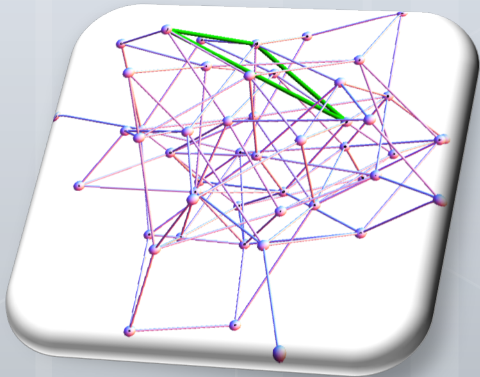
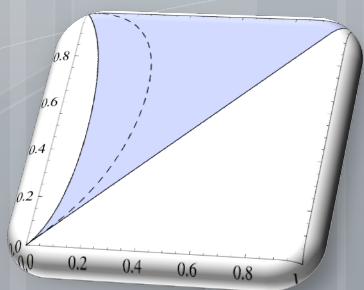
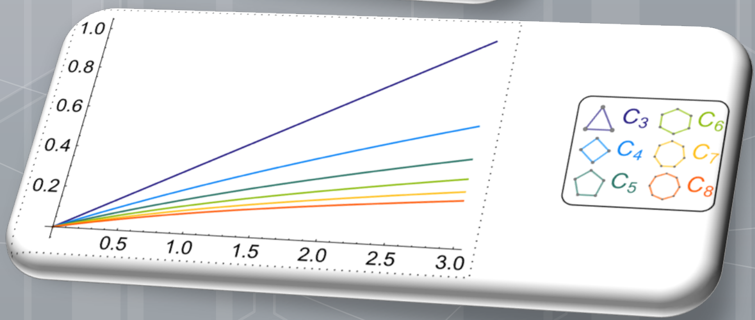


IAS CSDM seminar

Apr 2018

Large deviations in random graphs





Eyal Lubetzky

Courant Institute (NYU)

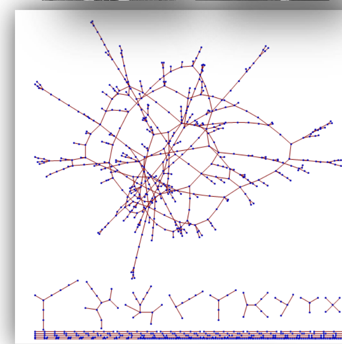
Based on joint works with
A. Dembo, B. Bhattacharya,
S. Ganguly and Y. Zhao

Subgraphs in the Erdős–Rényi RG

- $\mathcal{G}(n,p)$: indicators of $N = \binom{n}{2}$ edges: i.i.d. Bernoulli(p).

- Let
  - $G_n \sim \mathcal{G}(n,p)$ for fixed $0 < p < 1$.
  - $X_n = \#$ copies of a fixed graph H in G_n

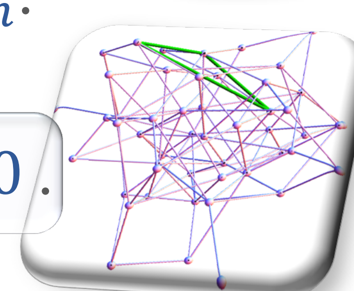
  [Ruciński '88]: $$\frac{X_n - \mathbb{E}X_n}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow[n\to\infty]{} \mathcal{N}(0,1)$$



$n = 1000$
$p = 1.5/n$

- Prototypical example: $X_n = \#$ triangles in G_n.
- Large deviations:

  estimate $\mathbb{P}(X_n \geq (1+\delta)\mathbb{E}X_n)$ for fixed $\delta > 0$.



Large deviations

- Underlying space: **i.i.d.** $Y_1, \dots, Y_N$ (e.g., edge indicators).
- **Cramér's Theorem**: address probability of rare events under mild assumption (on $\Lambda(\lambda) = \log \mathbb{E}\left[e^{\lambda Y_1}\right]$):

$$\lim_{n\to\infty} \frac{1}{N} \log \mathbb{P}\left(\frac{1}{N}\Sigma_i Y_i \geq (1+\delta)\mathbb{E}Y_1\right) = -I(\delta)$$

  with the **rate function** $I(r) := \sup_\lambda \{\lambda r - \Lambda(\lambda)\}$.

  - E.g.: if $Y_1 \sim \mathrm{Ber}(p)$ (the sum $\sim \mathrm{Bin}(N,p)$):

$$I(r) = r \log \frac{r}{p} + (1-r) \log \frac{1-r}{1-p}$$

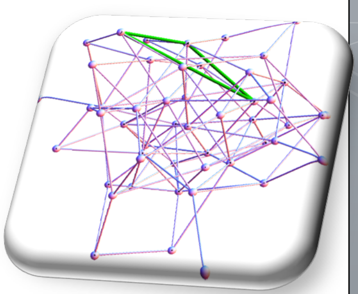
  - Hoeffding's inequality: for all $a > 0$,

$$\mathbb{P}(\Sigma_i Y_i \geq aN) \leq \inf_\lambda \left\{ e^{-(\lambda a - \Lambda(\lambda))N} \right\} = e^{-I(a)N}$$

    (optimizing the best λ in Hoeffding gives limit of the log-prob).

Large deviations

- Underlying space: **i.i.d.** $Y_1, \dots, Y_N$ (e.g., edge indicators).
- **Cramér's Theorem**: address probability of rare events under mild assumption (on $\Lambda(\lambda) = \log \mathbb{E}\left[e^{\lambda Y_1}\right]$):
$$\lim_{n\to\infty} \frac{1}{N} \log \mathbb{P}\left(\tfrac{1}{N}\Sigma_i Y_i \geq (1+\delta)\mathbb{E}Y_1\right) = -I(\delta)$$
with the **rate function** $I(r) \coloneqq \sup_\lambda\{\lambda r - \Lambda(\lambda)\}$.
- ***What about dependent random variables?***
  - One of simplest systems of dependent r.v.'s: $X_{ijk} = Y_i Y_j Y_k$ for $1 \leq i < j < k \leq n$.
  - ***Q1* ("how often"):** find the rate function
  - ***Q2* ("why"):** typical behavior cond on LD



Upper tails in random graphs

X=#triangles in $\mathcal{G}(n,p)$

- Upper tail **rate function**: $R(n,p,\delta)$ such that

$$\mathbb{P}(X \geq (1+\delta)\,\mathbb{E}X) = \exp[-R(n,p,\delta)]$$

  - The **infamous upper tail**
    S Janson, A Ruciński - Random Structures & Algorithms, 2002 - Wiley Online Library

