Uniformly $\mathbb{X}$ Intersecting Families

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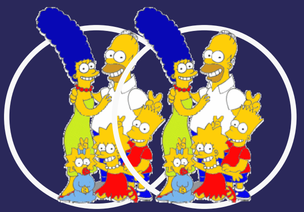


$\mathbb{X}$ Intersecting Families

- Let $\mathcal{A}$, $\mathcal{B}$ denote two families of subsets of $[n]$.
- The pair $(\mathcal{A},\mathcal{B})$ is called

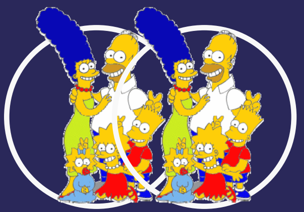
  "ℓ-cross-intersecting"

  iff

  $$|A \cap B| = \ell \text{ for all } A \in \mathcal{A}\ ,\ B \in \mathcal{B}.$$

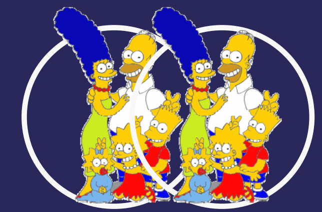
- Q: What is $P_\ell(n)$, the maximum value of

  $|\mathcal{A}||\mathcal{B}|$ over all ℓ-$\mathbb{X}$-intersecting pairs $(\mathcal{A},\mathcal{B})$?



Previous work: single family

- What is the maximal size of $\mathcal{F} \subset 2^{[n]}$ with given pair-wise intersections?
- Erdős-Ko-Rado '61: $|F \cap F'| \geq t$ and $|F| = k$ for all $F, F' \in \mathcal{F}$, then: $|\mathcal{F}| \leq \binom{n-t}{k-t}$
- Katona's Thm '64: no restriction on $|F|$.
- Additional examples:
  Ray-Chaudhuri-Wilson '75, Frankl-Wilson '81, Frankl-Füredi '85.

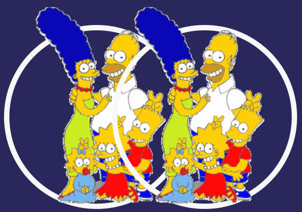


Previous work: two families

- <u>Conj</u> (Erdős '75): if $\mathcal{F} \subset 2^{[n]}$ has no pair-wise intersection of $\lfloor \frac{n}{4} \rfloor$ then $|\mathcal{F}| \leq (2-\varepsilon)^n$
- Settled by studying pairs of families:

  Frankl-Rödl '87: if $\mathcal{A},\mathcal{B} \subset 2^{[n]}$ have a forbidden $\mathbb{X}$-intersection $\eta n \leq l \leq (\frac{1}{2}-\eta)n$ then $|\mathcal{A}||\mathcal{B}| \leq (4-\varepsilon(\eta))^n$

  $\eta < 1/4$
- Frankl-Rödl '87 studied several notions of $\mathbb{X}$-intersecting pairs, including $P_\ell(n)$.



Previous work: $P_\ell(n)$

- Frankl-Rödl '87: $P_0(n) \leq 2^n$,
  and for all $\ell \geq 1$, $P_\ell(n) \leq 2^{n-1}$.
- Ahlswede-Cai-Zhang '89:
  lower bound: take $n \geq 2\ell$, and:

Linear algebra over $(\mathbb{Z}_p)^n$

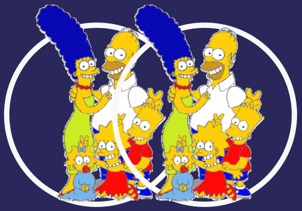
One set A_1 of 2ℓ elements

$\mathcal{A}$
$A_1=[2\ell]$

$\mathcal{B}$
$|A_1 \cap B| = \ell$

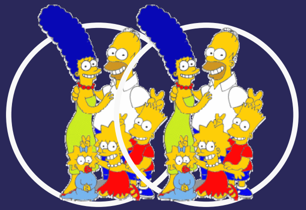
All sets containing ℓ elements of A_1

$$|\mathcal{A}||\mathcal{B}| = \binom{2\ell}{\ell} 2^{n-2\ell} = (1+o(1))\frac{2^n}{\sqrt{\pi \ell}}$$



Previous work: $P_\ell(n)$

- Conj (Ahlswede-Cai-Zhang '89): the above construction maximizes $P_\ell(n)$.
  - True for $\ell{=}0$ and $\ell{=}1$.
- For general ℓ: Gap $= [\ \Theta(\frac{2^n}{\sqrt{\ell}})\ ,\ \Theta(2^n)\ ]$
- Q (Sgall '99): does $P_\ell(n)$ decrease with ℓ?
- Keevash-Sudakov '06: the above conj is true for $\ell{=}2$ as well.



