Extrema of Potts interfaces

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* The \( q \)-state \textit{Potts model} \( (q \geq 2, \text{integer}) \)

- Underlying geometry:
  \( \Lambda_n \subset \mathbb{Z}^3 \).

Main focus:
\( \Lambda_n = [-n, n]^3 \times (\mathbb{Z} + \frac{1}{2}) \)

- Set of configurations:
  \( \{1, 2, \ldots, q\}^{\Lambda_n} \)

Probability of a configuration \( \Sigma \):
\[
\pi_n(\Sigma) = \frac{1}{Z_p(n, \beta)} \exp \left[ -\beta \sum_{x \sim y} 1{\{\Sigma(x) \neq \Sigma(y)\}} \right]
\]
for \( \beta > 0 \), the inverse temperature.

- Boundary conditions (b.c.): fixed coloring of \( \Lambda_n^c \)

- Main focus:
  - b.c. \( \mathbb{B} \) in lower half space \( \mathbb{H}_- \)
  - b.c. \( \mathbb{R} \) in upper half space \( \mathbb{H}_+ \)
  - Low temperature: \( \beta > \beta_0 \) fixed, large

Q: What can we say about the random interfaces between the \( \mathbb{B} \) and \( \mathbb{R} \) of the boundary?
Background: Ising model (the case $q=2$):

- Spins are either $B$ on $R$.
- Dobrushin, in the early 1970s, studied the Interface:
  - Take $F_\sigma = \{ f = (x,y)^* \text{ (dual) for } x \sim y \text{ s.t. } \sigma(x) \neq \sigma(y) \}$
  - $I$ is the 0-connected component of faces in $F_\sigma$ touching the boundary.

$[f,f']$ are 0-adjacent (or $x$-adjacent) if $f \cap f' = \emptyset$ (even via a corner)

* Theorem (Dobrushin, '72, '73) RIGIDITY OF THE INTERFACE

In 3D Ising on $\Lambda_n$ at $\beta > \beta_0$, for all $x = (x_1,x_2), h$

$$P((x_1,x_2,h) \in I) \leq e^{-\frac{1}{3} \beta h}.$$  

w.h.p.: $I$ flat at height 0 above $0.99 n^2$ faces $x \in [-n,n]^3$.

* Corollaries

1. There exist non-translation invariant $\mathbb{Z}^3$ Gibbs measures.
2. Max & Min height of $I$ are $\leq \frac{10}{\beta} \log n$ w.h.p.
Some of the follow-up works on Low TEMP ISING:

- [Beijeren '75]:
  alternative simpler argument for rigidity.

- [Bricmont, Lebowitz, Pfister, Olivieri '79a, '79b, '79c]
  extension of the rigidity argument to
  the Widom-Rowlinson model.

- [Gielis, Grimmet '02]:
  extension of rigidity argument to super-critical
  percolation/Random Cluster model conditioned to
  have interfaces.

- many other works on the Wulff shape,
  LD for magnetization, surface tension
  [Pisztora '96], [Bodineau '96], [Cerf, Pisztora '00]
  [Bodineau '05], [Cerf '06], ...
Recent progress: Ising model (the case $q=2$):

* Gheissari and L. ('21, '22) identified the correct exponential rate:

$$P [(x_1, x_2, h) \in I] = e^{-(\alpha + o(1)) h} \quad \text{as} \quad h \to \infty.$$ 

for an explicit $\alpha \in (0, 1)$. 

* Led to the following result on Max/Min of $I$:

**Theorem:** (Gheissari, L., '21, '22):

$M_n$, the max height of $I$, satisfies

$$M_n - \mathbb{E}M_n = O_p(1)$$

and

$$\mathbb{E}M_n = \left(\frac{2}{\alpha} + o(1)\right) \log n.$$ 

the interface $I$ a.s. separates the two b.c. phases (finite bubbles aren't drawn)

Q: Analogue in Potts??
* Interfaces in the Potts model:

\[ \exists \text{ (a.s.) unique if RED component:} \quad V_{\text{RED}} \]

\[ \exists \text{ (a.s.) unique if BLUE component:} \quad V_{\text{BLUE}} \]

* Equiv.: say 

\[ V_{\text{RED}} = \{ u: \exists \text{ path of red vertices connecting } u \text{ to } \cup_{n}^{-} \} \]

\[ V_{\text{BLUE}} = \{ u: \exists \text{ path of blue vertices connecting } u \text{ to } \cup_{n}^{-} \} \]