    [Janson, Oleszkiewicz, Rucinski '04], [Bollobás '81, '85], [Janson Luczak, Rucinski '00], [Janson, Rucinski '02, '04a, '04b], [Vu '01],[Kim, Vu '04], [Chatterjee-Dey '10], … , *via* Hoeffding-Azuma ineq./ Talagrand ineq./ Stein's method/…

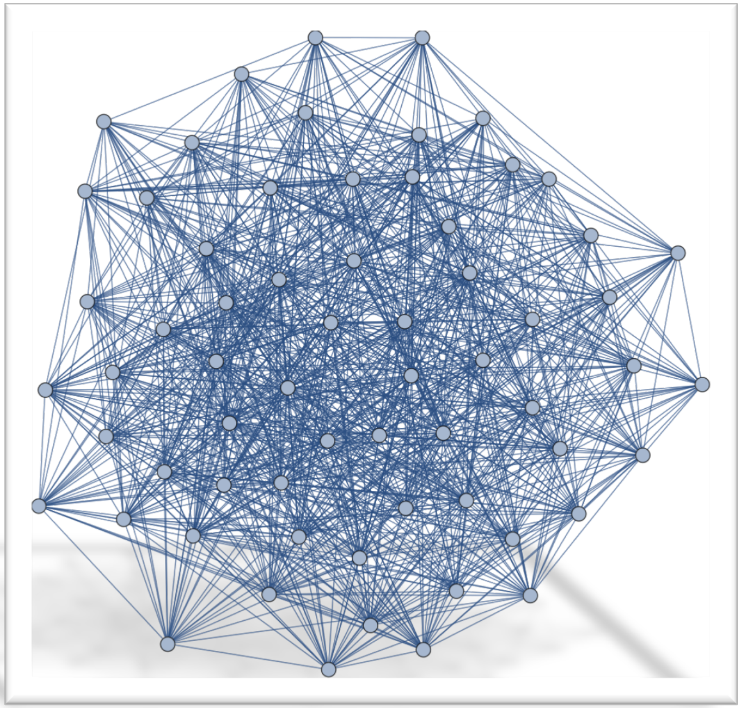
$$n^2p^2 \lesssim R(n,p,\delta) \lesssim n^2p^2\log(1/p)$$

  - **Order** of $R(n,p,\delta)$ finally resolved in [Chatterjee '12] and [DeMarco, Kahn '12], independently showing

$$R(n,p,\delta) \asymp n^2p^2\log(1/p)$$

  - leading order asymptotics?

Large deviations in $\mathcal{G}(n,p)$: the dense regime



$0 < p < 1$
fixed

Large deviations in random graphs

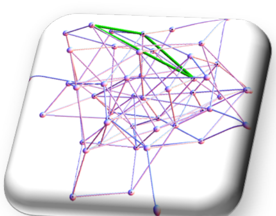
- <u>QUESTION</u> [Chatterjee and Varadhan (2011)]:
  - Fix $0 < p < r < 1$.
  - Take $G \sim \mathcal{G}(n,p)$ conditioned on having at least as many triangles as a typical $\mathcal{G}(n,r)$.
  - Is $G \approx \mathcal{G}(n,r)$, namely, are they close in cut-distance?

- Possibilities: extra triangles due to

replica symmetry

1. *(yes)* overwhelming # edges, uniformly distributed.
2. ***(no)*** fewer edges, arranged in a special structure.

symmetry breaking



cut distance between G_n and $\mathcal{G}(n,r)$:

$$\delta_\square(G_n, r) = \max_{A,B \subset V} \frac{1}{n^2} \left| e(A,B) - r|A||B| \right|$$

Upper tails of triangles in $\mathcal{G}(n,p)$

- Q: Let $G \sim \mathcal{G}(n,p)$ conditioned on $\geq \binom{n}{3} r^3$ triangles for $0 < p < r < 1$. Is $G \approx \mathcal{G}(n,r)$, namely, is $\delta_\square(G,r)$ small?
- A: depends on (p,r)…

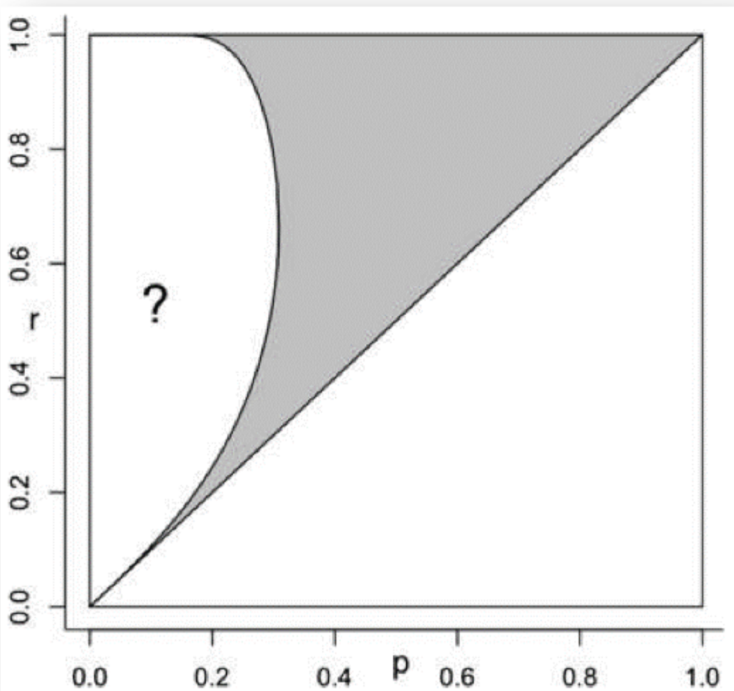


figure from [Chatterjee-Dey '10]

[Chatterjee-Dey '10]: Stein's method

[Chatterjee-Varadhan '11]: LDP via Szemerédi's regularity & graph limits.

