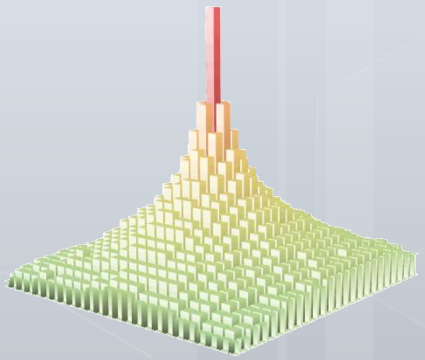
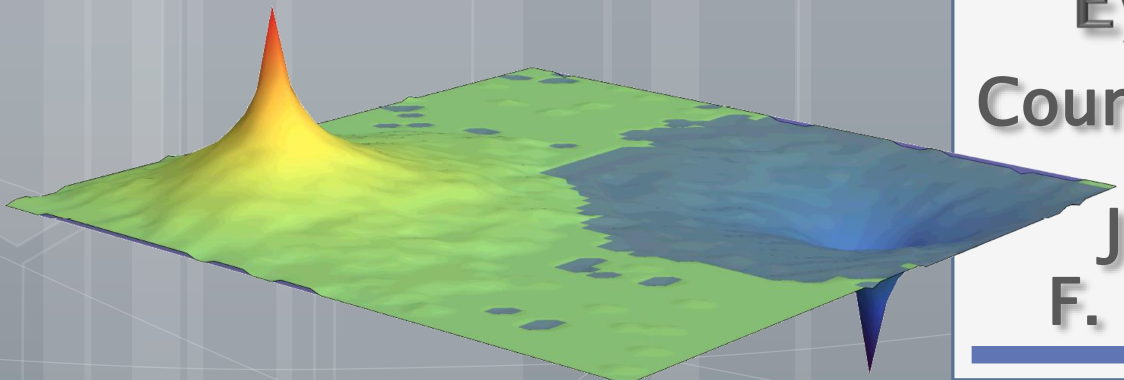


Feb 2015

ICERM, Providence

Harmonic Pinnacles in the Discrete Gaussian Model



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Joint work with
F. Martinelli, A. Sly

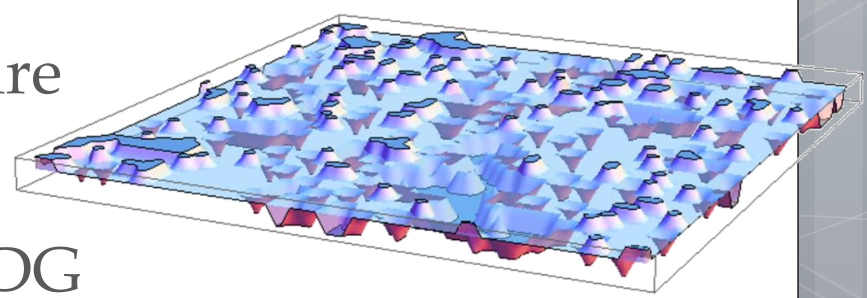
The Discrete Gaussian model

- DEFINITION: (*2D DG model*) probability measure on $\eta : \Lambda \to \mathbb{Z}$ for $\Lambda = \{1, \dots, L\}^2$ given by

$$\pi_\Lambda(\eta) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(-\beta \sum_{x\sim y} |\eta_x - \eta_y|^2\right)$$

where $\eta_x = 0$ for $x \notin \Lambda$ (0 boundary condition).

  - $\beta \geq 0$: inverse temperature
  - $Z_{\beta,\Lambda}$: partition function
  - $\pi = \lim_{L\to\infty} \pi_\Lambda$: ∞-volume DG



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- $\in$ family of surfaces models introduced in the 1950's
- dubbed *Discrete Gaussian* by [Chui-Weeks '76]
- dual of the Villain XY model [Villain '75]
- related by a duality trans. to the Coulomb gas model
- its $\mathbb{R}$-valued analogue: β scales out $\rightsquigarrow$ **DGFF**

Detour for the connoisseur

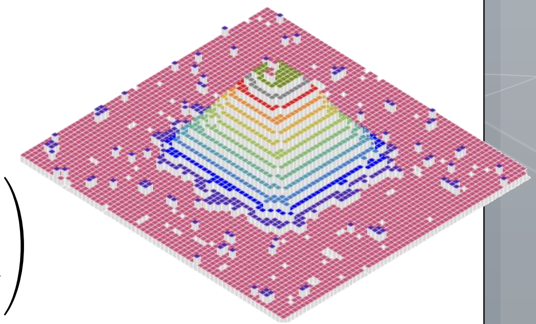
- DEFINITION: (*2D DG model*) probability measure on $\eta : \Lambda \to \mathbb{Z}$ for $\Lambda = \{1, \dots, L\}^2$ given by

$$\pi_\Lambda(\eta) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(-\beta \sum_{x\sim y} \left|\eta_x - \eta_y\right|^2\right)$$

where $\eta_x = 0$ for $x \notin \Lambda$ (0 boundary condition).

