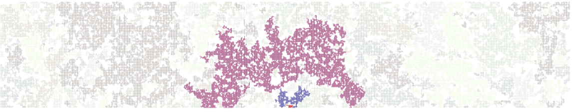
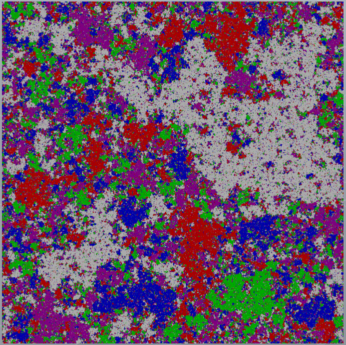


Dynamics for the critical 2D Potts/FK model: many questions and a few answers

Eyal Lubetzky

May 2018

Courant Institute, New York University



Outline

The models: static and dynamical

Dynamical phase transitions on $\mathbb{Z}^2$

Phase coexistence at criticality

Unique phase at criticality

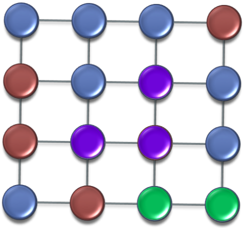
The models: static and dynamical

The (static) 2D Potts model

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{P}} = \{1,\ldots,q\}^{V(G)}$$

(each *site* receives a *color*).



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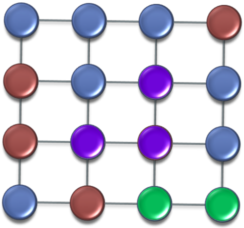

Definition (the q-state Potts model on G) [Domb '51]

Probability distribution μ_{P} on Ω_{P} given by the Gibbs measure:

$$\mu_{\mathrm{P}}(\sigma) = \frac{1}{Z_{\mathrm{P}}} \exp\left(\beta \sum_{x \sim y} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}\right)$$

($\beta \geq 0$ is the inverse-temperature; Z_{P} is the partition function)

Glauber dynamics for the Potts model

Recall: $$\mu_{\mathrm{P}}(\sigma) = \frac{1}{Z_{\mathrm{P}}} \exp\Big(\beta \sum_{x\sim y} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}\Big)$$

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Time-dependent statistics of the Ising model

RJ Glauber – Journal of Mathematical Physics, 1963 Cited by 3545

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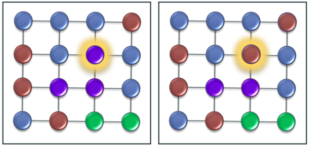

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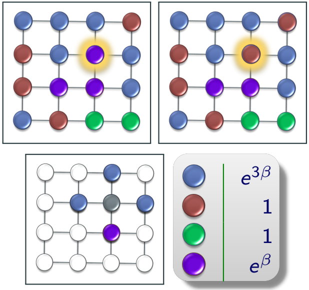

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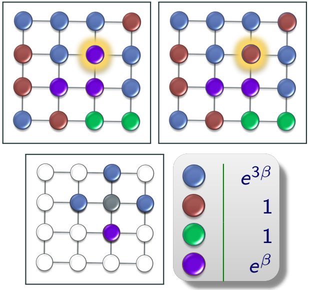
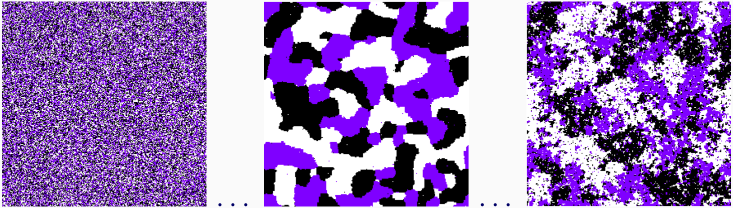


META QUESTION: *How long does it take to converge to μ?*

Glauber dynamics for the 2D Potts model

Glauber dynamics, 3-color Potts model on a 250×250 torus for $\beta = 0.5 \rightsquigarrow \beta = 2.01 \rightsquigarrow \beta = 1.01$.