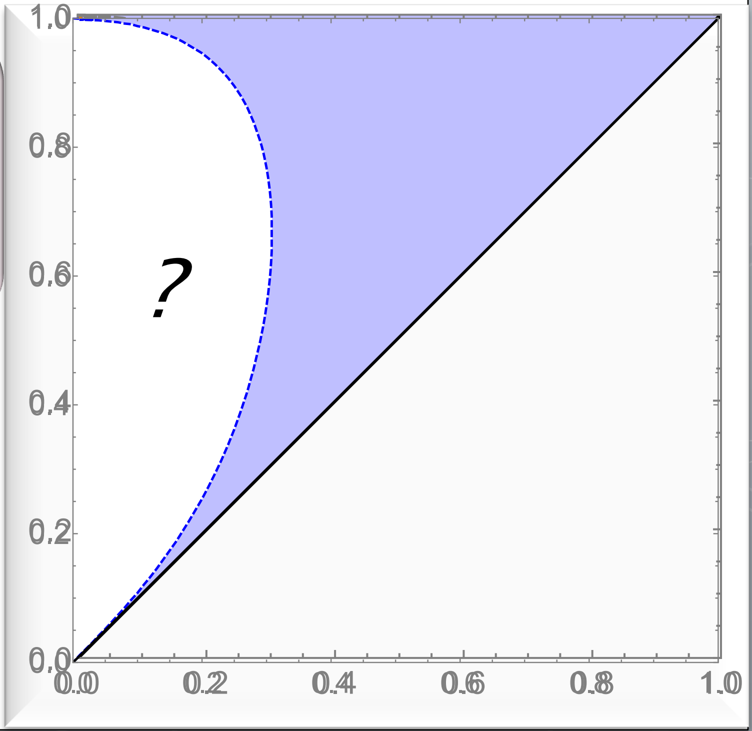
- $p \geq \frac{2}{2+e^{3/2}} \approx 0.31$: always symmetric.
- ≥ 2 phase transitions for small p.
- *e.g.*, $p = 1/4$ and $r = 1/2$?

Phase diagram for triangles

- [Chatterjee-Dey '10, Chatterjee-Varadhan '11]:
  - Replica sym. if $(r^3, I_{p(r)}) \in$ convex-minorant of $x \mapsto I_p(x^{1/3})$.
  - Full phase diagram? One or more phase transitions?
- THEOREM: ([L., Zhao '15])

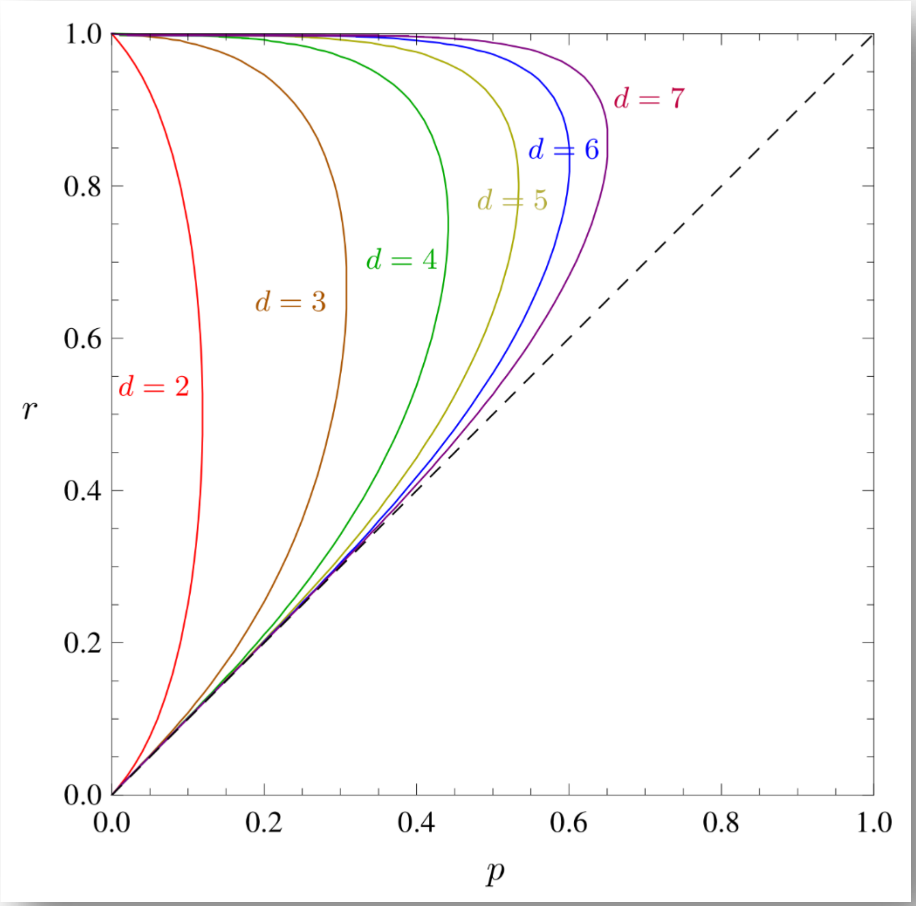
Symmetry replica for upper tails of triangles occurs iff
$$p < \left[1 + (r^{-1} - 1)^{1/(1-2r)}\right]^{-1}$$

  - Coincides with the convex minorant of $x \mapsto I_p(\sqrt{x})$



Phase diagram for regular graphs

- More generally:
  - Fix $0 < p < r < 1$ and a d-regular graph H.
  - Minimizer is $f \equiv r \iff (r^d, I_p(r))$ lies on the convex-minorant of $x \mapsto I_p(x^{1/d})$.



Variational problem (triangles)

- For each pair of vertices (i, j):
  adjust its probability to $\omega_{ij} \geq p$ at a cost of $I_p(\omega_{ij})$.
- Optimization problem:

Minimize	$\sum_{i,j} I_p(\omega_{ij})$
subject to	$\sum_{i,j,k} \omega_{ij}\omega_{jk}\omega_{ik} \geq r^3$

  OPT = rate function $R(n, p, r) \sim -\log \mathbb{P}\left(X \geq \binom{n}{3} r^3\right)$
  - [Chatterjee-Varadhan '11]: dense RG (fixed p).
  - [Chatterjee-Dembo '16]: sparse RG ($p \geq n^{-(1-o(1))/42}$).
    - [Eldan '18+]: extended region ($p \geq n^{-(1-o(1))/18}$).
    - **(very) slowly** decaying p: weak regularity.

Example: slowly decaying p

- For each pair of vertices (i, j):
  adjust its probability to $\omega_{ij} \geq p$ at a cost of $I_p(\omega_{ij})$.
- Optimization problem:

Minimize	$\sum_{i,j} I_p(\omega_{ij})$
subject to	$\sum_{i,j,k} \omega_{ij}\omega_{jk}\omega_{ik} \geq r^3$

  OPT = rate function $R(n,p,r) \sim -\log \mathbb{P}\left(X \geq \binom{n}{3} r^3\right)$

- PROPOSITION:

Let $0 < \eta < \delta < 1$ *and* $0 < p < 1$. *Then*

$$\mathbb{P}(t(K_3, \mathcal{G}(n,p)) \geq (1+\delta)p^3) \leq M^n \epsilon^{-M^2} e^{-\phi(n,p,\delta-\eta)}$$

where $\epsilon = \eta p^3/6$ *and* $M = 4^{1/\epsilon^2}$.

