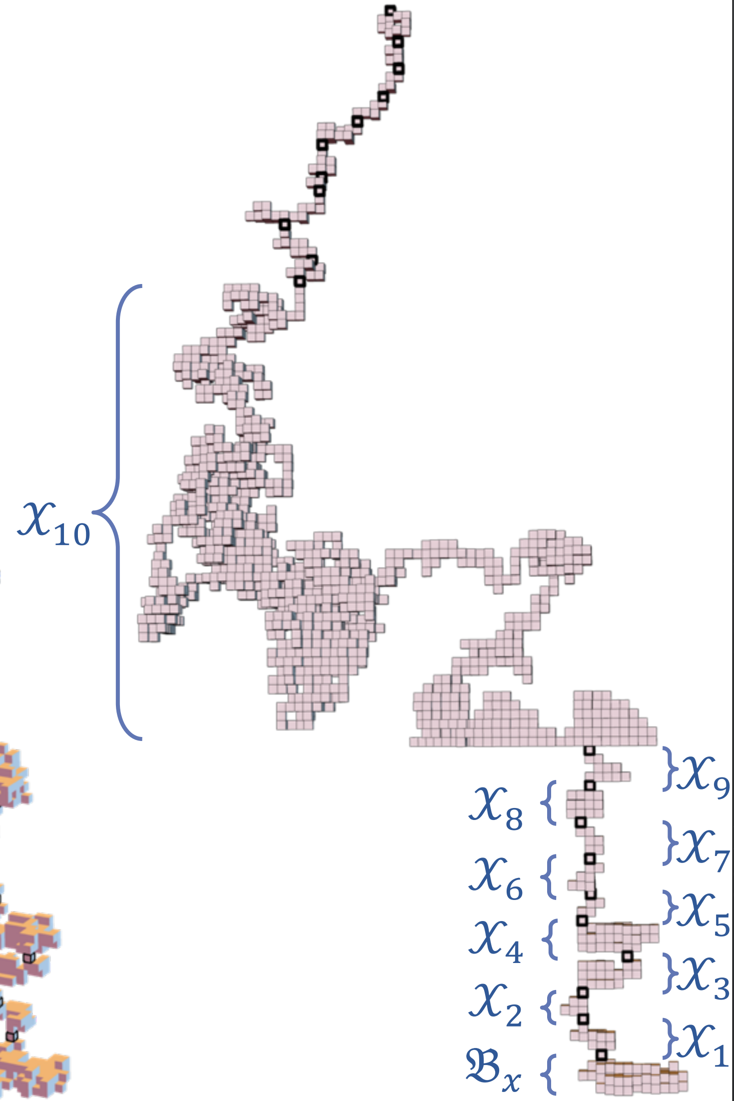


Decomposition of pillars

- Typical increments are perturbations (with exponential tails) of the trivial increment 

- But: (rarely) they can be quite complex…





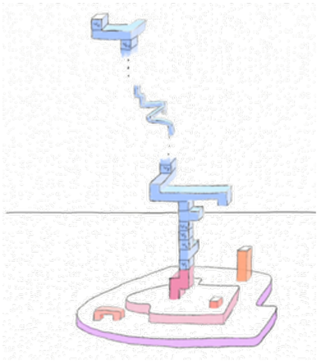
Key ingredient: shape of tall pillars

- THEOREM: ([Gheissari, L. '19a,'19b])

$\exists\, \beta_0$ s.t. for $\forall \beta > \beta_0$ and every $x = (x_1, x_2, 0)$ is in the bulk (distance $\geq h^2$ from $\partial\Lambda$), conditional on $\mathrm{ht}(\mathcal{P}_x) \geq h$,

1. W.h.p. $\mathcal{P}_x$ has at least $(1 - \epsilon_\beta)h$ increments.
2. $\forall t$, the size of the increment $\mathcal{X}_t$ has an exponential tail.
3. Base $\mathfrak{B}_x$ has an exponential tail on its diameter, height.

- Used to decorrelate $\mathrm{ht}(\mathcal{P}_x)$ and $\mathrm{ht}(\mathcal{P}_y)$ as part of the 2nd moment argument.



