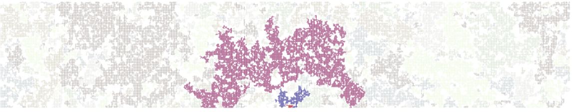
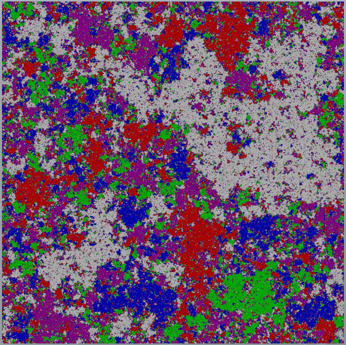


Dynamics for the critical 2D Potts/FK model: many questions and a few answers

Eyal Lubetzky

October 2018

Courant Institute, New York University



Outline

The models: static and dynamical

Dynamical phase transitions on $\mathbb{Z}^2$

Phase coexistence at criticality

Unique phase at criticality

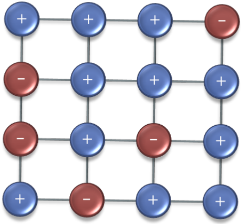
The models: static and dynamical

The (static) 2D Ising model

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{I}} = \{-1, 1\}^{V(G)}$$

(each *site* receives a plus/minus *spin*).



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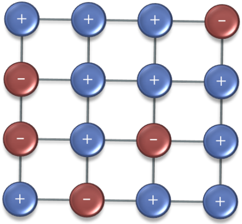


Definition (the Ising model on G) [Lenz 1920]

Probability distribution μ_{I} on Ω_{I} given by the Gibbs measure:

$$\mu_{\mathrm{I}}(\sigma) = \frac{1}{Z_{\mathrm{I}}} \exp\Big(\beta \sum_{x \sim y} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}\Big)$$

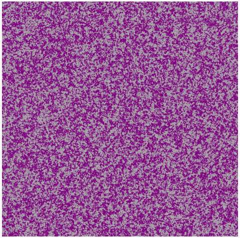
($\beta \geq 0$ is the inverse-temperature; Z_{I} is the partition function)

The (static) 2D Ising model: phase transition

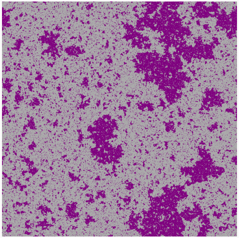
- Underlying graph: G = finite 2D grid.
- Set of possible configuration: $\Omega_{\mathrm{I}} = \{-1, 1\}^{V(G)}$
- Probability of a configuration: $\mu_{\mathrm{I}}(\sigma) \propto \exp\left(\beta \sum_{x \sim y} \delta_{\sigma(x),\sigma(y)}\right)$

Local (nearest-neighbor) interactions can have macroscopic effects:

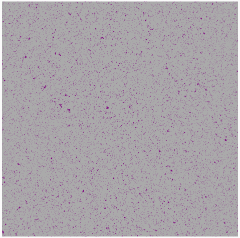
Ising model on a 1000×1000 torus



$\beta = 0.75$



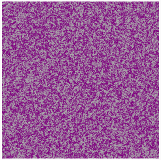
$\beta = 0.88$



$\beta = 1$

The (static) 2D Ising model: phase transition (ctd.)

Ising model on a $2D$ torus



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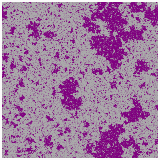

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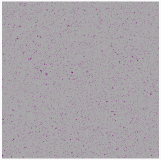
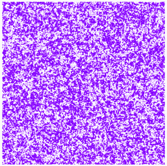
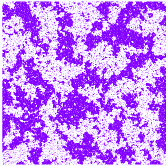


$\beta = 1$

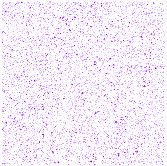
Noisy majority model on a $2D$ torus: Ising universality class?