  - useful for $p \gg (\log n)^{1/6}$.

Variational problem (triangles)

- *Graphons:* symmetric measurable $W: [0,1]^2 \to [0,1]$.
- Optimization problem

$$\begin{aligned} &\textbf{\textit{Min}} \quad \textstyle\sum_{i,j} I_p(\omega_{ij}) \\ &\textit{subj to } \textstyle\sum_{i,j,k} \omega_{ij}\omega_{jk}\omega_{ik} \geq r^3 \end{aligned}$$

  reformulated [CV'11] as

$$\begin{aligned} &\textbf{\textit{Min}} \quad \int_{[0,1]^2} I_p(W(x,y))dxdy \\ &\textit{subj to } \int_{[0,1]^3} W(x,y)W(y,z)W(x,z)dxdydz \geq r^3 \end{aligned}$$

  - minimum achieved by compactness (Lovász-Szegedy).
- [CV'11]: solution gives the rate function; moreover, w.h.p. $(\mathcal{G}(n,p) \mid t(H,\cdot) \geq r)$ close (in $\delta_\square$) to minimizer.
  - in general, intractable…

Variational problem (general H)

- The LDP is reduced to a variational problem on *graphons* $f:[0,1]^2 \to [0,1]$ (symmetric measurable):
  - Set:
    - $I_p(f) = \int_{[0,1]^2} I_p\big(f(x,y)\big)dxdy.$
    - Subgraph count (H with $V(H) = [m]$) in f:

$$t(H,f) = \int_{[0,1]^m} \prod_{ij \in E(H)} f\big(x_i, x_j\big) dx_1 \cdots dx_m$$

    - Variational problem for upper tails:

$$\phi(p,r) = \inf\{\, I_p(f) : t(H,f) \geq r \,\}.$$

Phase diagram for triangles

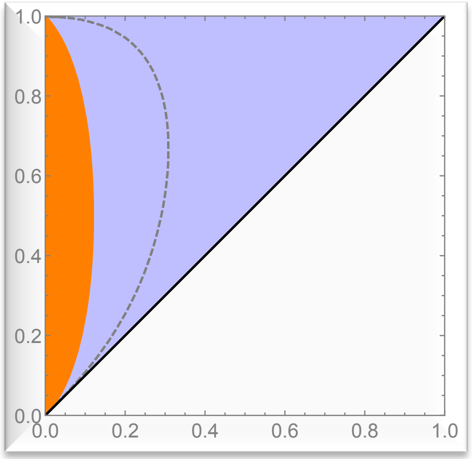
- THEOREM: ([L., Zhao '15])

Let $0 < p < r < 1$ *. The constant graphon* $W \equiv p$ *minimizes* $\int_{[0,1]^2} I_p\big(W(x,y)\big)dxdy$ *subject to*

$$\int_{[0,1]^3} W(x,y)W(y,z)W(x,z)dxdydz \geq r^3$$

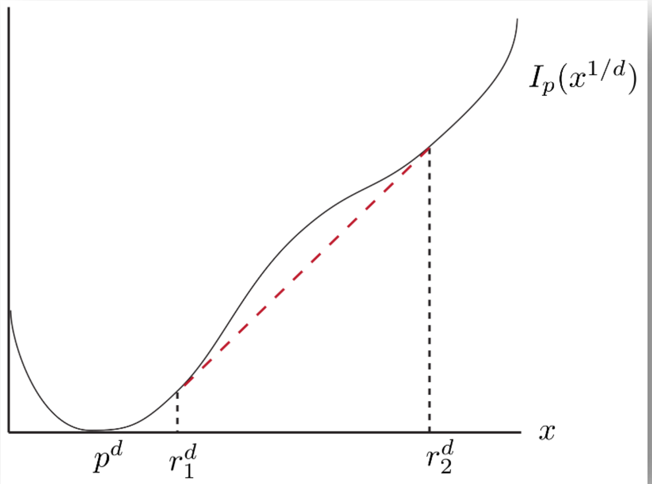
iff $\left(r^2, I_p(r)\right)$ *lies on the convex minorant of* $x \mapsto I_p(\sqrt{x})$

- Symmetry breaking phase: perturbative analysis…
- Where does the convex-minorant enter?