Cluster expansion & Dobrushin's approach

- Peierls' classical phase transition argument eliminates bubbles, but is not enough to "flatten" the interface.
- Instead: do Peierls on *interfaces* via *cluster expansion*:

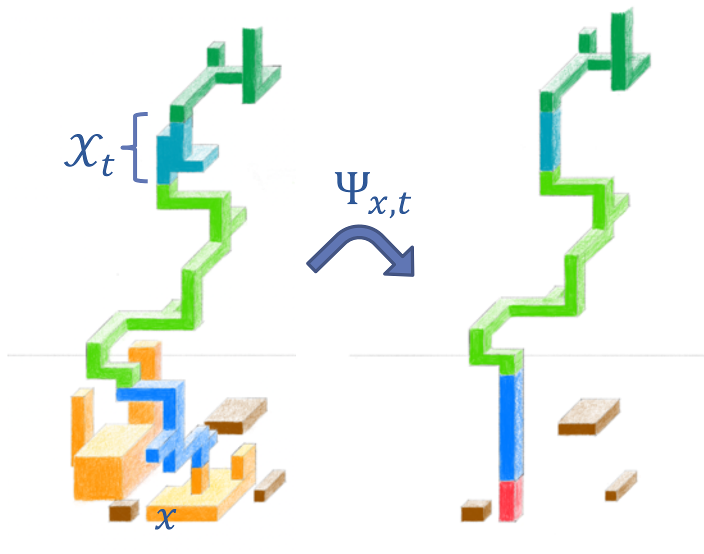
THEOREM: ([Minlos, Sinai '67], [Dobrushin '72])

$$\mu(\mathcal{I}) \propto \exp\left[-\beta|\mathcal{I}| + \sum_{f\in\mathcal{I}} \mathbf{g}(f,\mathcal{I})\right]$$

where $\mathbf{g} \le K_0$ and $|\mathbf{g}(f,\mathcal{I}) - \mathbf{g}(f',\mathcal{I}')| \le e^{-\bar{c}\,\mathbf{r}(f,\mathcal{I},f',\mathcal{I}')}$.

- [Dobrushin '72] decomposed $\mathcal{I}$ into groups of walls & ceilings, then defined a map that deletes a wall around x, flattening $\mathcal{I}$ (2D analysis).

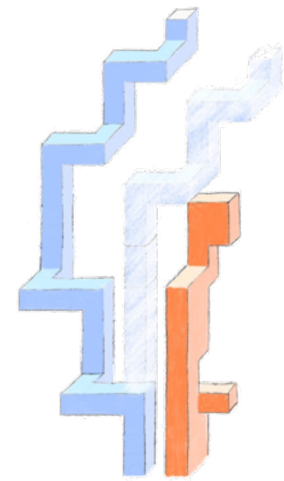
The interface map $\Psi_{x,t}$



$\Psi_{x,t}$: $\{\mathcal{I} : \mathrm{ht}(\mathcal{P}_x) \geq h, |\mathfrak{B}_x| \vee |\mathcal{X}_t| \geq r\} \to \{\mathcal{I} : \mathrm{ht}(\mathcal{P}_x) \geq h\}$ s.t.

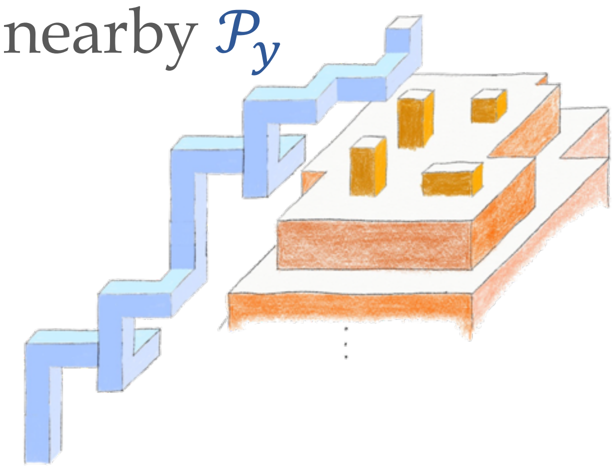
1. Energy control: $\mu(\mathcal{I}) \leq e^{-c\beta(|\mathcal{I}|-|\Psi_{x,t}(\mathcal{I})|)}\mu\left(\Psi_{x,t}(\mathcal{I})\right)$
2. Multiplicity control: at most $e^{c\ell}$ many $\mathcal{I} \in \Psi_{x,t}^{-1}(\mathcal{I}')$ such that $|\mathcal{I}| - |\mathcal{I}'| = \ell$.