$p = 0.25$



$p = 0.15$



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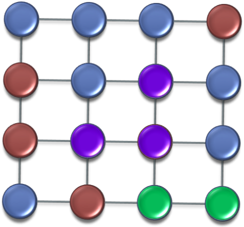
The (static) 2D Potts model

Generalizes the Ising model from 2-state spins to q-state spins:

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{P}} = \{1, \ldots, q\}^{V(G)}$$

(each *site* receives a *color*).



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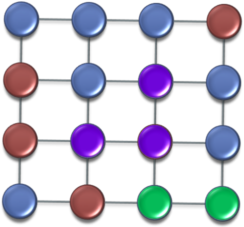

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($\beta \geq 0$ is the inverse-temperature; Z_{P} is the partition function)

Glauber dynamics for the Potts model

Recall: $$\mu_{\mathrm{P}}(\sigma) = \frac{1}{Z_{\mathrm{P}}} \exp\left(\beta \sum_{x \sim y} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}\right)$$

A family of MCMC samplers for spin systems due to Roy Glauber:

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RJ Glauber – Journal of Mathematical Physics, 1963   Cited by 3607

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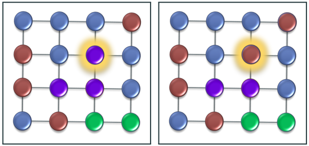

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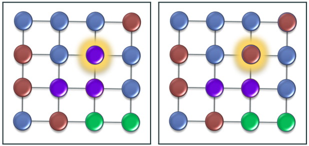

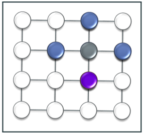

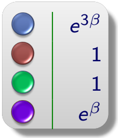

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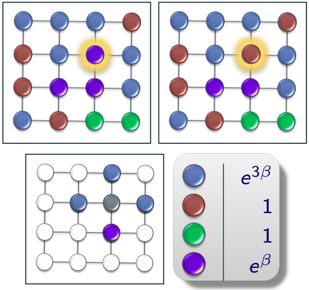
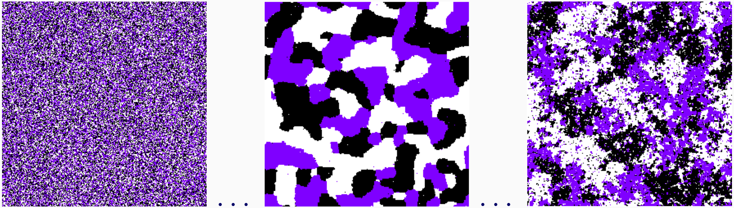


Meta question: *How long does it take to converge to μ?*

Glauber dynamics for the 2D Potts model

Glauber dynamics, 3-color Potts model on a 250×250 torus for $\beta = 0.5 \rightsquigarrow \beta = 2.01 \rightsquigarrow \beta = 1.01$.



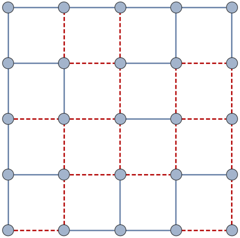
Q. 1 Fix $\beta > 0$ and $T > 0$. Does continuous-time Glauber dynamics $(\sigma_t)_{t \geq 0}$ for the 3-color Potts model on an $n \times n$ torus attain $\max_{\sigma_0} \mathbb{P}_{\sigma_0}(\sigma_T(x) = \textsc{blue})$ at σ_0 which is $\textsc{all-blue}$?

The (static) 2D Fortuin–Kasteleyn model

- Underlying geometry: G = finite 2D grid.
- Set of possible configuration:

$$\Omega_{\mathrm{FK}} = \{\omega : \omega \subseteq E(G)\}$$

(equiv., each edge is *open*/*closed*).



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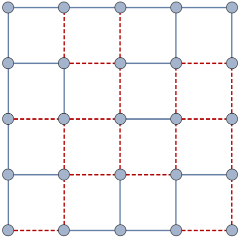

Definition (the (p,q)-FK model on G) [Fortuin, Kasteleyn '69]

Probability distribution μ_{P} on Ω_{FK} given by the Gibbs measure:

$$\mu_{\mathrm{FK}}(\omega) = \frac{1}{Z_{\mathrm{FK}}}\left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)}$$

(Z_{FK} is the partition function; $\kappa(\omega) = \#$ connected components in ω)

Well-defined for any real (not necessarily integer) $q \geq 1$.