- Suppose we restrict to $\left|\eta_x - \eta_y\right| \le 1$ for every $x \sim y$.
  - Which $\{\eta : \eta_0 = h\}$ *maximize* $\pi_V(\eta)$?
    (*i.e.*, what is the ground state of $\{\eta_0 = h\}$?)
  - Hint: *Alternating Sign Matrices*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



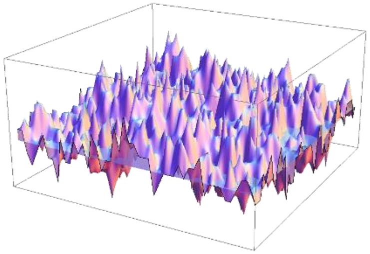
DG surface: basic questions

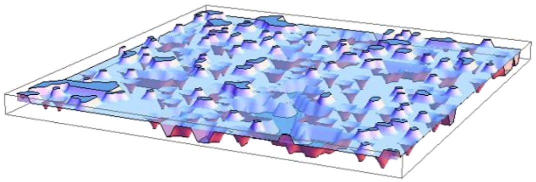
- ***Height profile:***
  I. What are the height fluctuations at the origin (say), e.g., what is $\mathbb{E}[\eta_0^2]$? Does it diverge with L?
  II. What is the maximum height $X_L = \max_x \eta_x$?
- ***The effect of a floor:***
  III. How are these affected by conditioning that $\eta \geq 0$?
- rigorously studied in breakthrough papers from the 80's
  [Fröhlich, Spencer '81a, '81b, '83], [Brandenberger, Wayne '82], [Bricmont, Fontaine, Lebowitz '82], [Bricmont, El-Mellouki, Fröhlich '86], …

DG surface: predicted behavior

- *Roughening phase transition* at a critical $\beta_{\mathrm{R}} \approx 0.665$:

$\beta < \beta_R$	$\beta > \beta_R$
rough (delocalized)	*rigid* (localized)
• fluctuations at origin $\xrightarrow[L\to\infty]{} \infty$	• $O(1)$ fluctuations at origin
• discreteness minor, $\approx$ DGFF	• discreteness relevant





- Transition exclusive to dimension $d = 2$: surface is rough for $d = 1$ and rigid for $d \geq 3$ [Temperley '52, '56] [Bricmont, Fontaine, Lebowitz '82] via [Fröhlich, Simon, Spencer '75]

High temperature DG vs. the DGFF

- **DGFF profile:** What is $\mathbb{E}[\eta_0^2]$? What is $X_L = \max_x \eta_x$?
  - $\mathrm{Var}(\eta_0) \sim \frac{2}{\pi}\log L$, $\mathbb{E}X_L \sim 2\sqrt{2/\pi}\log L$, concentration

    [Bolthausen, Deuschel, Giacomin '01], [Bolthausen, Deuschel, Zeitouni '11], [Bramson, Zeitouni '12], [Ding, Bramson, Zeitouni '15+], …

- **DGFF above a floor:** (conditioning that $\eta \geq 0$)
  - Surface bulk concentrates around $\mathbb{E}X_L$ and behaves ≈ shifted DGFF:

    $\mathbb{E}[\, X_L \mid \eta \geq 0 \,] \sim 2\,\mathbb{E}X_L \sim 4\sqrt{2/\pi}\log L$, concentration

    [Bolthausen, Deuschel, Giacomin '01]

    Analogue for $\mathbb{Z}^3$ due to [Bolthausen, Deuschel, Zeitouni '95]

- **DG for small enough β:**

  $\beta \ll \beta_R$

  - Indeed $\mathrm{Var}(\eta_0) \asymp \log L$ [Fröhlich, Spencer '81a,'81b]