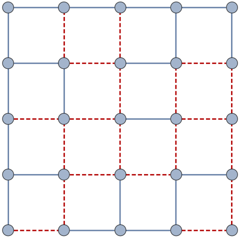
Q. 1 Fix $\beta > 0$ and $T > 0$. Does continuous-time Glauber dynamics $(\sigma_t)_{t\geq 0}$ for the 3-color Potts model on an $n \times n$ torus attain $\max_{\sigma_0} \mathbb{P}_{\sigma_0}(\sigma_T(x) = \textsc{blue})$ at σ_0 which is $\textsc{all-blue}$?

The (static) 2D Fortuin–Kasteleyn model

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{FK}} = \{\omega : \omega \subseteq E(G)\}$$

(equiv., each edge is *open*/*closed*).



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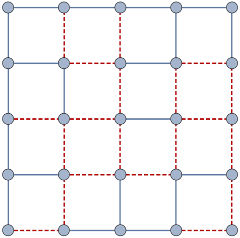

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{FK}} = \{\omega : \omega \subseteq E(G)\}$$

(equiv., each edge is *open*/*closed*).

Definition (the (p,q)-FK model on G) [Fortuin, Kasteleyin '69]

Probability distribution μ_{P} on Ω_{FK} given by the Gibbs measure:

$$\mu_{\mathrm{FK}}(\omega) = \frac{1}{Z_{\mathrm{FK}}}\left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)}$$

(Z_{FK} is the partition function; $\kappa(\omega) = \#$ connected components in ω)

Well-defined for any real (not necessarily integer) $q \geq 1$.

Glauber dynamics for the FK model

Recall: $$\mu_{\rm FK}(\omega) = \frac{1}{Z_{\rm FK}}\left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)}$$

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- An update at $e \in E$ replaces $\mathbb{1}_{\{e\in\omega\}}$ by a new spin $\sim \mu_{\mathrm{FK}}(e \in \omega \mid \omega \setminus \{e\})$.

Glauber dynamics for the FK model

Recall: $\mu_{\mathrm{FK}}(\omega) = \frac{1}{Z_{\mathrm{FK}}}\left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)}$

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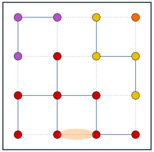

edge prob

p

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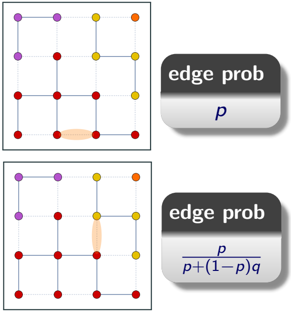
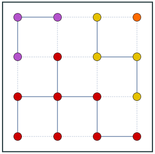


Coupling of (Potts,FK), Swendsen–Wang dynamics

[Edwards–Sokal '88]: coupling of $(\mu_{\mathrm{P}}, \mu_{\mathrm{FK}})$:

$$\Psi_{G,p,q}(\sigma,\omega) = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{|\omega|} \prod_{e=xy\in E} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}\,.$$



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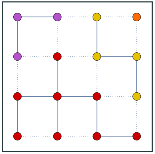


SW dynamics: $\equiv$ to $\sigma \mapsto (\sigma,\omega) \mapsto (\sigma',\omega) \mapsto \sigma'$ via this coupling:

Definition (*Swendsen–Wang dynamics*) [Swendsen–Wang '87]

- Take $p = 1 - e^{-\beta}$.
- Given $\sigma \in \Omega_{\mathrm{P}}$, generate a configuration $\omega \in \Omega_{\mathrm{FK}}$ via bond percolation on $\{e = xy \in E : \sigma(x) = \sigma(y)\}$ with parameter p.
- Color $\forall$ connected component of ω by an IID uniform color out of $\{1,\ldots,q\}$ to form $\sigma' \in \Omega_{\mathrm{P}}$.