Glauber dynamics for the FK model

Recall: $\mu_{\mathrm{FK}}(\omega) = \frac{1}{Z_{\mathrm{FK}}} \left(\frac{p}{1-p} \right)^{|\omega|} q^{\kappa(\omega)}$ where $\kappa(\omega)$ is the # conn. comp. in ω.

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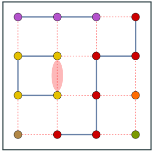

edge prob

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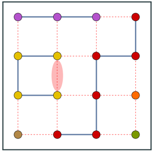
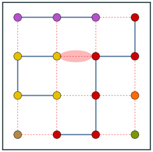


edge prob p



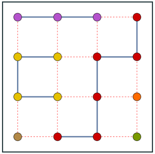
edge prob $\frac{p}{p+(1-p)q}$

Coupling of the Potts and FK models

[Edwards–Sokal '88]: coupling of $(\mu_{\mathrm{P}}, \mu_{\mathrm{FK}})$ for $p = 1 - e^{-\beta}$:

$$\Psi_{G,p,q}(\sigma,\omega) = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{|\omega|} \prod_{e=xy\in E} \mathbb{1}_{\{\sigma(x)=\sigma(y)\}}$$

$$\left(\mu_{\mathrm{P}}(\sigma) \propto \left(\tfrac{1}{1-p}\right)^{\#\{x\sim y:\sigma(x)=\sigma(y)\}} , \mu_{\mathrm{FK}}(\omega) \propto \left(\tfrac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)} \right)$$



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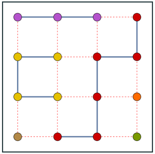

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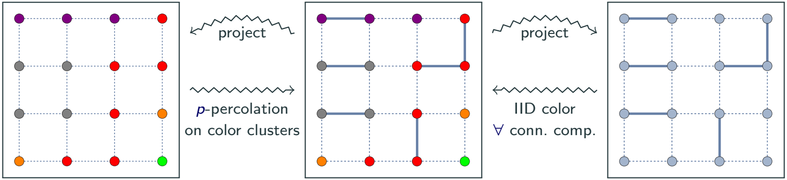
Simple method to move between Potts & FK: [Swendsen–Wang '87]



$\sigma \sim \mu_{\mathrm{P}}$ $(\sigma,\omega) \sim \Psi$ $\omega \sim \mu_{\mathrm{FK}}$

Coupling of the Potts and FK models

[Edwards–Sokal '88]: coupling of $(\mu_{\mathrm{P}}, \mu_{\mathrm{FK}})$ for $p = 1 - e^{-\beta}$:

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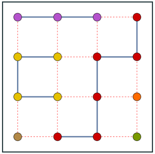

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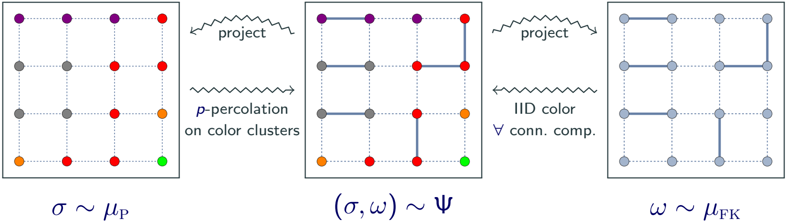


Continuum analog [Miller, Sheffield, Werner '17]: CLE percolations.

Measuring convergence to equilibrium in Potts/FK

Measuring convergence to the stationary distribution π of a discrete-time reversible Markov chain with transition kernel P:

- Spectral gap / relaxation time:

$$\text{gap} = 1 - \lambda_2 \quad \text{and} \quad t_{\text{rel}} = \text{gap}^{-1}$$

  where the spectrum of P is $1 = \lambda_1 > \lambda_2 > \ldots$.