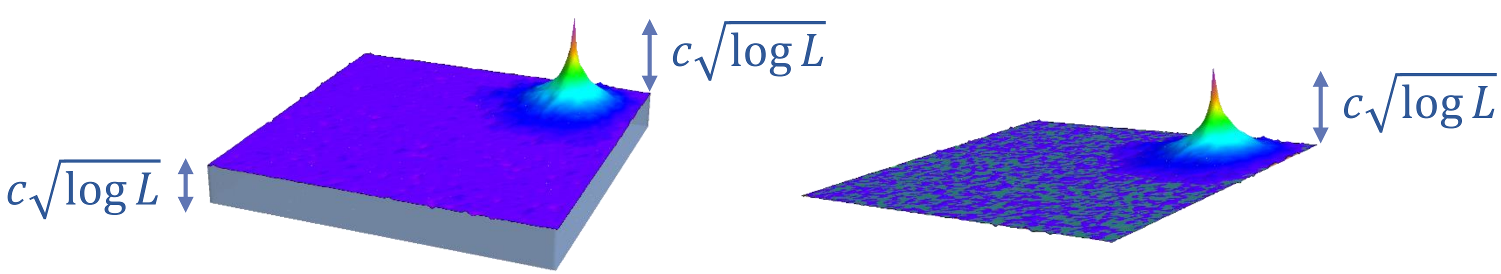
    (proof via Coulomb gas model analysis)

Low temperature DG

- Large enough β: surface is *rigid* by a Peierls argument ([Gallavotti, Martin-Lōf, Miracle-Solé '73] [Brandenberger, Wayne '82])
- [Bricmont, El-Mellouki, Fröhlich '86]:

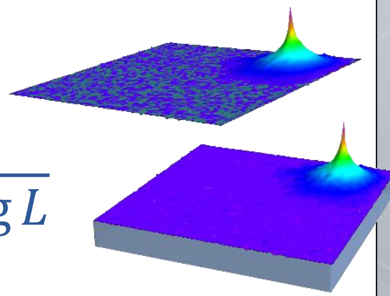
$\beta \gg \beta_R$

  - maximum: $\mathbb{E}[X_L] \asymp \sqrt{\beta^{-1}\log L}$
  - average with floor: $\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_x \eta_x \;\middle|\; \eta \geq 0\right] \asymp \sqrt{\beta^{-1}\log L}$
  - analogous results for the Absolute-Value SOS model (Hamiltonian: $\mathcal{H}(\eta) = \sum_{x\sim y}|\eta_x - \eta_y|$) with order $\beta^{-1}\log L$

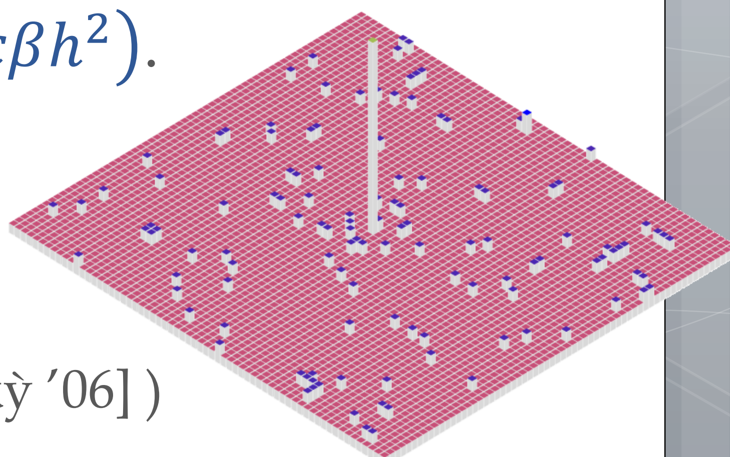


Intuition to the BEF'86 results

- [Bricmont, El-Mellouki, Fröhlich '86]:
  - maximum: $\mathbb{E}[X_L] \asymp \sqrt{\beta^{-1}\log L}$
  - average with floor: $\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_x \eta_x \;\middle|\; \eta \geq 0\right] \asymp \sqrt{\beta^{-1}\log L}$