Challenges due to interacting pillars



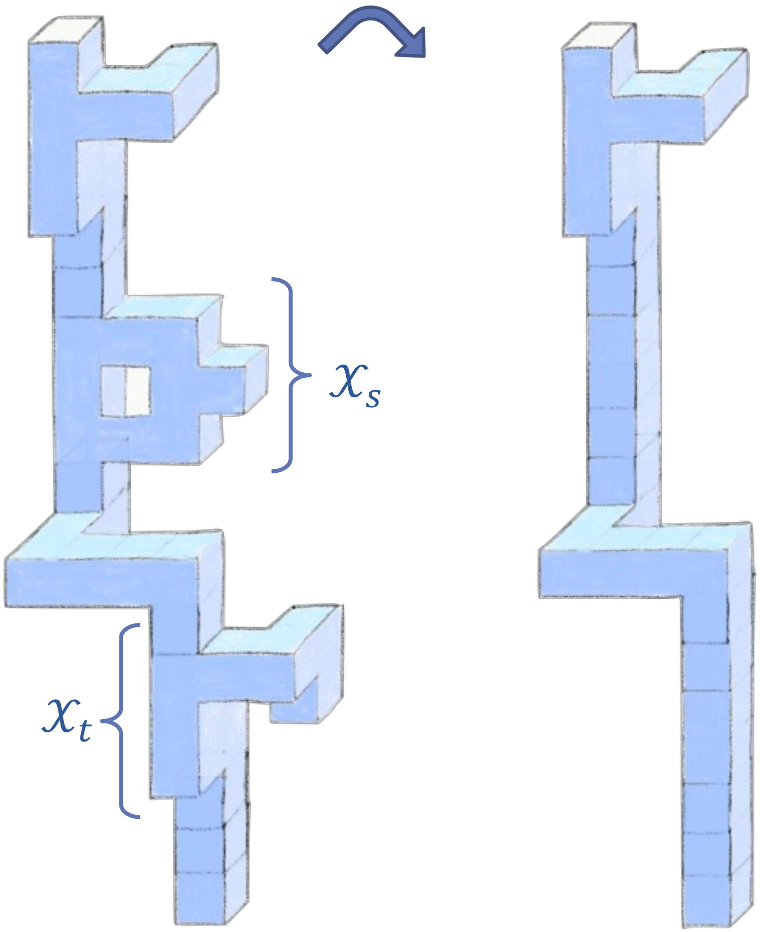
- The map $\Psi_{x,t}$ induces
  1. horizontal shifts
  2. vertical shifts (down & up)

- The pillar $\mathcal{P}_x$ to hit a nearby $\mathcal{P}_y$ (possibly making the map not well-defined)
- The pillar may get very close to a nearby $\mathcal{P}_y$ and heavily interact with it (destroying the energy control).