- Mixing time (in total variation):

$$t_{\text{mix}} = \inf\left\{ t : \max_{\sigma_0 \in \Omega} \|P^t(\sigma_0, \cdot) - \pi\|_{\text{TV}} < 1/(2e) \right\}$$

(Continuous time (heat kernel $H_t = e^{t\mathcal{L}}$): gap in $\text{spec}(\mathcal{L})$, and replace P^t by H_t.)

For most of the next questions, these will be equivalent.

Measuring convergence to equilibrium in Potts/FK (ctd.)

[Ullrich '13, '14]: related gap of discrete-time Glauber dynamics for Potts and FK on any graph $G = (V, E)$ with maximal degree Δ:

$$\mathrm{gap}_{\mathrm{FK}}^{-1} \leq C_{\beta,\Delta,q}\, \mathrm{gap}_{\mathrm{P}}^{-1}\, |E| \log |E| \,.$$

- Glauber for FK is as fast as for Potts up to polynomial factors.
- Glauber for FK can be exponentially faster (in $|V|$) than Potts.

When are the Glauber dynamics for Potts and FK both fast on $\mathbb{Z}^2$? both slow? FK fast and Potts slow?

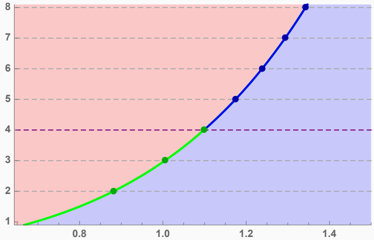
Q. 2 Is $\mathrm{gap}_{\mathrm{FK}}^{-1} \leq C_{\beta,\Delta,q}\, \mathrm{gap}_{\mathrm{P}}^{-1}$ on $\forall\ G$ with max degree Δ?

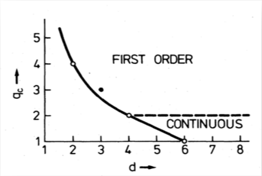
(NB: gap of Swendsen–Wang is comparable up to poly factors to $\mathrm{gap}_{\mathrm{FK}}$.)

Dynamical phase transitions on $\mathbb{Z}^2$

Dynamical phase transition

Prediction for Potts Glauber dynamics on the torus





Fast at high temperatures, exponentially slow at low temperatures.

Critical slowdown: as a power law or exponentially slow?

Dynamical phase transition

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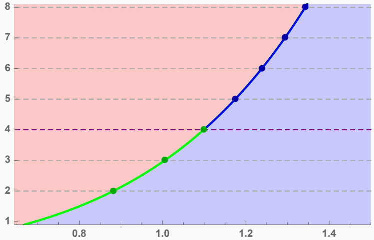
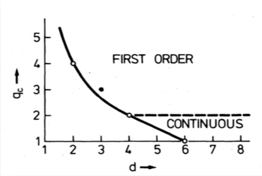




Fast at high temperatures, exponentially slow at low temperatures.

Critical slowdown: as a power law or exponentially slow?

FK Glauber: expected to be fast also when $\beta > \beta_c$.

- [Guo, Jerrum '17]: for $q = 2$: fast on any graph G at any β.

Results on $\mathbb{Z}^2$ off criticality

- **High temperature**:
  - [Martinelli,Olivieri '94a,'94b],[Martinelli,Olivieri,Schonmann '94c]: $\text{gap}_{\text{P}}^{-1} = O(1)$ $\forall \beta < \beta_c$ at $q = 2$; extends to $q \geq 3$ via [Alexander '98], [Beffara,Duminil-Copin '12].
  - [Blanca, Sinclair '15]: rapid mixing for FK Glauber $\forall \beta < \beta_c, q > 1$ ($t_{\text{mix}} = O(\log n)$ and $\text{gap}_{\text{FK}}^{-1} = O(1)$).