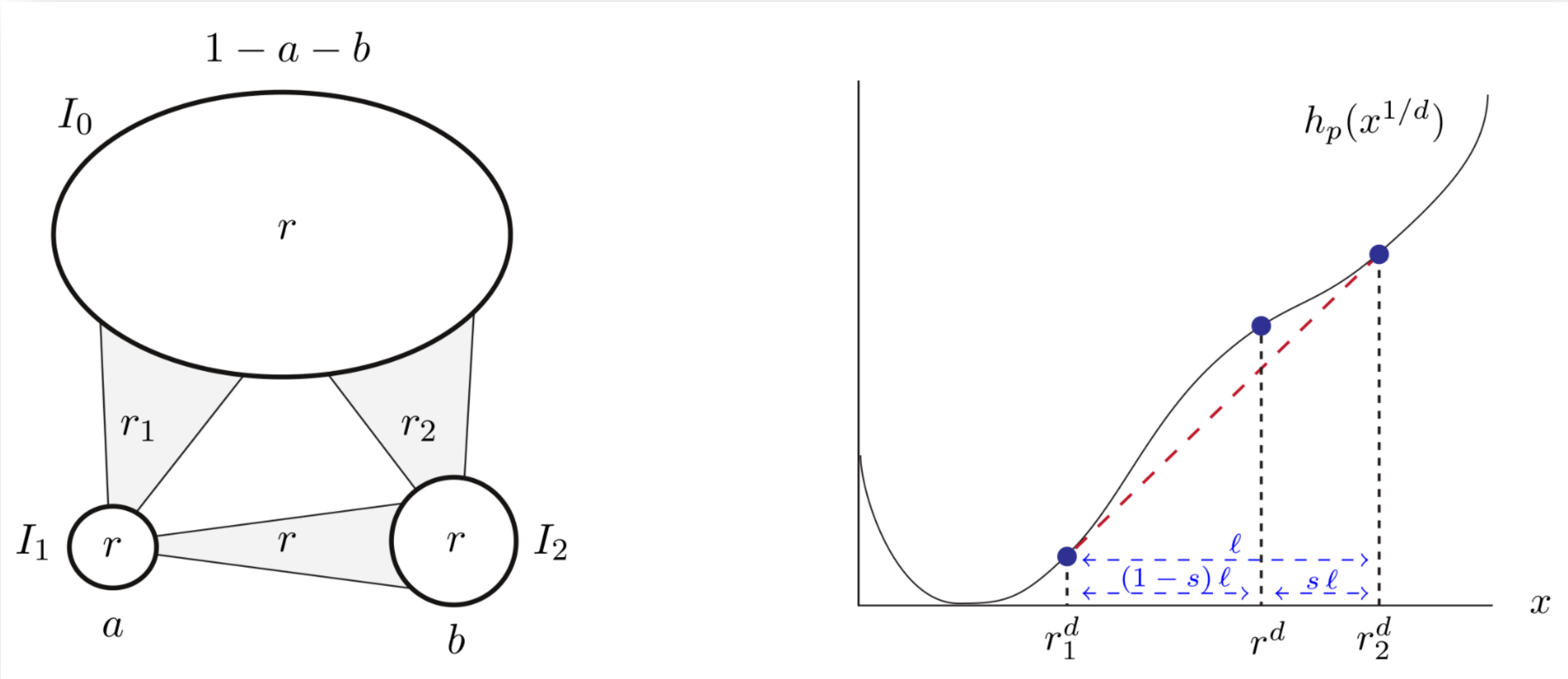
Key to sym. replica phase

- Where does the convex-minorant enter?
  - Let $\psi(x) = I_p(x^{1/k})$ and $\hat{\psi}$ be its convex-minorant.
  - Then by Jensen: $I_p(f) = \int \psi(f^k)dxdy \geq \int \hat{\psi}(f^k)dxdy \geq \hat{\psi}(\int f^k)$.
  - So, if $\int f^k \geq r^k$ and $\psi(r^k) = \hat{\psi}(r^k)$ then $I_p(f) \geq \psi(r^k) = I_p(r)$
- For example, if $t(K_3, f) \geq r^3$:
  - [CD'10],[CV'11]: $\int f^3 \geq r^3$ by Hölder.
  - One can exploit subgraph structure: generalized Hölder [Finner '92] gives $\int f^2 \geq r^2$: **correct phase**.



Matching the sym. breaking phase

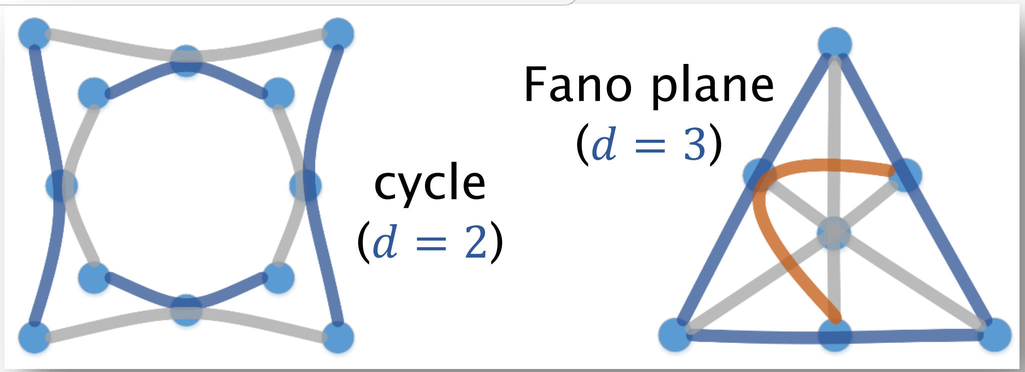
- Tri-partite construction:



  - Choice of $a = s\varepsilon^2$ and $b = (1-s)\varepsilon^2 + \varepsilon^3$ for small enough ε beats the constant function $f \equiv r$.

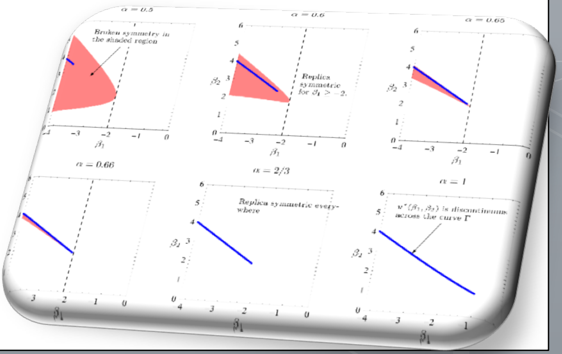
Analogs of phase-diagram result

- d-regular linear hypergraphs:



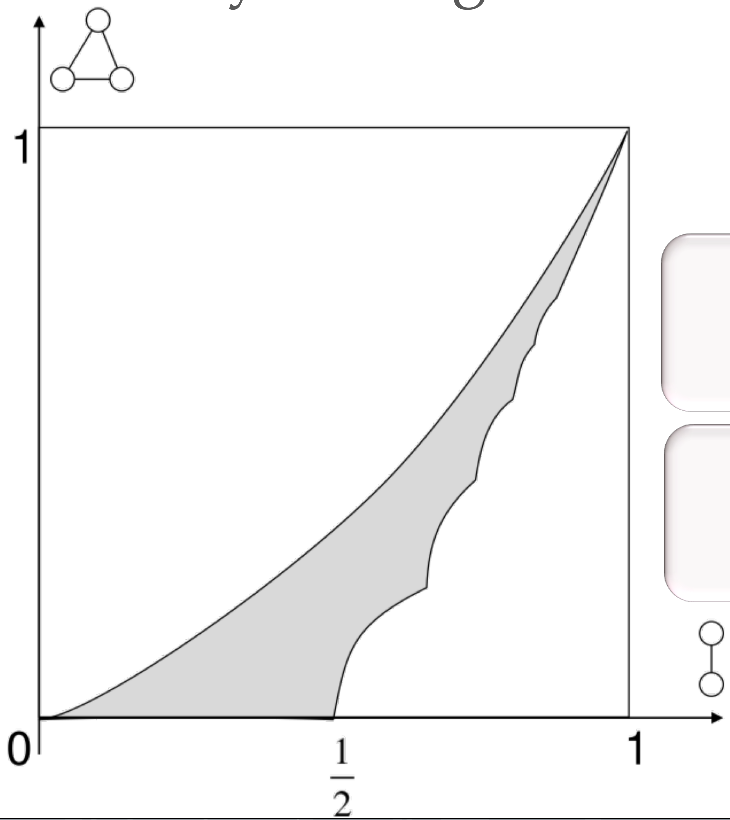
- Leading eigenvalue:
  $G \sim \mathcal{G}(n,p)$ conditioned on $\lambda_1(G) \geq nr$.
  - phase diagram coincides with $d = 2$.