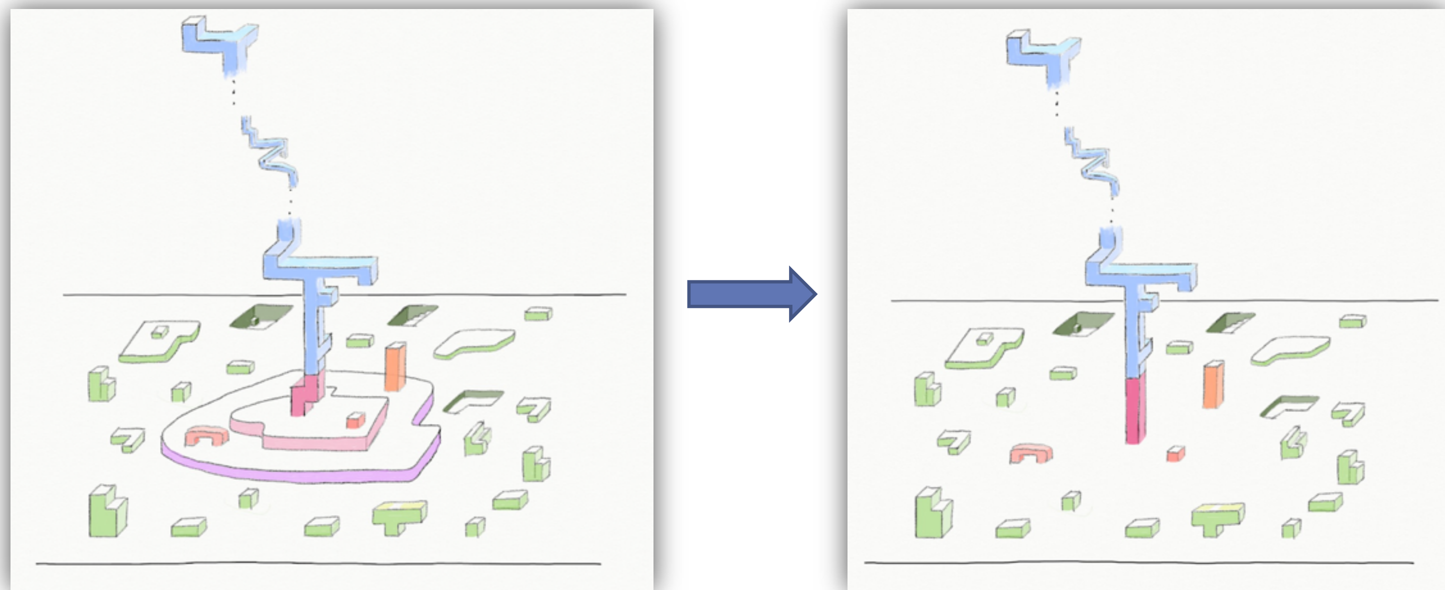
A basic $\Psi_{x,t}$ for controlling increments

- Target the structure of the increment $\mathcal{X}_t$ by:
  - straightening $\mathcal{X}_t$ if its size is too large.
  - straightening any other increment $\mathcal{X}_s$ for $s \geq t$ whose size is at least
    $$e^{c|s-t|}$$
    (*too large w.r.t.* $\mathcal{X}_t$).



A basic $\Psi_{x,t}$ for controlling increments

- Base is delicate: incorporates interaction with other nearby pillars in the interface…
- Trying to extend the definition of the base so as to rule out such interactions gives an $O(\log h)$ error on its size: sufficient for LLN but *not for tightness*.



An algorithmic procedure to define $\Psi_{x,t}$

- Defining $\Psi_{x,t}$:
  - $\forall\, j \geq 1$, determine whether to straighten $\mathcal{P}_x$ at the increment $\mathcal{X}_j$. If so:
    - $\forall y \neq x$, determine whether this action may cause $\mathcal{P}_x$ to draw to closely to $\mathcal{P}_y$. If so, delete $\mathcal{P}_y$ as well.
- Delicate balance between deleting too little (energy control) and deleting too much (multiplicity control).

Algorithm 1: The map $\Psi_{x,t}$

1 Let $\{\tilde{W}_y : y \in \mathcal{L}_{0,n}\}$ be the standard wall representation of the interface $\mathcal{I} \setminus \mathcal{S}_x$. Also let $\mathcal{O}_{v_1}$ be the nested sequence of walls of v_1, so that $\theta_{\mathrm{ST}}\mathcal{O}_{v_1} = \tilde{\mathfrak{W}}_{v_1}$.

`// Base modification`

2 Mark $[x] = \{x\} \cup \partial_0 x$ and $\rho(v_1)$ for deletion (where $\partial_0 x$ denotes the four faces in $\mathcal{L}_0$ adjacent to x).

3 **if** *the interface with standard wall representation* $\tilde{\mathfrak{W}}_{v_1}$ *has a cut-height* **then**
- Let $h^\dagger$ be the height of the highest such cut-height.
- Let $y^\dagger$ be the index of a wall that intersects $(\mathcal{P}_x \setminus \mathcal{O}_{v_1}) \cap \mathcal{L}_{h^\dagger}$ and mark $y^\dagger$ for deletion.

