ON LEVEL LINE FLUCTUATIONS OF SOS SURFACES ABOVE A WALL

PATRIZIO CADDEO, YUJIN H. KIM, AND EYAL LUBETZKY

ABSTRACT. We study the low temperature $(2 + 1)$D Solid-On-Solid model on $[1, L]^2$ with zero boundary conditions and non-negative heights (a floor at height 0). Caputo et al. (2016) established that this random surface typically admits either a flat or a many nested macroscopic level loops $\{L_i\}_{i \geq 0}$ for an explicit $h \approx \log L$, and its top loop $L_0$ has cube-root fluctuations: e.g., if $\rho(x)$ is the vertical displacement of $L_0$ from the bottom boundary point $(x, 0)$, then $\max_{x \in L_0} \rho(x) = L^{1/3 + o(1)}$ over $x \in I_0 := L/2 + [-L^{2/3}, L^{2/3}]$. It is believed that rescaling $\rho$ by $L^{1/3}$ and $I_0$ by $L^{2/3}$ would yield a limit law of a diffusion on $[-1, 1]$. However, no nontrivial lower bound was known on $\rho(x)$ for a fixed $x \in I_0$ (e.g., $x = \frac{L}{2}$), let alone on $\min_{x \in I_0} \rho(x)$ in $I_0$, to complement the bound on $\max \rho(x)$.

Here we show a lower bound of the predicted order $L^{1/3}$: for every $\epsilon > 0$ there exists $\delta > 0$ such that $\min_{x \in I_0} \rho(x) \geq \delta L^{1/3}$ with probability at least $1 - \epsilon$. The proof relies on the Ornstein–Zernike machinery due to Campanino–Ioffe–Velenik, and a result of Ioffe, Shlosman and Toninelli (2015) that rules out pinning in Ising polymers with modified interactions along the boundary. En route, we refine the latter result into a Brownian excursion limit law, which may be of independent interest.

1. INTRODUCTION

We consider the Solid-On-Solid (SOS) model on $\Lambda_L = [1, L]^2$, an $L \times L$ square in $\mathbb{Z}^2$, at large inverse-temperature $\beta > 0$, with zero boundary conditions and a floor at height 0: denoting by $x \sim y$ a pair of adjacent sites $x, y \in \mathbb{Z}^2$, and setting $\varphi_x = 0$ for all $x \notin \Lambda_L$, the model assigns a height function $\varphi : \Lambda_L \to \mathbb{Z}_{\geq 0}$ (taking nonnegative integer heights) the probability

$$\pi^0_{\Lambda_L}(\varphi) \propto \exp \left(-\beta \sum_{x \sim y} |\varphi_x - \varphi_y| \right). \quad (1.1)$$

The model was introduced in the early 1950’s (see [5, 38]) to approximate the formation of crystals and the interface separating the plus and minus phases in the low temperature 3D Ising model.

While of interest in any dimension $d$, the study of the model on $\mathbb{Z}^2$ has special importance, as it is the only dimension associated with the roughening phase transition. For the low temperature 3D Ising model, which the $(2 + 1)$D SOS model approximates for large $\beta$, rigorously establishing the roughening phase transition is a tantalizing open problem which has seen very little progress since being observed some 50 years ago (numerical experiments suggest it takes place at $\beta_R \approx 0.408$, compared to the critical 3D Ising temperature $\beta_c \approx 0.221$). The corresponding phase transition for the $(2 + 1)$D SOS with no floor $\hat{\pi}$ (where $\varphi$ can be negative) was rigorously confirmed as follows: (i) (localization) for $\beta$ large enough, the surface is rigid, in that $\mathrm{Var}(\varphi_x) = O(1)$ at $x$ in the bulk, and furthermore $|\varphi_x|$ has an exponential tail [3]; (ii) (delocalization) for $\beta$ small enough, Fröhlich and Spencer [25, 26] famously showed that $\mathrm{Var}(\varphi_x) \propto \log L$, just as in the case where $\varphi$ takes values in $\mathbb{R}$. (iii) Very recently, Lammers [36] showed the phase transition in $\mathrm{Var}(\varphi_x)$ is sharp: there exists $\beta_R > 0$ such that $\mathrm{Var}(\varphi_x) \to \infty$ for all $\beta \leq \beta_R$ whereas it is $O(1)$ for all $\beta > \beta_R$; numerical experiments suggest that $\beta_R \approx 0.806$. (See [36] for additional details on the recent developments in the SOS model with no floor and related models of integer-valued height functions.)

Our setting is the low-temperature regime ($\beta$ large), yet with the restriction that the surface must lie above above a hard wall (the assumption $\varphi \geq 0$). Bricmont, El-Mellouki and Fröhlich [4] showed that this induces entropic repulsion, regarded as a key feature of the physics of random surfaces: the restriction $\varphi \geq 0$ propels the surface (despite the energy cost) so as to gain entropy. Namely, it was shown in [4] that $\frac{\epsilon}{\beta} \log L \leq \mathbb{E}[\varphi_x \mid \varphi \geq 0] \leq \frac{C}{\beta} \log L$ for absolute constants $c, C > 0$.