Results on $\mathbb{Z}^2$ off criticality

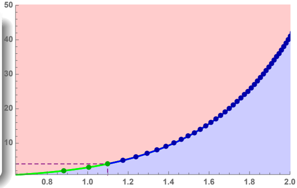
- **Low temperature**:
  - [Chayes, Chayes, Schonmann '87], [Thomas '89], [Cesi, Guadagni, Martinelli, Schonmann '96]: $\text{gap}_{\text{P}}^{-1} = e^{(c_\beta + o(1))n}$ $\forall \beta > \beta_c$, $q = 2$.
  - [Blanca, Sinclair '15]: result that $\text{gap}_{\text{FK}}^{-1} = O(1)$ $\forall \beta < \beta_c, q > 1$ transfers to $\forall \beta > \beta_c$ by duality.
  - [Borgs, Chayes, Frieze, Kim, Tetali, Vigoda, Vu '99] and [Borgs, Chayes, Tetali '12]: $\text{gap}_{\text{P}}^{-1} \gtrsim e^{cn}$ at $\beta > \beta_c$ and large q.
    (Result applies to the d-dimensional torus for any $d \geq 2$ provided $q > Q_0(d)$.)

Phase coexistence at criticality

Dynamics on an $n \times n$ torus at criticality for $q > 4$

Prediction: ([Li, Sokal '91],...)

Potts Glauber and FK Glauber on the torus each have $t_{\mathrm{mix}} \asymp \exp(c_q\, n)$ at β_c if the phase-transition is discontinuous.



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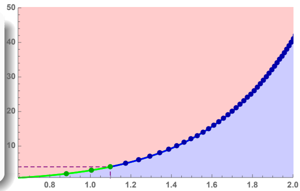
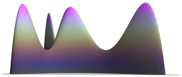


Intuition: FK does not suffer from the "predominantly one color" bottleneck (has only one *ordered phase*), yet it does have an *order/disorder* bottleneck.



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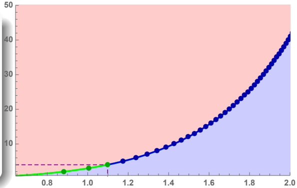

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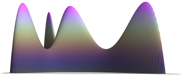


Rigorous bounds: [Borgs, Chayes, Frieze, Kim, Tetali, Vigoda, Vu '99], followed by [Borgs, Chayes, Tetali '12], showed this for q large enough:

Theorem

If q is sufficiently large, then Glauber dynamics for both the Potts and FK models on an $n \times n$ torus have $\mathrm{gap}^{-1} \geq \exp(cn)$ at $\beta = \beta_c$.

Slow mixing in coexistence regime on $(\mathbb{Z}/n\mathbb{Z})^2$

Building on the work of [Duminil-Copin, Sidoravicius, Tassion '15]:

Theorem (Gheissari, L. '18)

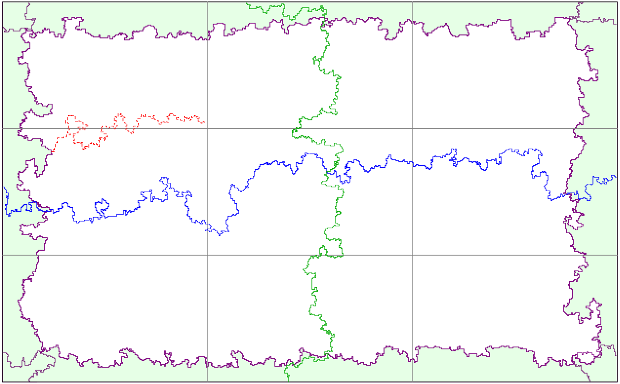
For any $q > 1$, if $\exists$ multiple infinite-volume FK measures at $\beta = \beta_c$ on an $n \times n$ torus then $\mathtt{gap}_{\mathrm{FK}}^{-1} \geq \exp(c_q n)$.

In particular, via [Duminil-Copin,Gagnebin,Harel,Manolescu,Tassion]:

Corollary

For any $q > 4$, both Potts and FK on the $n \times n$ torus at $\beta = \beta_c$ have $\mathtt{gap}^{-1} \geq \exp(c_q\, n)$.

Proof sketch: an exponential bottleneck



The torus vs. the grid (periodic vs. free b.c.)