`// Spine modification (A): the 1st increment`

4 Set $\mathfrak{s}_1 \leftarrow 0$ and $y^*_A \leftarrow \emptyset$.

for $j = 1$ **to** $\mathscr{T} + 1$ **do**
- Let $s \leftarrow \mathfrak{s}_j$ and $\mathfrak{s}_{j+1} \leftarrow \mathfrak{s}_j$.
- **if** $\mathtt{m}(\mathscr{X}_j) \geq j - 1$ **then** `// (A1)`
  - Let $\mathfrak{s}_{j+1} \leftarrow j$.
- **if** $\mathfrak{D}_x(\tilde{W}_y, j, -v_{s+1}, 0) \leq \mathtt{m}(\tilde{W}_y)$ *for some* y **then** `// (A2)`
  - Let $\mathfrak{s}_{j+1} \leftarrow j$ and mark for deletion every y for which (A2) holds.
- **if** $\mathfrak{D}_x(\tilde{W}_y, j, -v_{s+1}, 0) \leq (j-1)/2$ *for some* y **then** `// (A3)`
  - Let $\mathfrak{s}_{j+1} \leftarrow j$ and let y^*_A be the minimal index y for which (A3) holds.

Let $j^* \leftarrow \mathfrak{s}_{\mathscr{T}+2}$ and mark y^*_A for deletion.

`// Spine modification (B): the t-th increment`

5 **if** $t > j^*$ **then**
- Set $\mathfrak{s}_t \leftarrow t - 1$ and $y^*_B \leftarrow \emptyset$.
- **for** $k = t$ **to** $\mathscr{T} + 1$ **do**
  - Let $s \leftarrow \mathfrak{s}_k$ and $\mathfrak{s}_{k+1} \leftarrow \mathfrak{s}_k$.
  - **if** $\mathtt{m}(\mathscr{X}_k) \geq k - t$ **then** `// (B1)`
    - Let $\mathfrak{s}_{k+1} \leftarrow k$.
  - **if** $\mathfrak{D}_x(\tilde{W}_y, j, -v_{s+1}, v_t - v_{j^*+1}) \leq \mathtt{m}(\tilde{W}_y)$ *for some* y **then** `// (B2)`
    - Let $\mathfrak{s}_{k+1} \leftarrow k$ and mark for deletion every y for which (B2) holds.
  - **if** $\mathfrak{D}_x(\tilde{W}_y, j, -v_{s+1}, v_t - v_{j^*+1}) \leq (k-t)/2$ *for some* y **then** `// (B3)`
    - Let $\mathfrak{s}_{k+1} \leftarrow k$ and let y^*_B be the minimal index y for which (B3) holds.
- Let $k^* \leftarrow \mathfrak{s}_{\mathscr{T}+2}$ and mark y^*_B for deletion.

else
- Let $k^* \leftarrow j^*$.

6 **foreach** *index* $y \in \mathcal{L}_{0,n}$ *marked for deletion* **do** delete $\tilde{\mathfrak{F}}_y$ from the standard wall representation $(\tilde{W}_y)$.

7 Add a standard wall $W_x^{\mathcal{J}}$ consisting of $\mathrm{ht}(v_1) - \frac{1}{2}$ trivial increments above x.

8 Let $\mathcal{K}$ be the (unique) interface with the resulting standard wall representation.

9 Denoting by $(\mathscr{X}_i)_{i\geq 1}$ the increment sequence of $\mathcal{S}_x$, set

$$\mathcal{S} \leftarrow \begin{cases} (\underbrace{X_\varnothing, X_\varnothing, \ldots, X_\varnothing}_{\mathrm{ht}(v_{j^*+1})-\mathrm{ht}(v_1)}, \mathscr{X}_{j^*+1}, \ldots, \mathscr{X}_{t-1}, \underbrace{X_\varnothing, X_\varnothing, \ldots, X_\varnothing}_{\mathrm{ht}(v_{k^*+1})-\mathrm{ht}(v_t)}, \mathscr{X}_{k^*+1}, \ldots) & \text{if } t > j^*, \\ (\underbrace{X_\varnothing, X_\varnothing, \ldots, X_\varnothing}_{\mathrm{ht}(v_{j^*+1})-\mathrm{ht}(v_1)}, \mathscr{X}_{j^*+1}, \ldots) & \text{if } t \leq j^*. \end{cases}$$

10 Obtain $\Psi_{x,t}(\mathcal{I})$ by appending the spine with increment sequence $\mathcal{S}$ to $\mathcal{K}$ at $x + (0, 0, \mathrm{ht}(v_1))$.