The gap between these bounds was closed in [13], where it was established that $\mathbb{E}[\varphi_x \mid \varphi \geq 0]$ is $\frac{1}{4\beta} \log L + O(1)$, and moreover, $(1 - \epsilon_\beta)L^2$ sites are at such a height with high probability (w.h.p.).
The following intuition explains the height asymptotics: if the surface lies rigid about height \( h \), then the cost of raising every site by 1 is \( 4\beta L \) (incurred at the sites along the boundary); the benefit in doing so would be to gain the ability to feature spikes of depth \( h+1 \) (forbidden at level \( h \) due to the restriction \( \varphi \geq 0 \)), and as such a spike has an energetic cost of about \( e^{-4\beta h} \), the entropy gain is about \( H(e^{-4\beta h})L^2 \) where \( H(\cdot) \) is the Shannon entropy; the two terms are equated at \( h \sim \frac{1}{4\beta} \log L \).

Significant progress in the understanding of the shape of the SOS surface above a hard wall was obtained in the sequel by the same authors [14]. The height-\( h \) level lines of the surface are the loops formed by placing dual-bonds between every pair \( x \sim y \) such that \( \varphi_x < h \) and \( \varphi_y \geq h \). To account for local thermal fluctuations, call a loop macroscopic if its length is at least \((\log L)^2\). With this notation, (a more detailed version of) the following theorem was given in [14] (see also [12]):

**Theorem** ([14, Thms. 1,2,3 and Rem. 1.3]). For \( \beta \) large enough, the \((2+1)D\) SOS model with zero boundary conditions on a square \( \Lambda_L = [1,L]^2 \) above the wall \( \varphi \geq 0 \), satisfies the following w.h.p.:

(i) **Shape:** At least \((1-\epsilon_\beta)\) of the sites \( x \in \Lambda_L \) have height \( \varphi_x = h^* \), where the random \( h^* \) is either \( \left\lfloor \frac{1}{4\beta} \log L \right\rfloor \) or \( \left\lfloor \frac{1}{4\beta} \log L \right\rfloor - 1 \). Moreover, there is a unique macroscopic loop at each height \( 0,1,\ldots,h^* \), and none above height \( h^* \). Further, for a diverging sequence\(^1\) \( L \)'s, the sequence of nested loops \( \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \), when rescaled to \([0,1]^2\), converges in probability in Hausdorff distance to a deterministic limit defined by a Wulff shape \( W \), which is the convex body of area 1 minimizing the line integral of a surface tension \( \tau_\beta(\cdot) \) along its perimeter \( \partial W \); the scaling limit of \( \mathcal{L}_k \) (where \( \mathcal{L}_0 \) is the top level line at height \( h^* \)) is given by the union of all possible translates of \( W \), rescaled by an explicit radius \( r_k \) that is decreasing in \( k \).

(ii) **Fluctuations:** For a diverging sequence\(^2\) \( L \)'s, the maximum displacement of the top level line \( \mathcal{L}_0 \) from the boundary segment \( I \times \{0\} \) for \( I = [\epsilon_\beta L, (1-\epsilon_\beta)L] \) is \( L^{1/3+o(1)} \). That is, if \( \mathcal{P}(x) = \max\{y \leq L/2 : (x,y) \in \mathcal{L}_0\} \) is the maximum y-coordinate of a point \((x,y)\) visited by \( \mathcal{L}_0 \) in the bottom-half of \( \Lambda_L \), then

\[
\max_{x \in I} \mathcal{P}(x) \leq L^{1/3+\epsilon},
\]

for any fixed \( \epsilon > 0 \), whereas for every interval \( I' \subset I \) of length \( L^{2/3-\epsilon} \),

\[
\max_{x \in I'} \mathcal{P}(x) \geq L^{1/3-\epsilon}.
\]

Consider the distance of the top level line loop \( \mathcal{L}_0 \) from a point \((x_0,0)\) on the bottom boundary of the box, where the scaling limit is flat—e.g., the center-side \( x_0 = L/2 \) (see Fig. 1 for a depiction).

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\(^1\)Namely, for any sequence of \( L \)'s where \( a_L \), the fractional part of \( \frac{1}{4\beta} \log L \), does not converge to an explicit \( \lambda_c(\beta) \).