* Augment the components:

\[ \hat{V}_{\text{RED}} = V_{\text{RED}} \cup \{ \text{finite components of } V_{\text{RED}}^{c} \} \]

\[ \hat{V}_{\text{BLUE}} = V_{\text{BLUE}} \cup \{ \text{finite components of } V_{\text{BLUE}}^{c} \} \]

* Interface:

\[ I_{\text{BLUE}} = \{ f = (u, v)^{+} \text{ for } u \in \hat{V}_{\text{BLUE}} \} \]

\[ \text{for } u \in \hat{V}_{\text{BLUE}} \]

\[ I_{\text{RED}} = \{ f = (u, v)^{+} \text{ for } u \in \hat{V}_{\text{RED}} \} \]

\[ \text{for } v \in \hat{V}_{\text{RED}} \]
\[ I_{\text{RED}} = \left\{ f = (u,v)^z \text{ for } u \in \hat{\mathcal{V}}_{\text{RED}} \right\} \]

\[ I_{\text{BLUE}} = \left\{ f = (u,v)^z \text{ for } u \in \hat{\mathcal{V}}_{\text{BLUE}} \right\} \]
Theorem 1: \([Chen, L.]\)

Consider \(q\)-state Potts on \(\Lambda_n = [-L, L]^2 \times (\mathbb{Z} + \frac{1}{2})\)
with Dobrushin b.c., \(q > 2\) and \(\beta > \beta_0\) (fixed).

Let
\[
M_n = \text{\underline{MIN}} \text{ height of } \mathsf{I}_{\text{Blue}}
\]
\[
M'_n = \text{\underline{MAX}} \text{ height of } \mathsf{I}_{\text{Blue}}
\]

Then:
\[
M_n - \mathbb{E}[M_n] = O_\mathcal{P}(1) \quad \text{(Tightness)}
\]
\[
M'_n - \mathbb{E}[M'_n] = O_\mathcal{P}(1)
\]

Moreover, \(\exists \ \bar{\sigma}, \bar{\sigma}' > 0\) s.t.
\[
\mathbb{E} M_n = \left( \frac{2}{\bar{\sigma}} + o(1) \right) \log L, \quad \mathbb{E} M'_n = \left( \frac{2}{\bar{\sigma}'} + o(1) \right) \log L
\]

and \(\bar{\sigma}' \geq \bar{\sigma}\) for \(q \neq 2\). \(\bar{\sigma} = \bar{\sigma}'\) for \(q = 2\).
The Random Cluster model: \((q \geq 1, 0 < p < 1)\)

- **Underlying geometry:**
  \[ \Lambda_n \subset \mathbb{Z}^3 \]
  
  **Main focus:**
  \[ \Lambda_n = [-n,n]^3 \times (\mathbb{Z} + \frac{1}{2}) \]

- **Set of configurations:**
  \[ \{ \omega : \omega \in E(\Lambda_n) \} \]

- **Probability of a configuration** \(\omega:\)
  \[
  \mu_n(\omega) = \frac{1}{Z_n(p,q)} \prod_{E(\omega)} (1-p)^q p^{1-E(\omega)} 
  \]

  \# open edges \ # closed edges \ in \(\omega\)

- **Boundary conditions:** a partition of the vertices of \(\partial \Lambda_n\) to conn. comp. (conn. "outside" of \(\Lambda_n\)).

- **Main focus:** all conn.
  - b.c. WIREd in lower half space \(H_-\)
  - b.c. WIREd in upper half space \(H_+\)
  - Low temperature: \(p > 1 - \epsilon_0\), fixed, large.
* [Edwards - Sokal ’88] coupling $\phi = (\pi, \mu):$

$$p = 1 - e^{-\beta}$$

$$\phi_{p, \mathbf{q}, \mathbf{t}}(\omega, \omega') = \frac{1}{Z_{ES}(n,p,q,t)} \prod \left( 1 - p \right)^{M(\omega)} \prod_{{(x,y)} \in \omega} 1_{\{\sigma(x) = \sigma(y)\}}$$

$$\pi(\omega) \propto (1 - p)^{M(\omega)}$$

$$\mu(\omega) \propto p^{1 - \omega}$$

- moving between

\[ \xrightarrow{\text{percolation on color clusters}} (\omega, \omega') \xrightarrow{\text{color & conn comp}} \text{IID } \text{Unif}(1, \ldots, q) \]

* Via this coupling: Potts

corresponds to RC wired $\forall \mathbf{t}$ wired $\forall \mathbf{q}$

conditioned on

$\forall \mathbf{q} \in \Lambda^+, \mathbf{t} \in \Lambda^+$

is an interface $I.$
* Interfaces in the RC model:

\[ \exists \text{(a.s.) unique left Top component: } V_{\text{Top}} \]

\[ \exists \text{(a.s.) unique left Bot component: } V_{\text{BoT}} \]