LLN: sub/super-multiplicativity?

- Important ingredient for the LLN: establishing

$$\exists \lim_{h\to\infty} -\frac{1}{h}\log \mu_\Lambda^{\mp}(\mathrm{ht}(\mathcal{P}_x) \geq h)\,.$$

- Natural route: establish sub/super-multiplicativity:

1. Move from $\{\mathrm{ht}(\mathcal{P}_x) \geq h\}$ to a comparable event in $\mathbb{Z}^3$:

$$A_h = \left\{x \overset{+}{\longleftrightarrow} \mathbb{R}^2 \times \{h\} \text{ in } \mathbb{R}^2 \times [0,\infty)\right\}.$$

2. If translation invariant, FKG can typically give

$$\mu_{\mathbb{Z}^3}^{\mp}\left(0 \overset{+}{\longleftrightarrow} (0,0,h_1+h_2)\right) \geq \mu_{\mathbb{Z}^3}^{\mp}\left(0 \overset{+}{\longleftrightarrow} (0,0,h_1)\right) \mu_{\mathbb{Z}^3}^{\mp}\left(0 \overset{+}{\longleftrightarrow} (0,0,h_2)\right).$$

3. **But** $\mu_{\mathbb{Z}^3}^{\mp}$ is more negative at height h_1 then at height 0 !

LLN: sub-multiplicativity

- To show that $\mu^{\mp}_{\mathbb{Z}^3}(\mathrm{ht}(\mathcal{P}_x) \geq h)$ is sub-multiplicative:
  1. Move from $\{\mathrm{ht}(\mathcal{P}_x) \geq h\}$ to a comparable event in $\mathbb{Z}^3$:
  $$A_h = \left\{ x \overset{+}{\longleftrightarrow} \mathbb{R}^2 \times \{h\} \text{ in } \mathbb{R}^2 \times [0,\infty) \right\}.$$
  2. Condition on the $+$-cluster of x in $\mathbb{R}^2 \times [0,h_1]$.
  Note: this cluster contains ***positive*** *information, notably in its intersection with* $\mathbb{R}^2 \times \{0\}$ …
- Need to show:
  - *For LLN*: effect of this positive information is $e^{o(h_1)}$.
  - *For tightness:* effect of this positive information is $O(1)$!
- Key: structure of $\mathcal{P}_x$ conditioned on $\{\mathrm{ht}(\mathcal{P}_x) \geq h_1\}$.

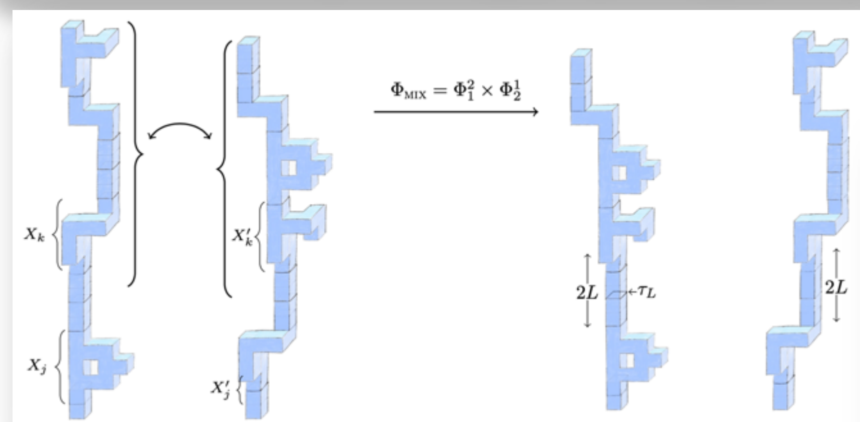
$2 \to 2$ maps: mixing and stationarity

- More refined info on shape conditional on $\mathrm{ht}(\mathcal{P}_x) \geq h$:

<u>THEOREM</u>: ([Gheissari, L. '19a])