- Exponential random graphs
  $\mathbb{P}(G) \propto e^{\binom{n}{2}(\beta_1 t(K_2,G)+\beta_2 t(K_3,G))}$
  (building on [Chatterjee-Diaconis'13]



Parallel: the edge-triangle model

- For fixed edge and triangle densities $(p, r) \in (0,1)^2$: what is the minimum entropy of such a graph?
  - already finding the feasible region is nontrivial:



upper curve: $r = p^{3/2}$
(special case of Kruskal-Katona)

lower curve: much harder;
[Razborov '08]: flag algebras.

figure from L. Lovász's book
"Large networks and graph limits"

Parallel: the edge-triangle model

- For fixed edge and triangle densities $(p, r) \in (0,1)^2$: what is the minimum entropy of such a graph?
  - Variational problem:

$$\psi(p,r) = \inf\{ I_{1/2}(f) : t(H,f) = r \,, t(K_2,f) = p \}.$$

  - Extension of [CV'11] (cf. [Radin, Sadun '13]): if

$\mathbb{Q}_\delta$ = uniform distribution over graphs G such that $|E(G) - m| < \delta n^2$ for $m = \lfloor \binom{n}{2} p \rfloor$

and X = # triangles in G then

note: $\mathbb{Q}_0 = \mathcal{G}(n,m)$

$$\lim_{\delta \downarrow 0} \lim_{n \to \infty} -\frac{1}{\binom{n}{2}} \log \mathbb{Q}_\delta\left(X \geq \tfrac{1}{6} n^3 r^3\right) = \psi(p,r)$$

Parallel: the edge-triangle model

- For fixed edge and triangle densities $(p, r) \in (0,1)^2$: what is the minimum entropy of such a graph?
  - Variational problem:

$$\psi(p,r) = \inf\{ I_{1/2}(f) : t(H,f) = r\, , t(K_2,f) = p\}.$$

  - [Kenyon, Radin, Ren, Sadun '16]: solution is bipodal for $r \in (p^3, p^3 + \delta)$ (more generally: cliques, stars).
  - See also: [Kenyon, Radin, Ren, Sadun '17a] on stars (M-podal for some finite M and solved for ≤ 30 nodes), [Kenyon, Radin, Ren, Sadun '17b] (numerics),…

Variational problem in $\mathcal{G}(n,m)$

- Edge-triangle variational problem:

$$\psi(p,r) = \inf\{\, I_p(f) : t(H,f) = r \,, t(K_2,f) = p \}.$$

$$\lim_{\delta \downarrow 0} \lim_{n\to\infty} -\frac{1}{\binom{n}{2}} \log \mathbb{Q}_\delta\left(X \geq \tfrac{1}{6} n^3 r^3\right) = \psi(p,r) + C_p$$

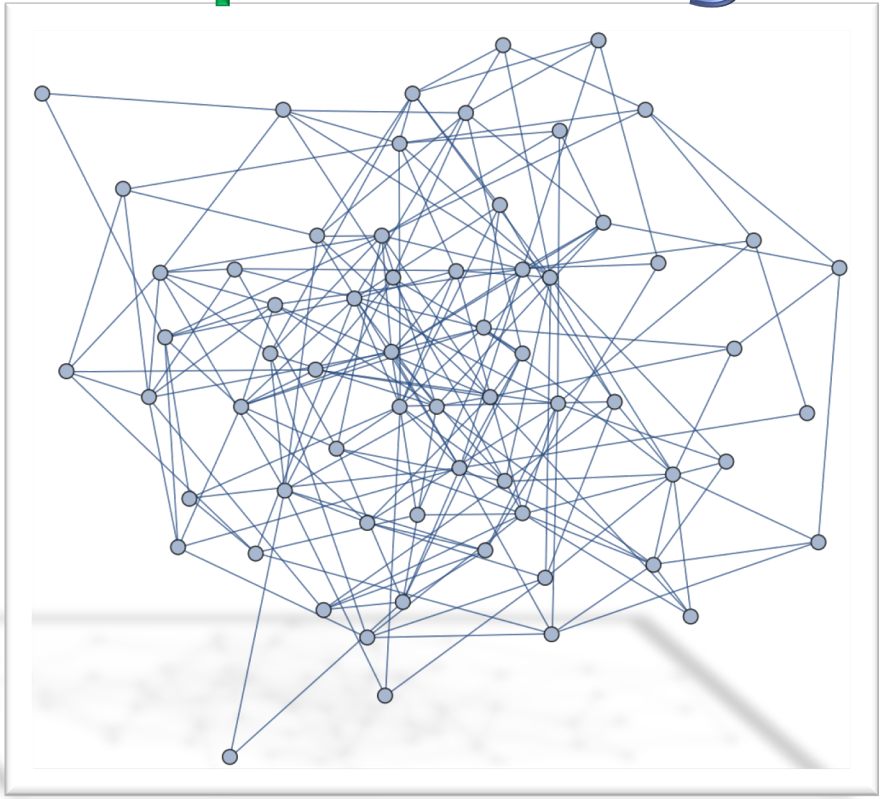
note: $\mathbb{Q}_0 = \mathcal{G}(n,m)$

- Natural guess: $\psi(p,r)$ is the rate function for $\mathcal{G}(n,m)$.
- <u>THEOREM</u>: ([Dembo, L. '18+])

For a.e. $0 < p < r < 1$ *and* H, *if* $m = (p + o(1))\binom{n}{2}$ *then*

1. *the rate function for* $\{t(H,\cdot) \geq r\}$ *in* $\mathcal{G}(n,m)$ *is* $\psi(p,r)$.
2. w.*h.p.* $(\,\mathcal{G}(n,m) \mid t(H,\cdot) \geq r\,)$ *close in* $\delta_\square$ *to OPT.*

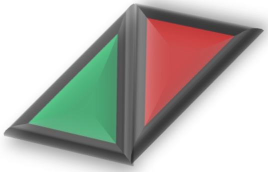
Large deviations in $\mathcal{G}(n, p)$: the sparse regime



$p_n \to 0$
as $n \to \infty$

Sparse random graphs

- Rate function in the *sparse* regime? e.g.,
  - Let $G \sim \mathcal{G}(n,p)$ for $p \ll 1$.
  - Let $X = \#$ triangles in G and write
    $$\mathbb{P}(X \geq 2\,\mathbb{E}X) = \exp[-R(n,p)]\,.$$
  - What is $R(n,p)$? [Generally, $R(n,p,\delta)$ for $X \geq (1+\delta)\,\mathbb{E}X$].
- For intuition, consider lower tails:
  - easy to see: $\mathbb{P}(X=0) \geq e^{-c\min\{n^2 p,\, n^3 p^3\}}$
  - [Janson (1990)]'s Poisson large deviation inequality:
    $$\mathbb{P}(X < (1-\delta)\mathbb{E}X) \leq e^{-c_\delta \frac{(\mathbb{E}X)^2}{\Delta + \mathbb{E}X}}$$
    matching upper bound!
  - $\Longrightarrow$ $R(n,p,\delta) \asymp \min\{n^2 p,\ n^3 p^3\}$ (transition at $p \asymp 1/\sqrt{n}$.)
- Similar treatment for upper tail?