* Augment the components:

\[ \hat{V}_{\text{Top}} = V_{\text{Top}} \cup \{ \text{finite components of } V_{\text{Top}}^{c} \} \]

\[ \hat{V}_{\text{BoT}} = V_{\text{BoT}} \cup \{ \text{finite components of } V_{\text{BoT}}^{c} \} \]

* Interfaces:

\[ I_{\text{Top}} = \{ f = (u,v)^{*} \text{ for } u \in \hat{V}_{\text{Top}} \} \]

\[ I_{\text{BoT}} = \{ f = (u,v)^{*} \text{ for } u \in \hat{V}_{\text{BoT}} \} \]

Analogously:

\[ I_{\text{BoT}} = \{ f = (u,v)^{*} \text{ for } u \in \hat{V}_{\text{BoT}} \} \]

Last but not least:

\[ I = \{ \text{1-connected comp of dual-closed faces touching the boundary} \} \]
\[ I_{T_{\rho}} = \left\{ f = (u, v)^z \; \text{for} \; u \in \hat{V}_{T_{\rho}} \; \text{and} \; v \notin \hat{V}_{T_{\rho}} \right\} \]

\[ I_{B_{\delta T}} = \left\{ f = (u, v)^z \; \text{for} \; u \in \hat{V}_{B_{\delta T}} \; \text{and} \; v \notin \hat{V}_{B_{\delta T}} \right\} \]
Theorem 2: [Chen, L.]
Consider the RC model on $\Lambda_n = [-L, L]^2 \times (2\mathbb{Z} + \frac{1}{2})$
with Dobrushin b.c., $q > 1$ and $\beta > \beta_0$ (fixed) condition on the existence of $I$.

Let $M_n = \text{MIN} \text{ height of } I_{\text{Bot}}$
$M'_n = \text{MAX} \text{ height of } I_{\text{Bot}}$

Then:

$$M_n - E[M_n] = O_p(1) \quad (\text{Tightness})$$

Moreover, $\exists \alpha, \alpha' > 0$ s.t.

$$E M_n = \left( \frac{2}{\alpha} + o(1) \right) \log n \quad , \quad E M'_n = \left( \frac{2}{\alpha'} + o(1) \right) \log n$$

and $\alpha' > \alpha$. 

\[ \hat{V}_{\text{Top}} \]
\[ \hat{V}_{\text{Bot}} \]
* A Tale of Four Rates:

We can compare the rates as follows:

Theorem: [Chen, L.]

The rates from Thms. 1, 2 satisfy:

\[ 4\beta - c \leq \alpha \leq 4\beta \]

\[ \sigma - \alpha = (1 \pm \epsilon_\beta) \, e^{-\beta} \]

\[ \sigma' - \alpha = (1 \pm \epsilon_\beta) \, (q-1) \, e^{-\beta} \]

\[ \alpha' - \alpha = (1 \pm \epsilon_\beta) \, q \, e^{-\beta} \]
How did the Ising pf work? (and why does it fail for Potts)

Step I: Cluster expansion:

\[ P(I) = \frac{1}{Z} e^{-\beta |I|} + \sum_{f \in I} g(f, I) \]

# faces in I

Interaction face function \( g \): uniformly odd, local

Step II: Dobrushin's rigidity framework:

- Classify conn. sets of "excess" faces in \( I \) as WALLS.
- Rest are CEILINGS.

- To show rigidity:
  - Attempt to delete a wall \( W \) gaining \( \beta |W| \).
  - The tricky part: controlling \( g \).

E.g.:

- Deleting \( W \) may shift other parts of \( I \) which accumulate interaction terms...
- Must continue deleting "nearby" walls.

Dobrushin grouped walls together via size vs. distance to make this arg work.
Step III: From WALLS to PILLARS:

Dobrushin’s deletion of complete groups of walls is too crude to recover LD rates.

Instead: [Gheissari, L.] looked at the PILLAR $P_x$:

the conn. comp in $H_+$ of $\bigcirc$ spins containing $x$

- Conditional on the event

$$E_h^x = \{ \text{hit } (P_x) \geq h \}$$

it should behave as a (directed) RW in $\mathbb{Z}^3$ with regeneration pts.

- Break it into increments

Goal:

(a) Show that a given increment tends to be “trivial”: a cube (4 side faces)

(b) Including 1st (exceptional) increment
- How do we show (a)?