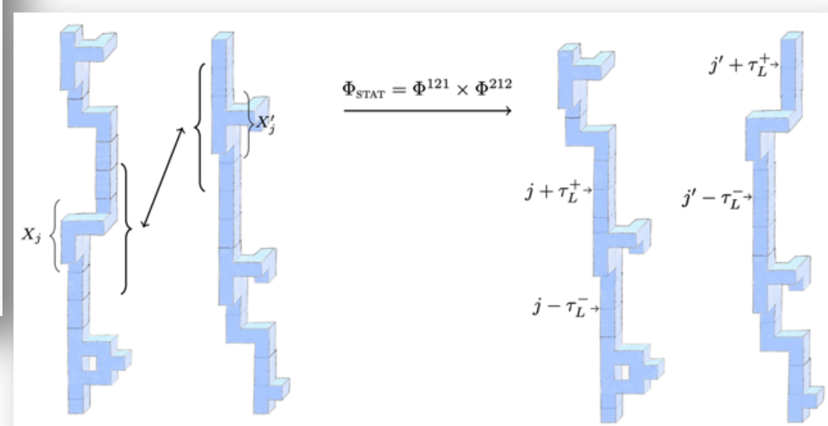
3. (mixing) $\forall i, j,\ \mathrm{Cov}(\mathcal{X}_i, \mathcal{X}_j) \leq C|j-i|^{-100}$.
4. (stationarity) $\exists$ stationary distribution ν on $\mathfrak{X}^{\mathbb{Z}}$ such that

$$\left(\ldots, \mathcal{X}_{h/2-1}, \mathcal{X}_{h/2}, \mathcal{X}_{h/2+1}, \ldots\right) \xrightarrow[h\to\infty]{} \nu$$



mixing $2 \to 2$ *map*

stationarity $2 \to 2$ *map*



CLT for pillar increments

- THEOREM: ([Gheissari, L. '19a])

For every κ there exists $\beta_0(\kappa)$ such that for all $\beta > \beta_0$:
If $f : \mathfrak{X} \to \mathbb{R}$ is a non-constant functional on increments s.t.

$$f(X) \le \exp[\kappa\,|X|] \quad \forall X$$

and $x = (x_1, x_2, 0)$ is in the bulk, then conditional on $\mathcal{P}_x$ having at least $1 \ll T_n \ll n$ increments,

$$\frac{1}{\sqrt{T_n}} \sum_{i \le T_n} (f(\mathcal{X}_i) - \mathbb{E} f(\mathcal{X}_i)) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \sigma)$$

for some $\sigma(\beta, f) > 0$.

- Proof uses a Stein's method treatment of stationary mixing sequences of random variables à la [Bolthausen '82].

CLT for location of tip, volume, surface area

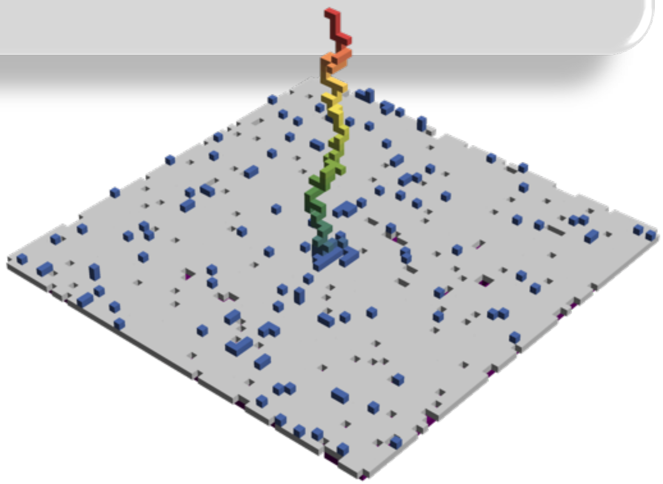
- COROLLARY: ([Gheissari, L. '19a])

Let $(Y_1, Y_2, \mathrm{ht}(\mathcal{P}_x))$ be the location of the tip of the pillar $\mathcal{P}_x$.
Conditional on $\mathcal{P}_x$ having at least $1 \ll T_n \ll n$ increments,

$$\frac{(Y_1, Y_2, \mathrm{ht}(\mathcal{P}_x)) - (x_1, x_2, \lambda T_n)}{\sqrt{T_n}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & (\sigma')^2 \end{pmatrix}\right)$$

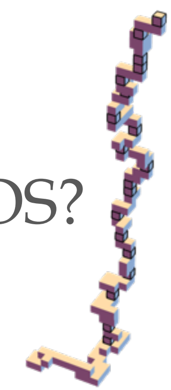
for some $\sigma, \sigma' > 0$.