Our results

- Confirmed the conj of Ahlswede-Cai-Zhang '89 for any sufficiently large ℓ.
- Characterized all the extremal pairs $\mathcal{A}$,$\mathcal{B}$ which attain the maximum of $|\mathcal{A}||\mathcal{B}| = \binom{2\ell}{\ell}2^{n-2\ell}$
- This also provides a positive answer to the Q of Sgall '99.

Main Theorem

- There exists some ℓ_0, such that, for all $\ell \geq \ell_0$, every ℓ-$\mathbb{X}$-intersecting pair $\mathcal{A},\mathcal{B} \subset 2^{[n]}$ satisfies:

$$|\mathcal{A}||\mathcal{B}| \leq \binom{2\ell}{\ell} 2^{n-2\ell}$$

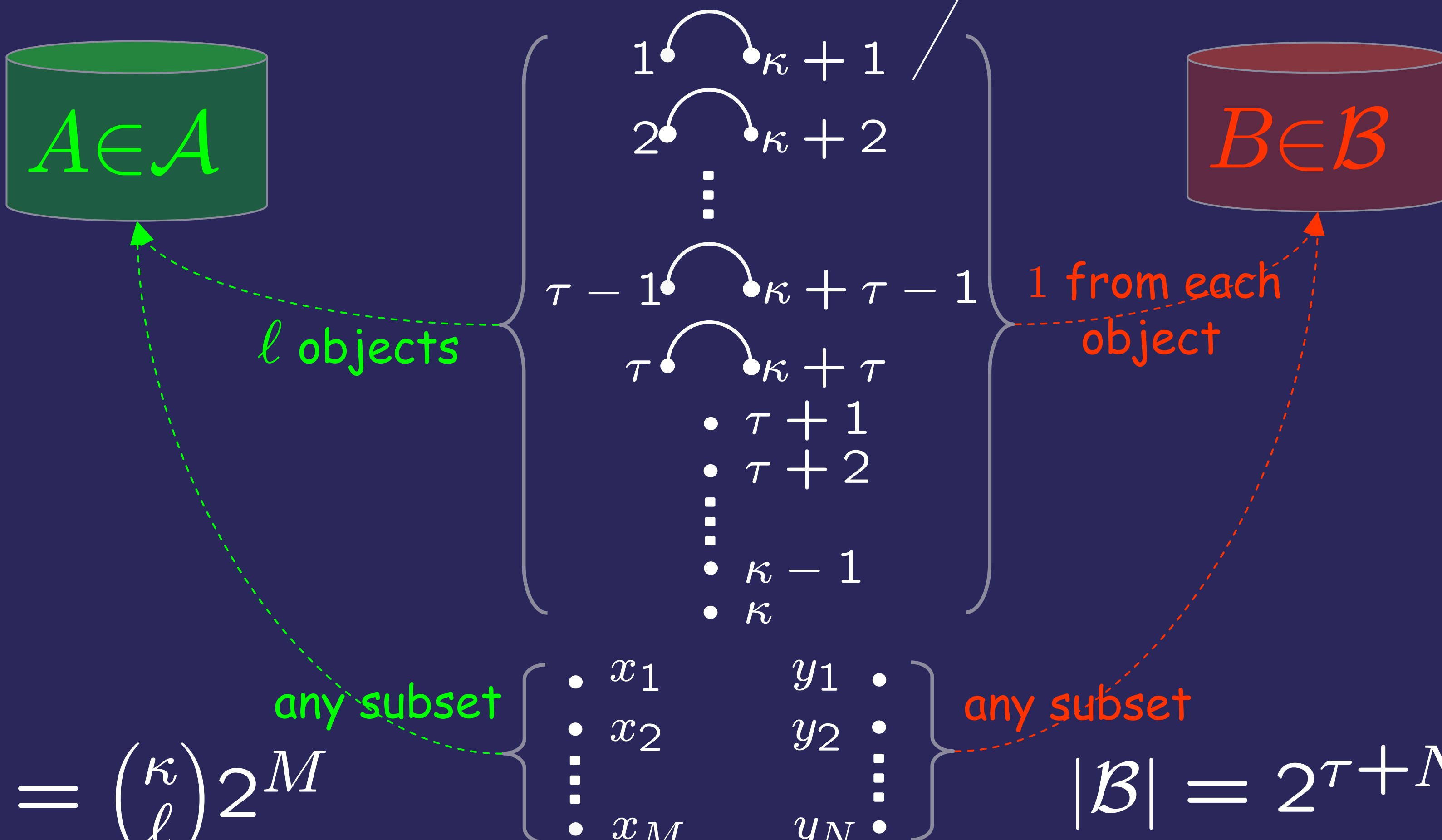
- Furthermore, equality holds iff the pair $\mathcal{A},\mathcal{B}$ is w.l.o.g. as follows:

The construction of Ahlswede et al. fits the special case $\tau = 0,\ \kappa = 2\ell$

$$|\mathcal{A}||\mathcal{B}| = \binom{\kappa}{\ell} 2^{n-\kappa} = \binom{2\ell}{\ell} 2^{n-2\ell}$$

Extremal pairs $\mathcal{A}$,$\mathcal{B}$

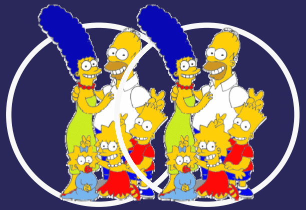
$\tau \leq \kappa$,
$\kappa \in \{2\ell - 1, 2\ell\}$



$$|\mathcal{A}| = \binom{\kappa}{\ell} 2^M$$

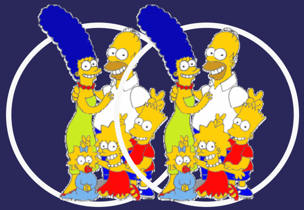
$$|\mathcal{B}| = 2^{\tau + N}$$

$$n = \tau + \kappa + M + N$$



Ideas used in the Proof

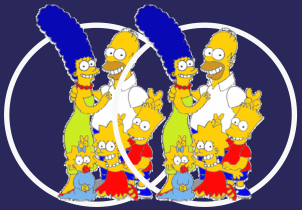
- Tools from Linear Algebra: study the vector spaces of the characteristic vectors of the sets in $\mathcal{A}$,$\mathcal{B}$ over $\mathbb{R}^n$.
- Techniques from Extremal Combinatorics, including:
  - The Littlewood-Offord Lemma.
  - Extensions of Sperner's Theorem
  - Large deviation estimates.
- Prove:



A weaker result

- Upper bound tight up to a constant:
- There exists some ℓ_0, such that, for all $\ell \geq \ell_0$, every ℓ-$\mathbb{X}$-intersecting pair $\mathcal{A},\mathcal{B} \subset 2^{[n]}$ satisfies:

$$|\mathcal{A}||\mathcal{B}| \leq \frac{2^{n+3}}{\sqrt{\ell}}$$



Vector spaces over $\mathbb{R}^n$

- Define:

  $\mathcal{F}_{\mathcal{A}}=\text{span}\ (\{\chi_A : A\in\mathcal{A}\})$ over $\mathbb{R}$,

  $\mathcal{F}_{\mathcal{B}}=\text{span}\ (\{\chi_B : B\in\mathcal{B}\})$ over $\mathbb{R}$.

- Set:

  $\mathcal{F}'_{\mathcal{B}}=\text{span}\ (\{\chi_B - \chi_{B_1} : B\in\mathcal{B}\})$ over $\mathbb{R}$,

  $k=\dim(\mathcal{F}_{\mathcal{A}})$, $h=\dim(\mathcal{F}'_{\mathcal{B}})$.

- $\mathcal{A}$,$\mathcal{B}$ are ℓ-$\mathbb{X}$-inter $\rightarrow \mathcal{F}_{\mathcal{A}} \perp \mathcal{F}'_{\mathcal{B}}$.
- $\Rightarrow k + h \leq n$.

Vector spaces over $\mathbb{R}^n$

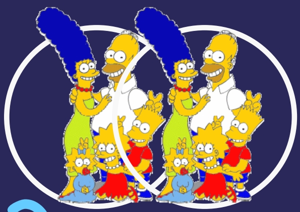
- Let $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ denote the matrices of bases for $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}'_{\mathcal{B}}$ after performing Gauss elimination:

$$M_{\mathcal{A}} = \left(\; I_k \; \middle| \; * \; \right) \; , \; M_{\mathcal{B}} = \left(\; I_h \; \middle| \; * \; \right)$$

- Since target vectors are in $\{0,1\}^n$:

$$|\mathcal{A}||\mathcal{B}| \leq 2^{k+h} \leq 2^n.$$

- If, say, $M_{\mathcal{A}}$ can produce at most $|\mathcal{A}| < \frac{8}{\sqrt{n}} \cdot 2^k$ sets, we are done.