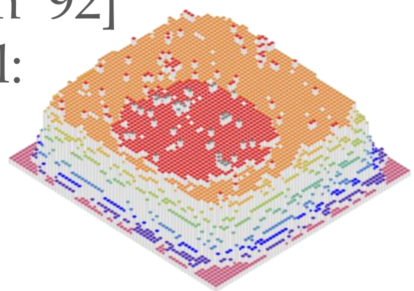


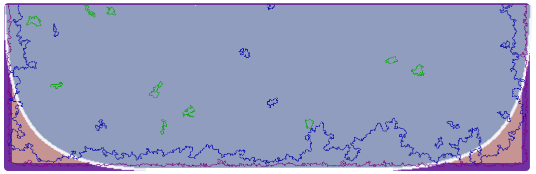
- Proof ideas:
  - *maximum*: LD governed by isolated spikes; a spike of height h costs $\exp(-c\beta h^2)$.
  - *surface height* above a floor: at most $2\mathbb{E}[X_L]$
  - *lower bound* on this height: Pirogov-Sinaï theory (see [Kotecký '06])



Progress for SOS in recent years

- [Caputo, L., Martinelli, Sly, Toninelli ’12, ‘14, ’15+]:
  Building on tools of [Dobrushin, Koteckỳ, Shlosman ‘92] and [Schonmann, Shlosman ‘95] for the Ising model:
  - maximum concentrates on $\frac{1}{2\beta}\log L$
  - average height above a floor $\sim \frac{1}{4\beta}\log L$
  - deterministic scaling limit of level-line.
  - $L^{1/3+o(1)}$ fluctuations of level-lines.





- [Ioffe, Shlosman, Velenik ‘15]:
  - Law of fluctuations ($L^{1/3} \times X$ involving Airy function)
- Central in SOS analysis:
  linearity of $\mathcal{H}(\eta) = \sum_{x\sim y} |\eta_x - \eta_y|$; what about DG?

Results: low temperature DG

Previous work: [Bricmont, El-Mellouki, Fröhlich '86]:

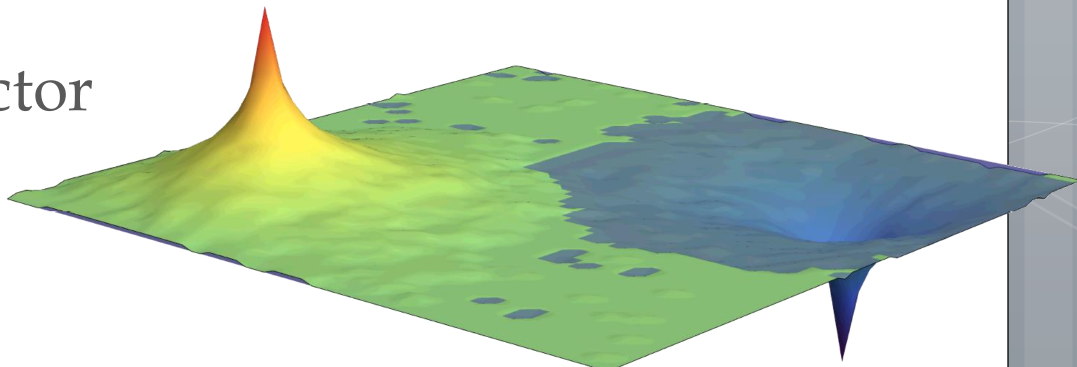
- maximum: $\mathbb{E}[X_L] \asymp \sqrt{\beta^{-1}\log L}$

- <u>THEOREM</u> [L., Martinelli, Sly]:

  $\exists M = M(L) \sim \sqrt{{}^{1}/_{2\pi\beta}\ \log L\ \log\log L}$ such that w.h.p. $X_L \in \{M, M+1\}$

- <u>REMARK</u>: *for a.e.* L *(log density)* $X_L = M$ *w.h.p.*

- Missing $\sqrt{\log\log L}$ factor due to nature of LD: *"harmonic pinnacles"* preferable to spikes.



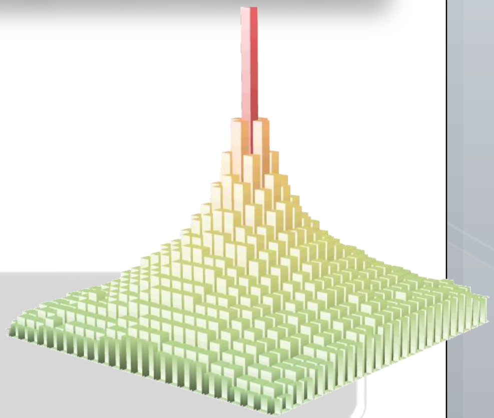
Results: low temperature DG

$M \sim \sqrt{{}^{1}/_{2\pi\beta}\ \log L\ \log\log L}$

- Central ingredient: LD estimate on ∞-volume DG:

  <ins>Proposition</ins> [L., Martinelli, Sly]:

$$\pi(\eta_0 \geq h) = \exp\left[-\left(2\pi\beta + o(1)\right)\frac{h^2}{\log h}\right]$$



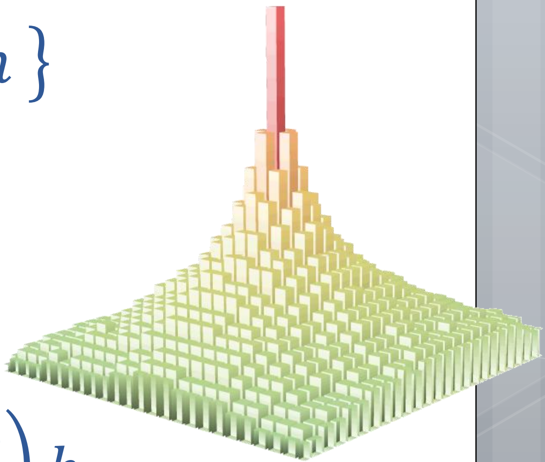
  (cf. $\exp[-c\beta h^2]$ for the prob. of a spike of height h.)

- M = max integer such that $\pi(\eta_0 \geq M) \geq L^{-2}\log^5 L$

Intuition: LD in DG

$$\log \pi(\eta_0 \geq h) \sim -2\pi\beta \frac{h^2}{\log h}$$

- LD dominated by "harmonic pinnacles", integer approximations to the discrete Dirichlet problem:
  - $I_r(h) = \inf\{\underbrace{\mathfrak{D}_{B_r}(\varphi)}_{\sum_{x\sim y}(\varphi_x - \varphi_y)^2} \;:\; \varphi|_{B_r^c} = 0\,, \varphi_0 = h\}$
  - real solution: harmonic function ϕ:



$$\phi_x = \mathbb{P}_x(\tau_0 < \tau_{\partial B_r})h = \left(1 - \frac{\log|x| + O(1)}{\log r}\right) h \quad ,$$

$$I_r(h) = 4h^2 \frac{\sum_x \mathbb{P}_x(\tau_0 < \tau_{\partial B_r})}{\mathbb{E}_0 \tau_{\partial B_r}} \sim 2\pi \frac{h^2}{\log r} \quad .$$

Intuition: LD in DG

$\log \pi(\eta_0 \geq h) \sim -2\pi\beta \frac{h^2}{\log h}$

- Real solution: $\phi_x \approx \left(1 - \frac{\log|x|}{\log r}\right) h\,,\ \ I_r(h) \sim 2\pi \frac{h^2}{\log r}$
- Discrete approximation (rounding) ends once $\phi_x < 1$:
  - Solving $\phi_x = 1$ for $|x| = r - 1$ gives $r \sim h / \log h$
  - Substituting in $I_r(h)$ gives $2\pi \frac{h^2}{\log h}$ (the sought LD rate).
- The volume of B_r is $O(h^2 / \log^2 h)$, so the rounding cost (even when charging 2β per bond in B_r) is negligible.
  - exploit exact formulas (ϕ harmonic)
  - main part: there is *no benefit from larger domains*.
- Additional ingredients: control $\frac{\pi(\eta_0 = h)}{\pi(\eta_0 = h-1)}$ and $\pi(\eta_z = h \mid \eta_0 = h)$ …

Results: shape of low temp DG

Previous work: [Bricmont, El-Mellouki, Fröhlich '86]:

- average with floor: $\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_x \eta_x \,\middle|\, \eta \geq 0\right] \asymp \sqrt{\beta^{-1}\log L}$

- THEOREM [L., Martinelli, Sly]:

conditioned on $\eta \geq 0$:

$$\exists H = H(L) \sim \sqrt{{}^{1}/_{4\pi\beta} \log L \ \log\log L} \text{ so that w.h.p.}$$

$$\#\{x : \eta_x \in \{H, H+1\}\} \geq (1-\varepsilon_\beta)L^2$$

for an arbitrarily small ε_β as β increases;

(i) $\forall 1 \leq h \leq H-1$: single macro. loop; its area is $(1-o(1))L^2$

(ii) height H : single macro. loop; its area is at least $(1-\varepsilon_\beta)L^2$