By "straightening" $P_x$:

* replacing $i$-th increment $X_i$
  by a trivial one.
* doing so for any $j$-th incr $X_j$
  whose size is too large
  compared to $\text{dist}(X_i, X_j)$.

- How do we show (b)?

  Complicated algorithm for modifying $I$.

$\textbf{Step IV:}$ The LD rate $\alpha$:  

- $P_x$ concerns a component of $+$.

Can't we use FKG for \textbf{SUPER-MULTIPLICATIVITY}? 

$N_\alpha$: due to the b.c. at height $h$, we are more negative.

- Instead: \textbf{SUB-MULTIPLICATIVITY}: (à la "BK-inequality")

Use \textbf{monotonicity} and properties (a), (b).
Random Cluster to the rescue?

- The toolkit to handle pillars is robust, but without the sub-multiplicative argument: of no value...
- While $P(I \in \cdot)$ in Ising does not sat. FRG, the Ising dist on configurations does: monotonicity used in a crucial way.
- Standard remedy to Potts non-monotonicity: RC.

* [Gielis-Grimmett 10] extended the framework of Dobrushin to RC cond on an interface:
  call this measure
  \[ \bar{\mu}_n = \mu_n( \cdot | D_n) \]

* Still no monotonicity because of the cond.
on the (exponentially unlikely) event $D_n$.
* However: at least the RC measure $\mu_n$ is monotone.

* Cluster expansion and rigidity pf give us the foundations for studying $I$ in $\bar{\mu}_n$. 
The RC interface $I$:

$[f = e^z \text{ s.t. } e \not\in \omega]$

$I = \{ 1$-connected comp of dual-closed faces touching the boundary $\}$

Not the interface we'd want to study but the one [GG'02] developed tools for:

No longer just a surface

* BUT: many complications:

- Cluster Expansion:

\[
\mu_\beta(I) \propto (1 - e^{-\beta}) \sum_{f \subseteq I} k_f \frac{e^{-\beta|I|}}{1 + \sum_{f \subseteq I} g(f, I)}
\]

(dual open) faces
1-corn to $I$
but not in $I$

[In accordance with the RC $p \# \text{open} (1-p) \# \text{closed} \# \text{corn}$]

- Walls & Ceilings: done w.r.t.
  extending $I$ into $I^2$ via some open faces:

\[
I^* := I \cup \{ f \in e^2 \text{ horizontal} \}
\]
The RC PILLAR $P_x$:

- Recall: Ising PILLAR = the $\lambda$-conn. comp in $\lor_+$ of + spins containing $x$

RC PILLAR $P_x$: the $\lambda$-conn. comp in $\hat{V}_{\text{Top}} \cap \lor_+$ containing $x$

Its faces def. by taking

$$F = \{ f = (u,v)^2 : u \in P_x, \ v \in \lor_+ \setminus P_x \}$$

and adding to it any 1-conn. comp of faces $E$ in $I \setminus I_{\text{Top}}$ s.t. $E \cap P_x \cap \lor_+ \neq \emptyset$

- Added "hairs" necessary to deal with $\partial I$ in the [GG'02] cluster expansion.

- But now separate pillars can touch each other ...
* Suppose we could control the pillar $P_x$.
What about the **sub-multiplicativity** argument?

- The goal: show

$$\bar{\mu}_n(A_{h_1+h_0}) \leq \bar{\mu}_n(A_{h_1}) \bar{\mu}_n(A_{h_0})$$

- In Ising: we exposed a $+$ component, by def surrounded by $-$'s.

- Here: much more delicate to def faces of $I$ we expose to support a Domain Markov Property.

(starting from the open faces $\partial I$)

- Last but not least: **the missing bar**:

Even if this recipe gave

$$\bar{\mu}_n(A_{h_1+h_0}) \leq \bar{\mu}_n(A_{h_1}) \mu_n(A_{h_0})$$

then the last term on the RHS is in a graph with different b.c.
(no longer the $\bar{\mu}_n$ measure)
Some of the ideas to bypass these obstacles:

* $P_x$ offsets $P_y$ via "hairs": establish that "typically"
  $P_x \in$ cone
devoid of other walls

* Offset the new terms $(1 - e^{-\beta}) |I| \kappa$ in the [GG'02] cluster expansion
  via deleted faces in the "straightening" of $P_x$.

* Approximate the event $E_h = \{ ht(P_x) > h \}$ by a suitable $A_h$ that is
  amenable to exposing certain faces of $I$ forming a b.c.
on the graph above height $h$
  not very sensitive to $D_n$ at large $h$ then add it to RHS by
  monotonicity:
  $\mu_n(A_{n2}) = \overline{\mu}_n(A_{n2})$
  Only works for a DECREASING $A_h$!
Carrying out the program:

Max height of $I_{\text{Top}}:

\[
\alpha := \lim_{h \to \infty} -\frac{1}{h} \log \mathbb{P}_\nu (\text{ht}(P^\nu_x) \geq h)
\]

What about its Min height?