Measuring convergence to equilibrium in Potts/FK

Measuring convergence to the stationary distribution π of a discrete-time reversible Markov chain with transition kernel P:

- Spectral gap / relaxation time:

$$\mathrm{gap} = 1 - \lambda_2 \quad \text{and} \quad t_{\mathrm{rel}} = \mathrm{gap}^{-1}$$

  where the spectrum of P is $1 = \lambda_1 > \lambda_2 > \ldots$.

- Mixing time (in total variation):

$$t_{\mathrm{mix}} = \inf\left\{ t : \max_{\sigma_0 \in \Omega} \|P^t(\sigma_0, \cdot) - \pi\|_{\mathrm{TV}} < 1/(2e) \right\}$$

(Continuous time (heat kernel $H_t = e^{t\mathcal{L}}$): gap in $\mathrm{spec}(\mathcal{L})$, and replace P^t by H_t.)

For most of the next questions, these will be equivalent.

Measuring convergence to equilibrium in Potts/FK (ctd.)

[Ullrich '13, '14]: related gap of discrete-time Glauber dynamics for Potts/FK and Swendsen–Wang on any graph G with maximal degree Δ (no boundary condition):

$$c(q,\beta,\Delta)\,\mathrm{gap}_{\mathrm{P}} \le \mathrm{gap}_{\mathrm{SW}}\,,$$
$$c(p,q)\,\mathrm{gap}_{\mathrm{FK}} \le \mathrm{gap}_{\mathrm{SW}} \le C\,\mathrm{gap}_{\mathrm{FK}}\,|E|\log|E|\,.$$

(Extended to any real $q > 1$ by [Blanca, Sinclair '15].)

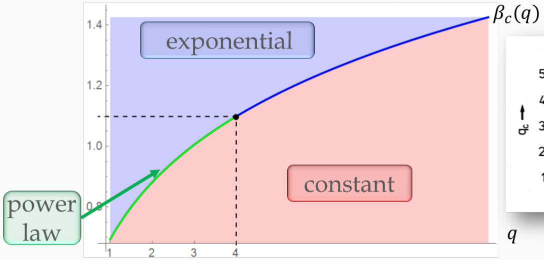
Up to polynomial factors: FK Glauber and SW are equivalent, and are at least as fast as Potts Glauber.

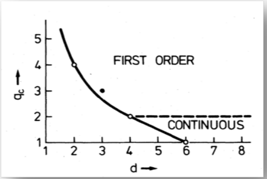
Q. 2 Is $\mathrm{gap}_{\mathrm{SW}} \ge c(q,\beta,\Delta)\,\mathrm{gap}_{\mathrm{P}}^{\mathrm{cont}}$ on $\forall$ G with max degree Δ?

Dynamical phase transitions on $\mathbb{Z}^2$

Dynamical phase transition

Prediction for Potts Glauber dynamics on the torus





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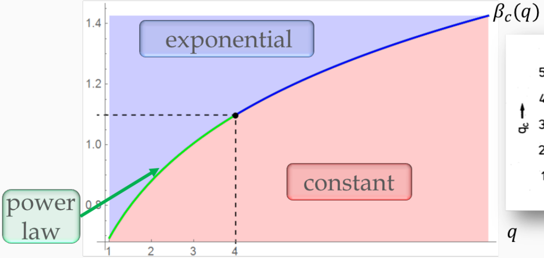
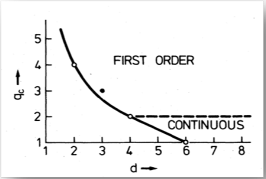




Swendsen–Wang: expected to be fast also when $\beta > \beta_c$.

- [Guo, Jerrum '17]: for $q = 2$: fast on any graph G at any β.
- [Gore, Jerrum '97]: for $q \geq 3$: slow on complete graph at β_c...