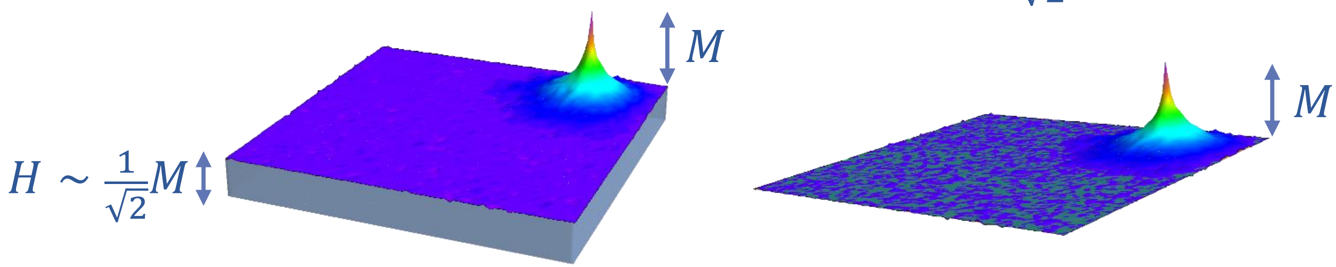
(iii) no $(H+2)$ macro. loops; no negative macro. loops.

- REMARK: *for a.e.* L *(log density) almost all sites are at level* H *w.h.p.*

Results: shape of low temp DG

- Roughly put: conditioned on $\eta \geq 0$, w.h.p.
  - DG surface is a *plateau* at height $H \sim (1/\sqrt{2})M$
  - Plateau is ≈ raised unconstrained surface.

(The floor effect increases X_L by factor of $\sim 1 + \frac{1}{\sqrt{2}}$.)



- <u>THEOREM</u> [L., Martinelli, Sly]:

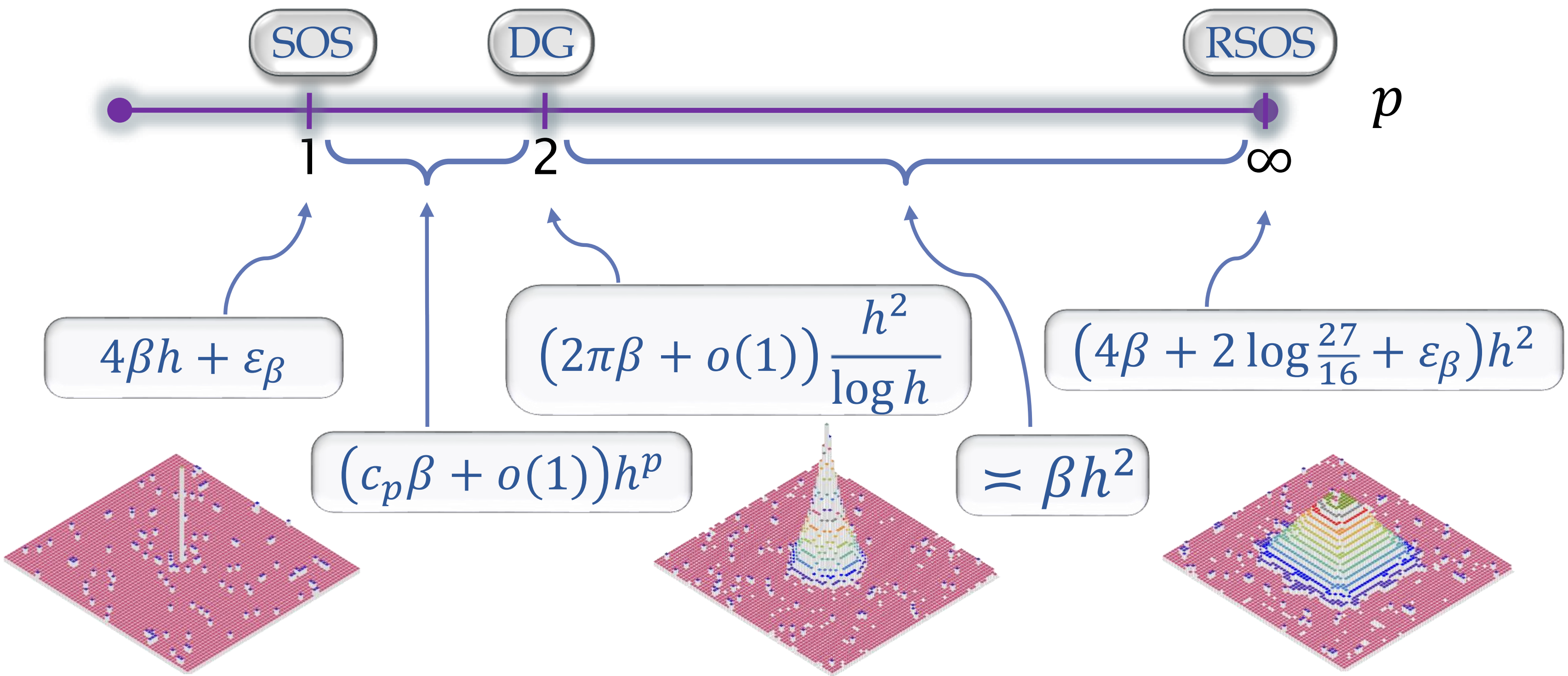
conditioned on $\eta \geq 0$:

$\exists M^* = M^*(L) \sim \frac{1+\sqrt{2}}{2\sqrt{\pi\beta}}\sqrt{\log L \ \log\log L}$ such that w.h.p.

$$X_L \in \{M^*,\ M^*+1,\ M^*+2\}$$

Generalizations to p–Hamiltonians

- Results extend to random surface models where $\mathcal{H}(\eta) = \sum_{x\sim y} |\eta_x - \eta_y|^p$ for any $p \in [1,\infty]$.



- Example: LD in ∞-volume: $-\log \pi(\eta_0 \geq h)$

Generalizations to p–Hamiltonians

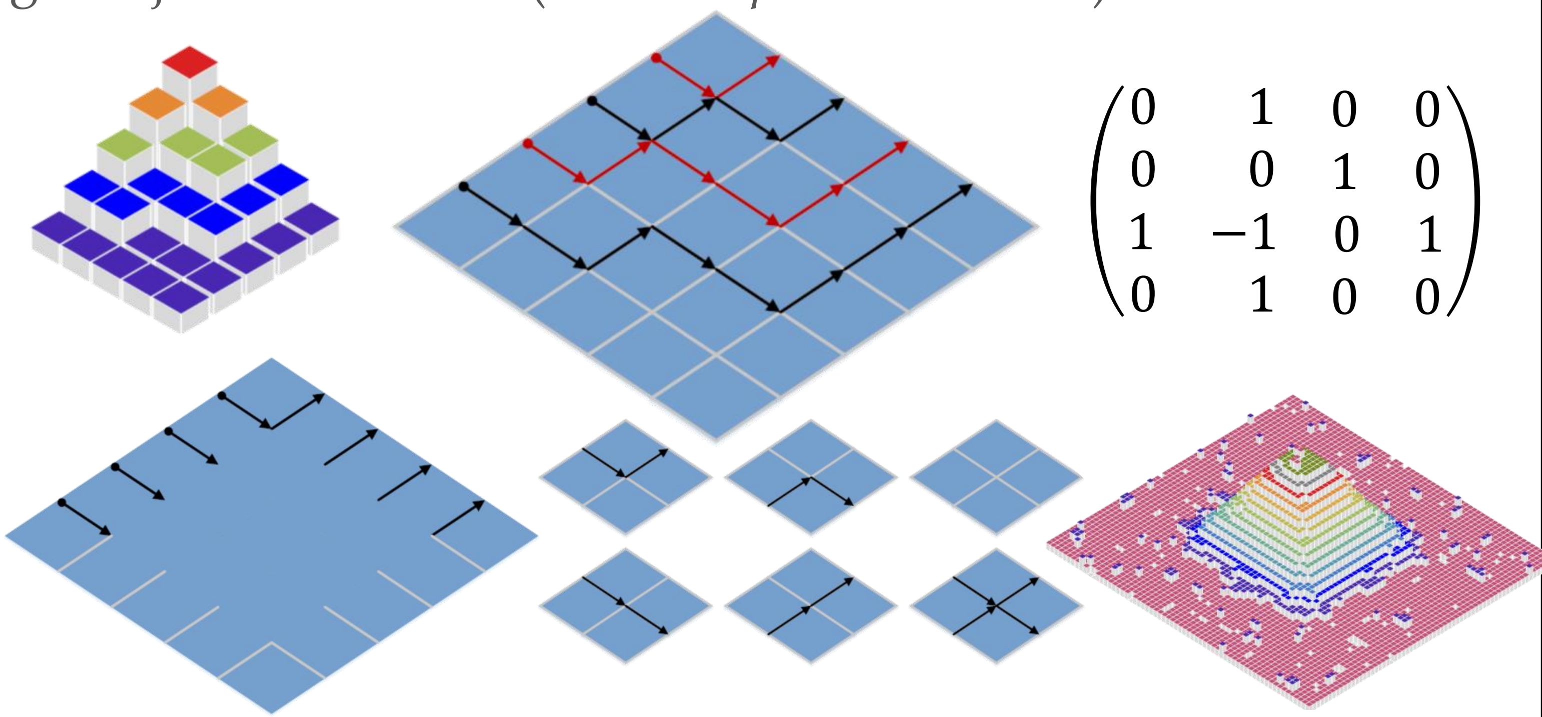
Model	Large deviation	Maximum		Height above floor	
	$-\log \pi(\eta_0 \geq h)$	center (M)	window	center (H)	window
$p=1$ (SOS)	$4\beta h + \varepsilon_\beta$	$\frac{1}{2\beta}\log L$	$O(1)$	$\lceil \frac{1}{4\beta}\log L \rceil$	± 1
$1<p<2$	$(c_p\beta + o(1))\, h^p$	$\left(\frac{2+o(1)}{c_p\beta}\log L\right)^{\frac{1}{p}}$	± 1	$\left(\frac{1+o(1)}{2}\right)^{\frac{1}{p}} M$	± 1
$p=2$ (DG)	$(2\pi\beta + o(1))\,\frac{h^2}{\log h}$	$\sqrt{\frac{1+o(1)}{2\pi\beta}\log L \log\log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1
$2<p<\infty$	$\asymp \beta h^2$	$\asymp \sqrt{\frac{1}{\beta}\log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1
$p=\infty$ (RSOS)	$\left(4\beta + 2\log\frac{27}{16} + \varepsilon_\beta\right) h^2$	$(1\pm\varepsilon_\beta)\sqrt{\frac{2}{c_\infty}\log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1