The sparse regime

$\mathbb{P}(X \geq (1+\delta)\,\mathbb{E}X) = \exp[-R(n,p,\delta)]$

- [Chatterjee, Dembo '16]: breakthrough result:
  for $p \gg n^{-\alpha}$ one has $R(n,p,\delta) \sim \phi(n,p,\delta)$ where

  $$\phi(n,p,\delta) = \inf\{\, I_p(G) : t(K_3,G) \geq (1+\delta)p^3 \,\}$$

  over $G \in \mathfrak{G}_n$, *weighted undirected graphs* on n vertices.
- Plausibly: extends throughout $\frac{\log n}{n} \ll p \ll 1$.
- (for $p \geq (\log n)^{-c}$: follows from weak regularity.)
- Opens the door to first asymptotic LDP results for the sparse random graph…
  - Recent: new alternative proof by [Eldan '16] with a better resulting constant $\alpha > 0$ (for triangles: $\alpha = \frac{1}{18}$).

Results in the sparse regime

- THEOREM: ([L., Zhao '17])

Fix $\delta > 0$. *If* $n^{-1/2} \ll p \ll 1$ *then*

$$\lim_{n\to\infty} \frac{\phi(n,p,\delta)}{n^2p^2\log(1/p)} = \min\left\{\frac{\delta^{2/3}}{2}, \frac{\delta}{3}\right\}$$

whereas if $n^{-1} \ll p \ll n^{-1/2}$ *then*

$$\lim_{n\to\infty} \frac{\phi(n,p,\delta)}{n^2p^2\log(1/p)} = \frac{\delta^{2/3}}{2}.$$

- COROLLARY: (with $\alpha = \frac{1}{42}$ [CD'16] or $\alpha = \frac{1}{18}$ [Eldan '16])

For any $\delta > 0$, *if* $n^{-\alpha}\log n \le p \ll 1$ *then*

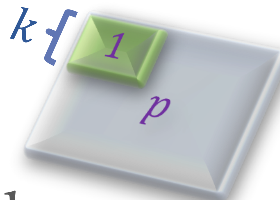
$$\mathbb{P}(X \ge (1+\delta)p^3) = e^{-(1-o(1))\min\left\{\frac{\delta^{2/3}}{2}, \frac{\delta}{3}\right\} n^2p^2\log\left(\frac{1}{p}\right)}$$

Ideas from the proofs

- For the lower bound on

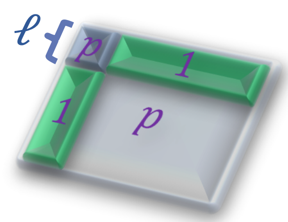
$$\mathbb{P}(X \geq (1+\delta)p^3) = e^{-(1-o(1))\min\left\{\frac{\delta^{2/3}}{2},\frac{\delta}{3}\right\}n^2p^2\log\left(\frac{1}{p}\right)}$$

  - Take an arbitrary set on $k = \delta^{1/3}np$ vertices and force it to be a *clique* :



$$p^{\binom{k}{2}} = p^{\left(\delta^{2/3}/2+o(1)\right)n^2p^2}$$

  - Or, a set of $\ell = \frac{1}{3}\delta np^2$ vertices and force it to be connected to all other vertices:



$$p^{\ell(n-\ell)} = p^{\left(\delta/3+o(1)\right)n^2p^2}$$

  - Latter is preferable iff $\delta < 27/8$.
- For the upper bound: reduce to a continuous variational problem; divide and conquer…

Extension to cliques

- THEOREM: ([L., Zhao '17])

Fix $\delta > 0$ *and* $k \geq 3$. *If* $n^{-1/(k-1)} \ll p \ll 1$ *then*

$$\lim_{n\to\infty} \frac{\phi_{K_k}(n,p,\delta)}{n^2 p^{k-1}\log(1/p)} = \min\left\{\frac{\delta^{2/k}}{2}, \frac{\delta}{k}\right\}$$

whereas if $n^{-2/(k-1)} \ll p \ll n^{-1/(k-1)}$ *then*

$$\lim_{n\to\infty} \frac{\phi_{K_k}(n,p,\delta)}{n^2 p^{k-1}\log(1/p)} = \frac{\delta^{2/k}}{2}.$$

- COROLLARY:

$\forall k \geq 3\ \exists \alpha_k > 0$: *For any* $\delta > 0$, *if* $n^{-\alpha_k} \leq p \ll 1$ *then*

$$\mathbb{P}\left(X_k \geq (1+\delta)p^{\binom{k}{2}}\right) = e^{-(1-o(1))\min\left\{\frac{\delta^{2/k}}{2}, \frac{\delta}{k}\right\} n^2 p^{k-1}\log\left(\frac{1}{p}\right)}$$

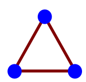
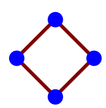
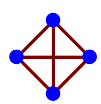
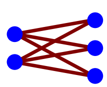
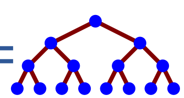
Upper tails for general graphs

- <u>THEOREM</u>: ([Bhattacharya, Ganguly, L., Zhao '17])

Fix $\delta > 0$ and H, and let $X = \#$ copies of H in $G \sim \mathcal{G}(n,p)$. If $n^{-1/(6|E(H)|)} \log n \le p \ll 1$ then

$$\mathbb{P}(X \ge (1+\delta)\mathbb{E}X) = p^{(c_H(\delta)+o(1))p^{\Delta} n^2}$$

with $\Delta =$ maximum degree of H and an explicit $c_H(\delta) > 0$.