Recall: for $q = 2$ and $\beta > \beta_c$:

- Glauber dynamics for the Ising model both on an $n \times n$ **grid** (**free** b.c.) and on an $n \times n$ **torus** has $\mathrm{gap}^{-1} \geq \exp(cn)$.

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- In contrast, on an $n \times n$ grid with **plus boundary conditions** it has $\mathtt{gap}^{-1} \leq n^{O(\log n)}$ [Martinelli '94], [Martinelli, Toninelli '10], [L., Martinelli, Sly, Toninelli '13].

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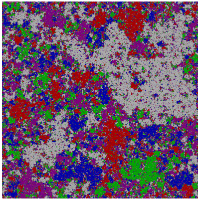
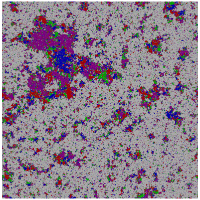
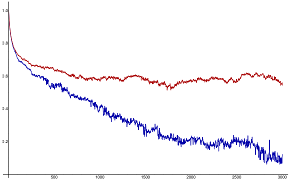
When the phase transition for Potts is discontinuous, at $\beta = \beta_c$: the dynamics under free boundary conditions is fast:







The torus vs. the grid (periodic vs. free b.c.)

On the grid, unlike the torus (where $\mathrm{gap}_{\mathrm{FK}}^{-1} \geq \exp(cn)$ at $\beta = \beta_c$):

Theorem (Gheissari, L. '18)

*For large q, FK Glauber on an $n \times n$ grid (**free** b.c.) at β_c has*

$$\mathrm{gap}_{\mathrm{FK}}^{-1} \leq \exp(n^{o(1)}) .$$

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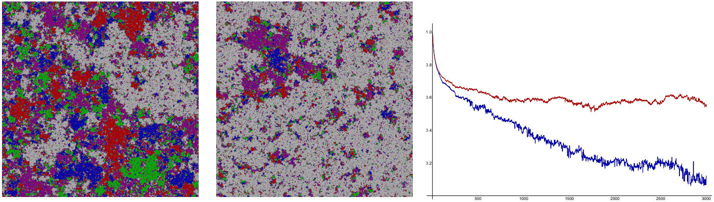
- Intuition: **free** b.c. destabilizes the wired phase (and bottleneck).
- Proof employs the framework of [Martinelli, Toninelli '10], along with cluster expansion.



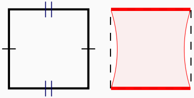
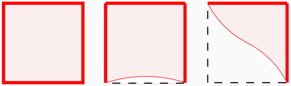
Sensitivity to boundary conditions

Toprid mixing on the torus; sub-exponential mixing on the grid.
Classifying boundary conditions that interpolate between the two?

Theorem ([Gheissari, L. 18’] (two of the classes, informally))

For large enough q, Swendsen–Wang satisfies:

1. *Mixed b.c. on 4 macroscopic intervals:* $\text{gap}^{-1} \geq \exp(cn)$.
2. *Dobrushin b.c. with a macroscopic interval:* $\text{gap}^{-1} = e^{o(n)}$.

Boundary		Swendsen–Wang
Periodic/Mixed		$\text{gap}^{-1} \geq e^{cn}$
Dobrushin		$\text{gap}^{-1} \leq e^{n^{1/2+o(1)}}$

Questions on the discontinuous phase transition regime

Q. 3 Let $q > 4$. Is FK Glauber on the $n \times n$ grid (**free** b.c.) quasi-polynomial in n? polynomial in n?

known: $\exp(n^{o(1)})$ for $q \gg 1$

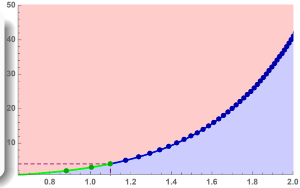
Q. 4 Let $q > 4$. Is Potts Glauber on the $n \times n$ grid (**free** b.c.) sub-exponential in n? quasi-polynomial in n? polynomial in n?