$\mathcal{H}(\eta) = \sum_{x\sim y} |\eta_x - \eta_y|^p$ for $p \in [1,\infty]$

Detour for the connoisseur: revisited

$$\log \pi(\eta_0 \geq h) = \left(4\beta + 2\log\tfrac{27}{16} + \varepsilon_\beta\right)h^2$$

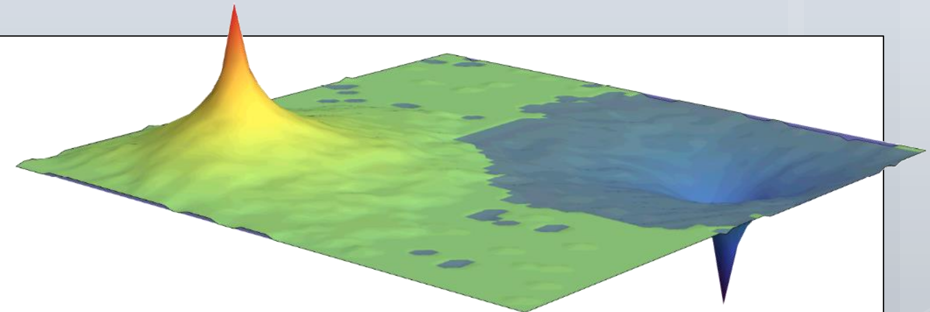
- Correspondence between RSOS optimal-energy surfaces, edge-disjoint walks and (via the *square ice* model) ASMs:



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- # of ASM's of order h is $\frac{1!\,4!\cdots(3h-2)!}{h!\,(h+1)!\cdots(2h-1)!} = \left(\frac{3\sqrt{3}}{4}\right)^{(1+o(1))h^2}$ [Zeilberger '96]
- translates into an entropy term of $\exp\left[2\log\left(\frac{27}{16}\right)h^2\right]$

Open problems



- Low temperature:
  - $L^{1/3+o(1)}$ fluctuations near center-sides?
  - Critical behavior (exceptional L's):
    - Wulff-shape scaling limit?
    - $L^{1/2+o(1)}$ fluctuations near corners?

$\beta \gg \beta_R$

- High temperature:
  - DG ≈ DGFF…; tightness of maximum? asymptotics?
  - p-Hamiltonian for $p \neq 1,2$: $\mathbb{E}[\eta_0^2] =$? Diverges?

$\beta \ll \beta_R$

- Understand β near β_R…

Thank you