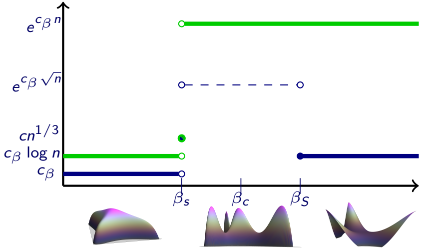
Intuition from the complete graph

Mixing time for Potts Glauber on the complete graph:

- [Ding, L., Peres '09a, '09b], [Levin, Luczak, Peres '10]: full picture for $q = 2$.
- [Cuff, Ding, L., Louidor, Peres, Sly '12]: full picture for $q \geq 3$.

Mixing time for Swendsen–Wang on the complete graph:

- [Cooper, Frieze '00], [Cooper, Dyer, Frieze, Rue '00], [Long, Nachmias, Ning, Peres '11]: full picture for Ising ($q = 2$).
- [Cuff, Ding, L., Louidor, Peres, Sly '12] full picture for $q \geq 3$.
- [Gore, Jerrum '99]: $e^{c\sqrt{n}}$ at β_c for $q \geq 3$; [Galanis, Štefankovic, Vigoda '15] and [Blanca, Sinclair '15]: picture of SW/CM for $\beta \notin (\beta_s, \beta_S)$, $e^{c\sqrt{n}}$ at $\beta \in (\beta_s, \beta_S)$; [Gheissari, L., Peres '18]: extended the $e^{c\sqrt{n}}$ to e^{cn}.



Results on $\mathbb{Z}^2$ off criticality

- **High temperature**:
  - [Martinelli,Olivieri '94a,'94b],[Martinelli,Olivieri,Schonmann '94c]: $\text{gap}_{\text{P}}^{-1} = O(1)$ $\forall \beta < \beta_c$ at $q = 2$; extends to $q \geq 3$ via [Alexander '98], [Beffara,Duminil-Copin '12].
  - [Blanca, Sinclair '15]: rapid mixing for FK Glauber $\forall \beta < \beta_c, q > 1$.
  - [Huber '99],[Cooper,Frieze '00]: SW is fast for $q = 2$ and β small.
  - [Blanca, Caputo, Sinclair, Vigoda '17]: $\text{gap}_{\text{SW}}^{-1} = O(1)$ $\begin{matrix} \forall \beta < \beta_c \\ \forall q \geq 2 \end{matrix}$.
  - [Nam, Sly '18]: cutoff for SW for small enough β.

Results on $\mathbb{Z}^2$ off criticality

- **Low temperature**:
  - [Chayes, Chayes, Schonmann '87], [Thomas '89], [Cesi, Guadagni, Martinelli, Schonmann '96]: $e^{(c_\beta + o(1))n}$ mixing $\forall \beta > \beta_c$ at $q = 2$.
  - [Martinelli '92]: $O(\log^\alpha n)$ mixing for SW at $q = 2$ and β large.
  - [Blanca, Sinclair '15]: rapid mixing for FK Glauber $\forall \beta > \beta_c, q > 1$; implies $t_{\mathrm{mix}} = n^{O(1)}$ for SW via [Ullrich '13,'14].
  - [Borgs, Chayes, Frieze, Kim, Tetali, Vigoda, Vu '99] and [Borgs, Chayes, Tetali '12]: e^{cn} mixing at $\beta > \beta_c$ and large q.
    (Result applies to the d-dimensional torus for any $d \geq 2$ provided $q > Q_0(d)$.)

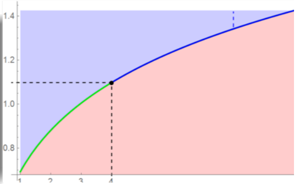
Q. 3 Show that $\mathrm{gap}_{\mathrm{SW}}^{-1} = O(1)$ and that Swendsen–Wang dynamics on an $n \times n$ torus has $t_{\mathrm{mix}} = O(\log n)$ $\forall \beta > \beta_c$ $\forall q \geq 2$.