- CLT also holds, e.g., for the surface area and volume of $\mathcal{P}_x$.

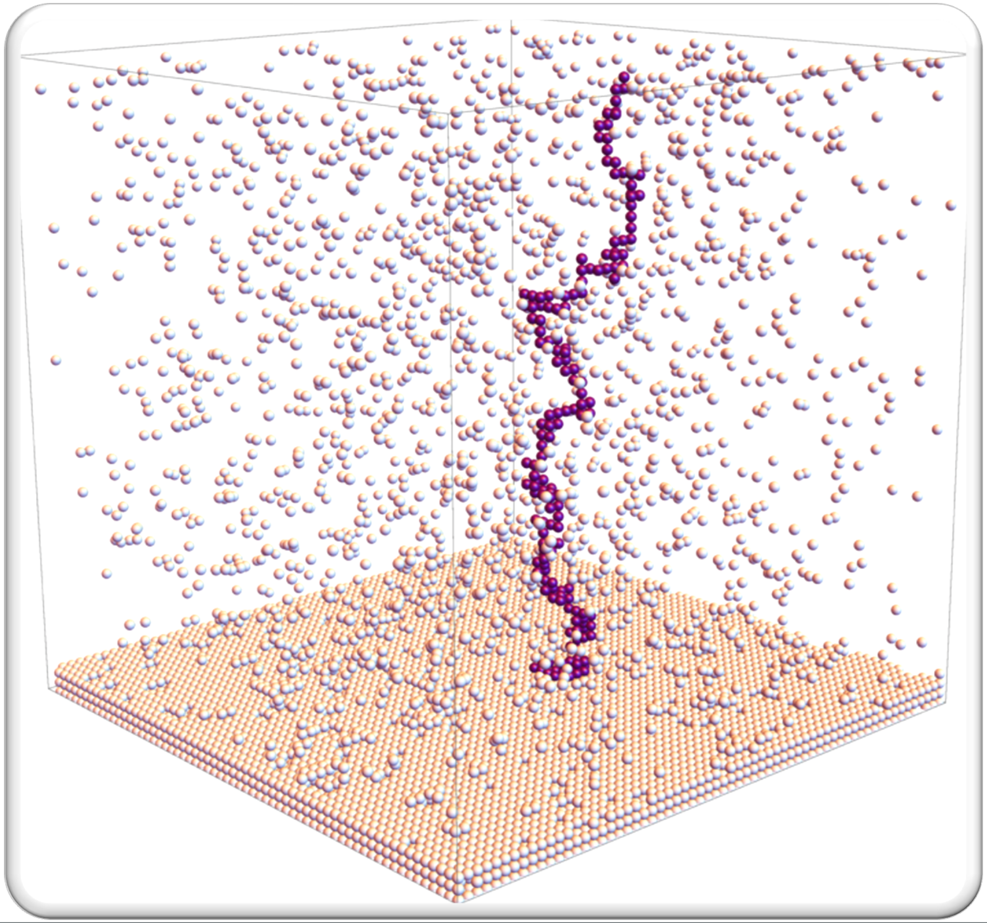


Open problems

- Open problems on M_n :
  - How does the LD quantity α depend on β?
    (*know:* $\alpha = \left(4 \pm o_\beta(1)\right)\beta$.)
    Is $\alpha < 4\beta$, so Ising interfaces are **rougher** than SOS?
  - Asymptotics of $\mathbb{E}[M_n]$?
    (*know:* $\frac{2}{\alpha_\beta}\log n + o_n(\log n)$.)



- Major open problems on the interface $\mathcal{I}$:
  - **Roughness** of ***tilted*** interfaces? (*conj.*: $\mathrm{Var}(\mathrm{ht}_x(\mathcal{I})) \asymp \log n$)
  - ***Roughening phase transition***? (*conj.*: $\beta_{\mathrm{R}} > \beta_c \Leftrightarrow d = 3$).



Thank you!