What about Potts?

Promising approach:

- Cond. on $\{\text{ht}(P^\nu_x) \geq h\}$ in the RC model, it behaves like a RW, in that its increments are asymp. stationary $\mathcal{Z}$ mixing.

- By the [ES] coupling, we need to consider the coloring of its interior.

- $\log \mathbb{P}(\exists B \in 2^\mathcal{B} \text{ s.t. } \text{ht}(P^\nu_x) \geq h)$ will be approx a sum of $110$ r.v.'s: $\log \mathbb{P}_\nu (\exists B \in 2^\mathcal{B} \in \mathcal{X})$
* The (retrospectively obvious) fault:

A typical \( P_x \) in \( \bar{P}_n \) (let (\( P_x \)) \( \geq h \)) has the above structure.

But the Max of \( I_{\text{blue}} \) might (and will!) come from an atypical \( P_x \).
(Most increments should be trivial, still)

* Complicated optimization: shape of \( P_x \) wants to minimize surface area, but also give many options for blue paths climbing to \( h \).

* Solution: show existence of the rate (rather than what its value is)

by another SUBMULTIPLICATIVITY argument:

\[
\phi_n ( \text{let}(P_x^{\text{blue}}) \geq h_1 + h_2 \mid \text{let}(P_x) \geq h_1 + h_2) \\
\leq C \phi_n ( \text{let}(P_x^{\text{blue}}) \geq h_1 \mid \text{let}(P_x) \geq h_1) \\
\cdot \phi_n ( \text{let}(P_x^{\text{blue}}) \geq h_2 \mid \text{let}(P_x) \geq h_2)
\]

BUT How??
The 3-to-3 map:

\[
\begin{align*}
\pi_T & \quad \pi_B \\
\pi_T & \quad \pi_B \\
\pi_T & \quad \pi_B \\
\end{align*}
\]

want to show that:

\[
\nu(P_B \times P_T) \leq (1+\varepsilon) \nu_1(P_B) \nu_2(P_T)
\]

write

\[
\nu(P_B \times P_T) - \nu_1(P_B) \nu_2(P_T)
\]

\[
= \sum_{A, A_1, A_2} \left[ \nu(P_B \times P_T, A) - \nu_1(P_B, A_1) \nu_2(P_T, A_2) \right]
\]

\[
= \sum_{A, A_1, A_2} \left[ \nu(P_B \times P_T, A) \nu_1(Q_B, A_1) \nu_2(Q_T, A_2) \\
- \nu(Q_B \times Q_T, A) \nu_1(P_B, A_1) \nu_2(P_T, A_2) \right]
\]
\[
\sum_{A, A_1, A_2} \nu \left( Q_B \times Q_B^T, A \right) \nu \left( P_B, A_i \right) \nu \left( P^T, A_2 \right) \left[ \frac{\nu \left( P_B \times P_B^T, A \right) \nu \left( Q_B, A_i \right) \nu \left( Q^T, A_2 \right)}{\nu \left( Q_B \times Q_B^T, A \right) \nu \left( P_B, A_i \right) \nu \left( P^T, A_2 \right)} - 1 \right]
\]

Control via cluster expansion with the 3-to-3 map

\[
\varepsilon \nu \left( P_B \right) \nu \left( P^T \right)
\]
* Recovering the LD rates \( \alpha', \sigma', \tau \):

With the 3→3 map we can recover the rates relative to \( \alpha \):

- **BLUE** path dominated by

\[
P(\bullet?) = \frac{p}{p+(1-p)q} + \frac{(1-p)q}{p+(1-p)q} \cdot \frac{1}{q} \approx 1 - \left(q-1\right) e^{-\beta}
\]

[for Max of \( I_{\text{blue}} \)]

- **Non RED** path dominated by

\[
P(\bullet?) = \frac{p}{p+(1-p)q} + \frac{(1-p)q}{p+(1-p)q} \cdot \frac{e^{-1}}{q} \approx 1 - e^{-\beta}
\]

[for Min of \( I_{\text{blue}} \)]

- **w-Conn** path dominated by

\[
P(\bullet?) = \frac{p}{p+(1-p)q} \leq 1 - q e^{-\beta}
\]

[for Min of \( I_{\text{top}} \)]

modulo: \( p \) gives info on conf inside !!