Phase coexistence at criticality

Dynamics on an $n \times n$ torus at criticality for $q > 4$

Prediction: ([Li, Sokal '91],...)

Potts Glauber and Swendsen–Wang on the torus each have $t_{\text{mix}} \asymp \exp(c_q\, n)$ at β_c if the phase-transition is discontinuous.



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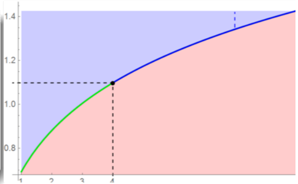
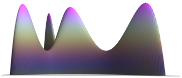


Intuition: Swendsen–Wang easily switches *between ordered phases*, yet the *order/disorder* transition is a bottleneck, as on the complete graph for $\beta \in (\beta_s, \beta_S)$.



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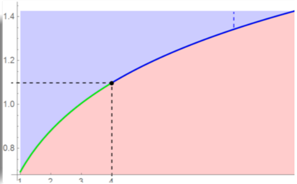

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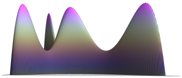


Rigorous bounds: [Borgs, Chayes, Frieze, Kim, Tetali, Vigoda, Vu '99], followed by [Borgs, Chayes, Tetali '12], showed this for q large enough:

Theorem

If q is sufficiently large, then both Potts Glauber and Swendsen–Wang on an $n \times n$ torus have $t_{\mathrm{mix}} \geq \exp(cn)$.

Slow mixing in coexistence regime on $(\mathbb{Z}/n\mathbb{Z})^2$

Building on the work of [Duminil-Copin, Sidoravicius, Tassion '15]:

Theorem (Gheissari, L. '18)

For any $q > 1$, if $\exists$ multiple infinite-volume FK measures then the Swendsen–Wang dynamics on an $n \times n$ torus has $t_{\mathrm{mix}} \geq \exp(c_q n)$.

In particular, via [Duminil-Copin,Gagnebin,Harel,Manolescu,Tassion]:

Corollary

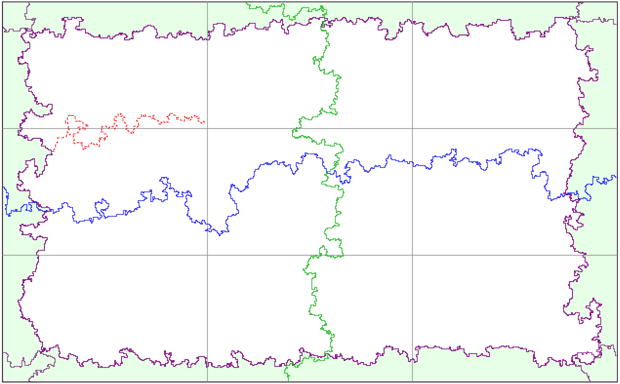
For any $q > 4$, Potts Glauber, FK Glauber and Swendsen–Wang on the $n \times n$ torus at $\beta = \beta_c$ all have $t_{\mathrm{mix}} \geq \exp(c_q\, n)$.

Proof of lower bound: an exponential bottleneck

Define the bottleneck set $S := \bigcap_{i=1}^{3} S_v^i \cap S_h^i$ where

$$S_v^i := \left\{ \omega : \exists x \text{ s.t. } (x,0) \longleftrightarrow (x,n) \text{ in } [\tfrac{(i-1)n}{3}, \tfrac{in}{3}] \times [0,n] \right\} ,$$

$$S_h^i := \left\{ \omega : \exists y \text{ s.t. } (0,y) \longleftrightarrow (n,y) \text{ in } [0,n] \times [\tfrac{(i-1)n}{3}, \tfrac{in}{3}] \right\} .$$



The torus vs. the grid (periodic vs. free b.c.)