Unique phase at criticality

Dynamics on an $n \times n$ torus at criticality for $1 < q < 4$

Prediction:

Potts Glauber and FK Glauber on the torus each have $\mathtt{gap}^{-1} \asymp n^z$ for a lattice-independent $z = z(q)$.



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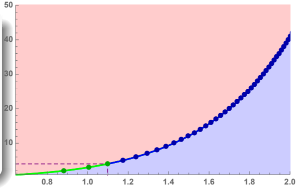

The exponent z is the *"dynamical critical exponent"*; various works in physics literature with numerical estimates, e.g., $z_{\mathrm{P}}(2) \approx 2.18$.

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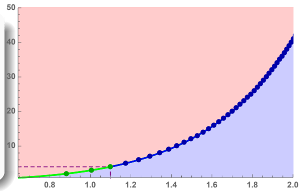

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Rigorous bounds:

Theorem (L., Sly '12)

Continuous-time Glauber dynamics for the Ising model ($q = 2$) on an $n \times n$ grid with arbitrary b.c. satisfies $n^{7/4} \lesssim \text{gap}^{-1} \lesssim n^c$.

Bound $\text{gap}^{-1} \lesssim n^c$ extends to FK Glauber via [Ullrich '13,'14].

Mixing of Critical 2D Potts Models

Theorem (Gheissari, L. '18)

Cont.-time Potts Glauber dynamics at $\beta_c(q)$ on an $n \times n$ torus has

1. *at $q = 3$:* $\Omega(n) \leq \mathrm{gap}^{-1} \leq n^{O(1)}$;
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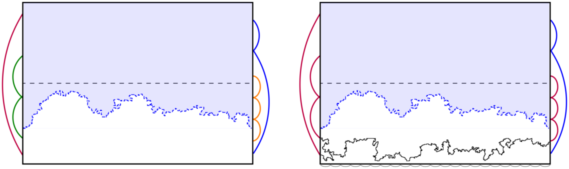
- The case $q = 4$ is subtle: crossing probabilities are believed to no longer be bounded away from 0 and 1 uniformly in the b.c.

FK Glauber for noninteger q

Obstacle in FK Glauber: macroscopic disjoint boundary bridges prevent coupling of configurations sampled under two different b.c.



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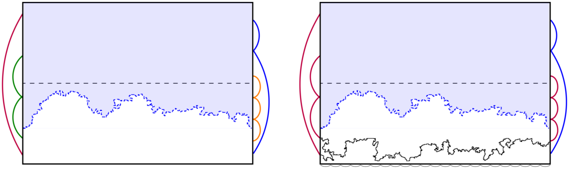
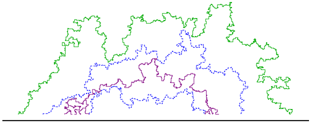


Theorem (Gheissari, L.)

For every $1 < q < 4$, *the FK Glauber dynamics at* $\beta = \beta_c(q)$ *on an* $n \times n$ *torus satisfies* $\mathrm{gap}^{-1} \leq n^{c \log n}$.

One of the key ideas: establish the exponential tail beyond some $c \log n$ for # of disjoint bridges over a given point.



Questions on the continuous phase transition regime

Q. 5 Let $q = 4$. Establish that Potts Glauber on an $n \times n$ **torus** (or a grid with free b.c.) satisfies $\mathrm{gap}^{-1} \leq n^c$.

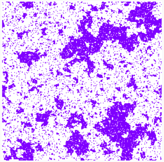
known: $n^{O(\log n)}$

Q. 6 Let $q = \pi$. Establish that FK Glauber on an $n \times n$ **torus** (or a grid with free b.c.) satisfies $\mathrm{gap}^{-1} \leq n^c$.

known: $n^{O(\log n)}$

Q. 7 Is $q \mapsto \mathrm{gap}_{\mathrm{P}}$ decreasing in $q \in (1, 4)$? Similarly for $\mathrm{gap}_{\mathrm{FK}}$?

And lastly: prove *something* at criticality or low temperature for the noisy majority model...



Thank you!