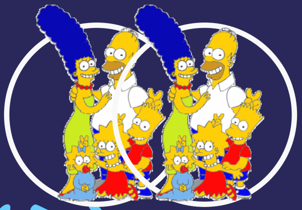


What do $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ look like?

- Can we indeed produce 2^d legal char. vectors from $M = \left(I_d \mid \overbrace{\;*\;}^{n-d} \right)$?
- We get constraints if there are:
  - Columns with many non-zero entry.
  - Rows not in $\{0, \pm 1\}^n \setminus \{0, 1\}^n$.

The Littlewood-Offord Lemma (1D)

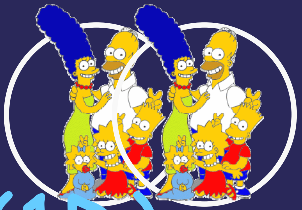
Families are antichains by induction



The Littlewood-Offord Lemma (1D)

- Q (Littlewood-Offord '43):
  Let $a_1, \ldots, a_n \in \mathbb{R}$ with $|a_i| > 1$ for all i.
  What is the max num of sub-sums $\sum_{i \in I} a_i$, $I \subset [n]$, which lie in a unit interval?
- Lemma (Erdős '45):
  Let $a_1, \ldots, a_n \in \mathbb{R} \setminus \{-\delta, \delta\}$ and let U denote an interval of length $\leq \delta$.
  Then the number of sub-sums $\sum_{i \in I} a_i$, $I \subset [n]$, which belong to U, is at most $\binom{n}{\lfloor n/2 \rfloor}$

Erdős's Pf of the L-O Lemma (1D)



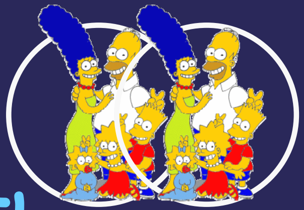
- Without loss of generality, all the a_i-s are positive (o/w, shift the interval U).
- A sub-sum which belongs to U is an antichain of $[n]$ and the result follows from Sperner's Thm.

What do $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ look like?

- Either $|\mathcal{A}||\mathcal{B}| \leq \frac{2^{n+3}}{\sqrt{\ell}}$, or w.l.o.g. :

$$M_{\mathcal{A}} = \left(\begin{array}{c|c|c} I_{k'} & -I_{k'} & 0 \\ \hline * & * & * \end{array}\right) \quad \} \; k' = \tfrac{2}{5}n - O(\log n)$$

$$M_{\mathcal{B}} = \left(\begin{array}{c|c|c} I_{h'} & -I_{h'} & 0 \\ \hline * & * & * \end{array}\right) \quad \} \; h' = \tfrac{13}{40}n - O(\log n)$$



Completing the proof of the Thm

- Recall: each row of $M_{\mathcal{A}}$ is orthogonal to each row of $M_{\mathcal{B}}$.
- Two (1,-1,0,…,0) rows are orthogonal only if the (1,-1) indices are disjoint.

- $M_{\mathcal{A}}$ gives $\frac{2}{5}n - O(\log n)$ pairs of indices.
- $M_{\mathcal{B}}$ gives $\frac{13}{40}n - O(\log n)$ pairs of indices.

- $\frac{4}{5}n + \frac{13}{20}n > n \Rightarrow$ contradiction.

■ Qed

Proof of Main result – some ideas

- If $M_{\mathcal{A}}$ is "far" from a structure which produces 2^k sets for $\mathcal{A}$, $M_{\mathcal{B}}$ must be "close" to a structure producing 2^h sets for $\mathcal{B}$.
- "Clean" the matrices gradually, using orthogonality to switch back and forth between $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$.
- An easy scenario to illustrate this:

Scenario: $\Omega(n)$ rows of $M_{\mathcal{A}}$ in $\{0,1\}^n$

$k \in \{2\ell - 1, 2\ell\}$

$$M_{\mathcal{A}} = \left(\begin{array}{c|c|c} I_h & 0 & I_h \\ \hline 0 & I_{k-h} & 0 \end{array}\right)$$

$A \in \mathcal{A}$: ℓ-subset of pairs and singles. $|\mathcal{A}| = \binom{k}{\ell}$

$$M_{\mathcal{B}} = \left(\begin{array}{c|c|c} -I_h & 0 & I_h \end{array}\right)$$

$B \in \mathcal{B}$: One of each pair and single. $|\mathcal{B}| = 2^h$

$$\chi_{B_1} = \left(1, \ldots, 1 \,|\, 1, \ldots, 1 \,|\, 0, \ldots, 0\right)$$

Optimal family with $\kappa = k$, $\tau = h$, $n = \kappa + \tau$.

Thank you!