Recall: for $q = 2$ and $\beta > \beta_c$:

- Glauber dynamics for the Ising model both on an $n \times n$ **grid** (**free** b.c.) and on an $n \times n$ **torus** has $t_{\mathrm{mix}} \geq \exp(cn)$.

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- In contrast, on an $n \times n$ grid with **plus boundary conditions** it has $t_{\mathrm{mix}} \leq n^{O(\log n)}$ [Martinelli '94], [Martinelli, Toninelli '10], [L., Martinelli, Sly, Toninelli '13].

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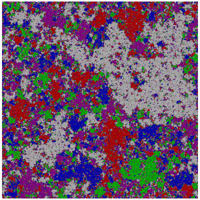
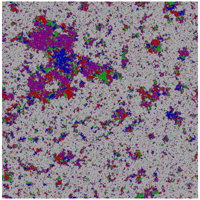
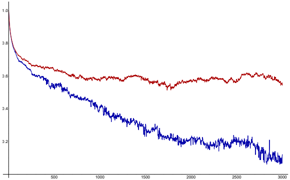
Recall: for $q = 2$ and $\beta > \beta_c$:

- Glauber dynamics for the Ising model both on an $n \times n$ **grid** (**free** b.c.) and on an $n \times n$ **torus** has $t_{\mathrm{mix}} \geq \exp(cn)$.
- In contrast, on an $n \times n$ grid with **plus boundary conditions** it has $t_{\mathrm{mix}} \leq n^{O(\log n)}$ [Martinelli '94], [Martinelli, Toninelli '10], [L., Martinelli, Sly, Toninelli '13].

When the phase transition for Potts is discontinuous, at $\beta = \beta_c$: the dynamics under free boundary conditions is fast:







The torus vs. the grid (periodic vs. free b.c.)

Unlike the torus, where $t_{\mathrm{mix}} \geq \exp(cn)$, on the grid SW is fast:

Theorem (Gheissari, L. '18)

*For large q, Swendsen–Wang on an $n \times n$ grid (**free** b.c.) at β_c has $t_{\mathrm{mix}} \leq \exp(n^{o(1)})$. The same holds for **red** b.c.*

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- To improve this to $\exp(n^{o(1)})$, one employs the framework of [Martinelli, Toninelli '10], along with cluster expansion.

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Toprid mixing on the torus; sub-exponential mixing on the grid.

Classifying boundary conditions that interpolate between the two?

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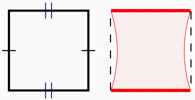
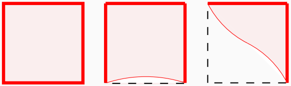
Classifying boundary conditions that interpolate between the two?

Theorem ([Gheissari, L. 18'+] (two of the classes, informally))

For large enough q, Swendsen–Wang satisfies:

1. *Mixed b.c. on 4 macroscopic intervals induce* $t_{\mathrm{mix}} \geq \exp(cn)$.
2. *Dobrushin b.c. with a macroscopic interval:* $t_{\mathrm{mix}} = e^{o(n)}$.

Boundary		Swendsen–Wang
Periodic/Mixed		$t_{\mathrm{mix}} \geq e^{cn}$
Dobrushin		$t_{\mathrm{mix}} \leq e^{n^{1/2+o(1)}}$

Questions on the discontinuous phase transition regime

Q. 4 Let $q > 4$. Is Swendsen–Wang (or FK Glauber) on the $n \times n$ grid (**free** b.c.) quasi-polynomial in n? polynomial in n?

known: $\exp(n^{o(1)})$ for $q \gg 1$

Q. 5 Let $q > 4$. Is Potts Glauber on the $n \times n$ grid (**free** b.c.) sub-exponential in n? quasi-polynomial in n? polynomial in n?

Q. 6 Let $q > 4$. Is Swendsen–Wang on the $n \times n$ grid with Dobrushin b.c. sub-exponential in n? quasi-poly(n)? poly(n)?