- e.g.:
  - $H =$  : $c_H(\delta) = \min\left\{\frac{1}{2}\delta^{2/3}, \frac{1}{3}\delta\right\}$
  - $H =$  : $c_H(\delta) = \min\left\{\frac{1}{2}\delta^{1/2}, \left(1+\frac{1}{2}\delta\right)^{1/2} - 1\right\}$
  - $H =$  : $c_H(\delta) = \min\left\{\frac{1}{2}\delta^{1/2}, \frac{1}{4}\delta\right\}$
  - $H =$  : $c_H(\delta) = (1+\delta)^{1/2} - 1$
  - $H =$  : $c_H(\delta) = \frac{1}{2}\sqrt{5+4\sqrt{1+\delta}} - \frac{3}{2}$

Upper tails for general graphs

- THEOREM: ([Bhattacharya, Ganguly, L., Zhao '17])

Fix $\delta > 0$ *and* H, *and let* $X = \#$ *copies of* H *in* $G \sim \mathcal{G}(n,p)$.
If $n^{-1/(6|E(H)|)} \log n \le p \ll 1$ *then*

$$\mathbb{P}(X \ge (1+\delta)\mathbb{E}X) = p^{(c_H(\delta)+o(1))p^{\Delta} n^2}$$

with $\Delta =$ *maximum degree of* H *and an explicit* $c_H(\delta) > 0$.

- Independence polynomial: $P_H(x) := \sum_{\substack{\text{indep} \\ \text{set } I}} x^{|I|}$.
  - H^* = induced subgraph of H on max-degree vertices
  - $\theta > 0$ is the solution to $P_{H^*}(\theta) = 1 + \delta$.

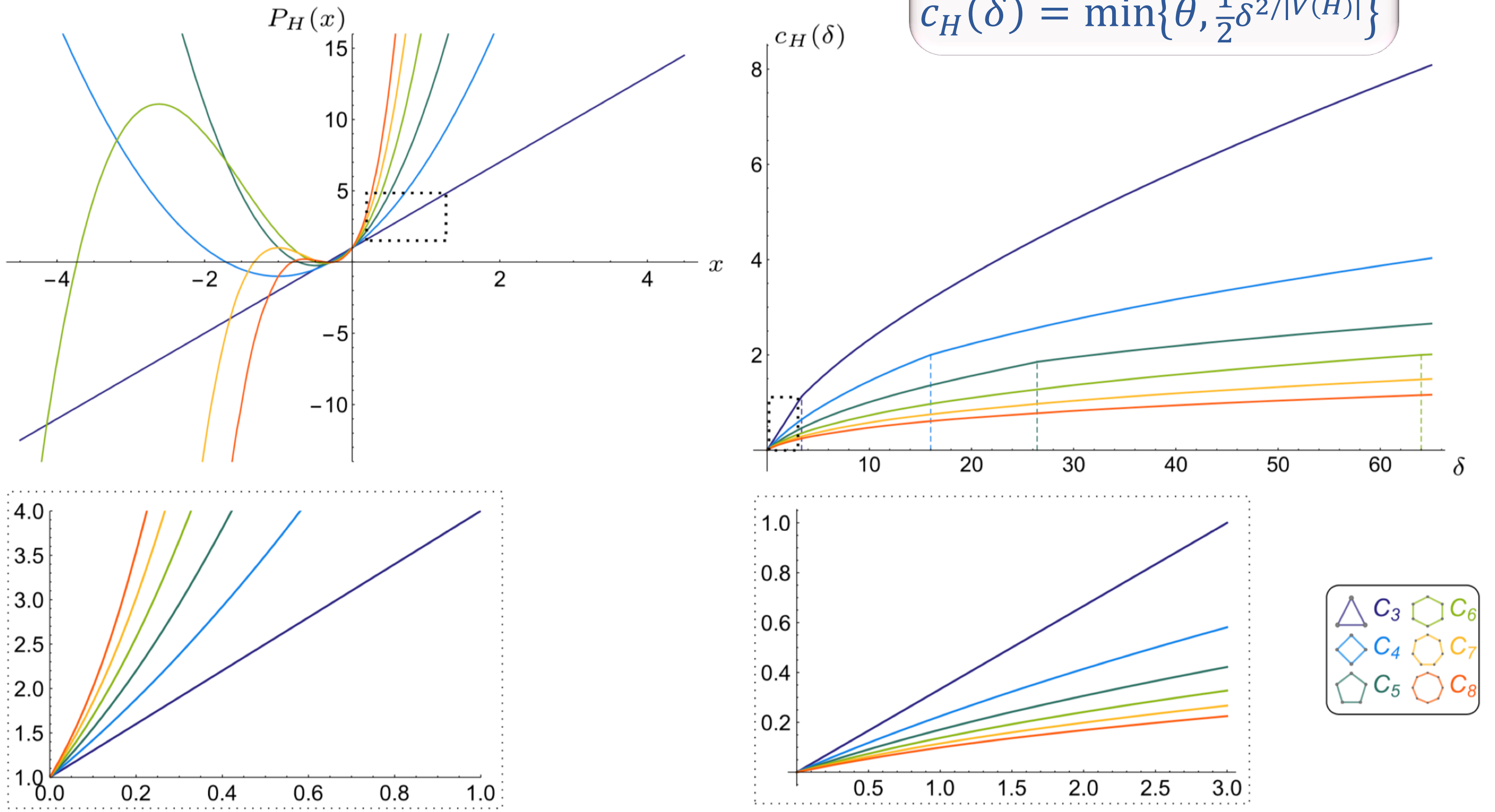
Then

$$c_H(\delta) = \begin{cases} \min\{\theta, \frac{1}{2}\delta^{2/|V(H)|}\} & H \text{ is regular} \\ \theta & H \text{ is irregular} \end{cases}$$

Upper tails for general graphs

- E.g.: LD for cycles:

$P_H(x) = \sum_{I \text{ indep.set}} x^{|I|}$
$\theta > 0$ s.t. $P_H(\theta) = 1 + \delta$
$c_H(\delta) = \min\{\theta, \frac{1}{2}\delta^{2/|V(H)|}\}$



Some upper tail open problems

- Dense regime:
  - Phase diagram for general (non-regular) graphs.
  - What is the solution in a single point within the symmetry breaking regime?
    (at no such pt. can we calculate the rate function…)
  - Uniqueness symmetry-breaking solution?
  - Are the symmetry-breaking solutions bipartite? or ∃ countable # phase transitions (# parts)?
- Sparse regime:
  - Push nonlinear large deviation results to $p \geq \frac{\log n}{n}$.

Thank you