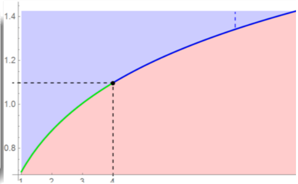
known: $\exp(n^{1/2+o(1)})$ for $q \gg 1$

Unique phase at criticality

Dynamics on an $n \times n$ torus at criticality for $1 < q < 4$

Prediction:

Potts Glauber and Swendsen–Wang on the torus each have $t_{\mathrm{mix}} \asymp n^z$ for a lattice-independent $z = z(q)$.



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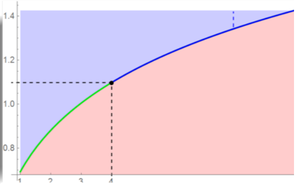

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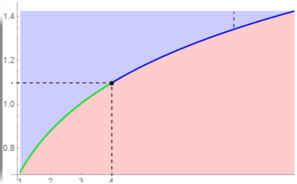

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Rigorous bounds:

Theorem (L., Sly '12)

Continuous-time Glauber dynamics for the Ising model ($q = 2$) on an $n \times n$ grid with arbitrary b.c. satisfies $n^{7/4} \lesssim t_{\mathrm{mix}} \lesssim n^c$.

Bound $t_{\mathrm{mix}} \lesssim n^c$ extends to Swendsen–Wang via [Ullrich '13,'14].

Mixing of Critical 2D Potts Models

Theorem (Gheissari, L. '18)

Cont.-time Potts Glauber dynamics at $\beta_c(q)$ on an $n \times n$ torus has

1. *at $q = 3$:* $\Omega(n) \le t_{\mathrm{mix}} \le n^{O(1)}$;
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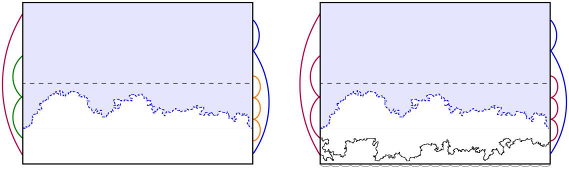
- The case $q = 4$ is subtle: crossing probabilities are believed to no longer be bounded away from 0 and 1 uniformly in the b.c.

FK Glauber for noninteger q

Obstacle in FK Glauber: macroscopic disjoint boundary bridges prevent coupling of configurations sampled under two different b.c.



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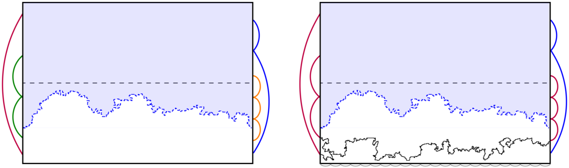
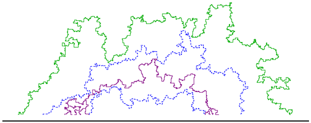


Theorem (Gheissari, L.)

For every $1 < q < 4$, the FK Glauber dynamics at $\beta = \beta_c(q)$ on an $n \times n$ torus satisfies $t_{\mathrm{mix}} \leq n^{c \log n}$.

One of the key ideas: establish the exponential tail beyond some $c \log n$ for # of disjoint bridges over a given point.



Questions on the continuous phase transition regime

Q. 7 Let $q = 4$. Establish that Potts Glauber on an $n \times n$ **torus** (or a grid with free b.c.) satisfies $t_{\mathrm{mix}} \leq n^c$.

known: $n^{O(\log n)}$

Q. 8 Let $q = 2.5$. Establish that FK Glauber on an $n \times n$ **torus** (or a grid with free b.c.) satisfies $t_{\mathrm{mix}} \leq n^c$.

known: $n^{O(\log n)}$

Q. 9 Is $q \mapsto \mathrm{gap}_{\mathrm{P}}$ decreasing in $q \in (1, 4)$? Similarly for $\mathrm{gap}_{\mathrm{SW}}$?

Thank you!