\(^2\)Namely, for any sequence of \( L \)'s such that, for the above \( a_L \)'s and \( \lambda_c(\beta) \), one has \( \liminf_{L \to \infty} a_L > \lambda_c \).
The above theorem shows that \( \rho(x_0) \leq L^{1/3+\epsilon} \) w.h.p., yet it gives no nontrivial lower bound on it. It is believed that \( \rho(x_0) \) should have order \( L^{1/3} \) (with no poly-log corrections); more precisely, one expects \( \rho(x_0) \asymp_L L^{1/3} \), where we write \( f \lesssim g \) if \( f/g \) is uniformly tight, and \( f \asymp_L g \) if \( f \lesssim_L g \lesssim_L f \).

Moreover, one expects that if one were to rescale \( \rho(x) \) by \( L^{1/3} \) along an interval of order \( L^{2/3} \) positioned on bottom boundary (within the flat portion of the scaling limit)—take, e.g.,

\[
I_0 := [L^{2/3} - L^{2/3}/2, L^{2/3}/2 + L^{2/3}]
\]

for concreteness—then, after rescaling said interval by \( L^{2/3} \) (in the concrete example, to \([-1, 1]\)), one would arrive at a limit law of a nontrivial diffusion, a variant of a Ferrari–Spohn diffusion [24]. (This prediction was stated here in terms of \( \rho(x) \), the maximal vertical displacement of \( \mathcal{L}_0 \), so as to be well-defined, as \( \mathcal{L}_0 \) can have many points with a given \( x \)-coordinate; the same statement is expected to hold for \( \rho(x) \) measuring the minimal displacement of \( \mathcal{L}_0 \), as defined in Theorem 1.1.)

To explain this prediction, note first that it is well-known that a Brownian excursion tilted (penalized) by \( \exp(-\lambda A) \), where \( A \) is the area under it, has the law of a Ferrari–Spohn diffusion. The entropic repulsion that propels the SOS level line loop \( \mathcal{L} \) to height \( h \) acts much like an area tilt: if a loop has internal area \( S \), then its probability (roughly) gains a tilt of \( \exp(\lambda S) \) as described earlier, or equivalently, a tilt of \( \exp(-\lambda A) \) where \( A = L^2 - S \) is the area exterior to it. Consider \( \mathcal{L}_k \), which is at height \( h^* - k \); there \( \lambda \approx e^{-4\beta(h^* - k)} \approx L^{-4\beta k} \) (recall \( h^* \approx \frac{1}{12} \log L \)), and we see that the rescaling of \( \rho(x) \) by \( L^{1/3} \) and \( I_0 \) by \( L^{2/3} \) cancels the \( L^{-1} \) factor in \( \lambda \) and translates into a tilt of \( \exp(-e^{4\beta k} \hat{A}) \) where \( \hat{A} \) is the rescaled area, as in the above continuous approximation (see also [32, 34]). Related to this, the famous problem of establishing a Ferrari–Spohn law for the 2D Ising interface under critical wetting (which may be seen as a version of the SOS problem only with a single contour as opposed to \( c \log L \) many) was finally settled in a recent seminal work by Ioffe, Ott, Shlosman and Velenik [29] (prior to that, the \( L^{1/3+o(1)} \) fluctuations were established by Velenik [39] and the tightness of the rescaled area was proved by Ganguly and Gheissari [27]). The challenges in handling a diverging number of interacting (non-crossing) contours with distinct area tilts (the \( k \)-th one is tilted by \( \approx \exp(-e^{4\beta k} \hat{A}) \) are such that the simplified problem that has Brownian excursions with area tilts is already nontrivial; see [9, 10, 11, 16] for recent progress on it.

In accordance with this prediction for the scaling limit of \( L^{-1/3} \rho(x) \) along \( I_0 \), one expects that both \( \max_{x \in I_0} \rho(x) \) and \( \min_{x \in I_0} \rho(x) \) would be of the same order as our rescaling factor \( L^{1/3} \); i.e., to be precise, that \( \max \rho(x) \gtrsim_L L^{1/3} \) and that \( \min \rho(x) \gtrsim_L L^{1/3} \). Our main result is the latter part (readily implying \( \rho(x_0) \gtrsim_L L^{1/3} \) at any given \( x_0 \leq (1 - \epsilon_3) L \), e.g., the center-side \( x_0 = L/2 \)).

**Theorem 1.1.** Fix \( \beta > \beta_0 \), and consider the \((2 + 1)D\) SOS model with zero boundary conditions on \( \Lambda_L = [1, L]^2 \) as per Eq. (1.1) above a wall \( \varphi \geq 0 \). Let \( \mathcal{L}_0 \) be the (w.h.p. unique) top macroscopic level line, consider the interval \( I_0 = [L^{2/3} - L^{2/3}/2, L^{2/3}/2 + L^{2/3}] \) centered on the bottom boundary, and let

\[
\rho(x) = \min \{ y \geq 0 : (x, y) \in \mathcal{L}_0 \}
\]

denote the minimum vertical displacement of \( \mathcal{L}_0 \) from the bottom boundary at the coordinate \( x \). Then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for large enough \( L \), with probability at least \( 1 - \epsilon \),

\[
\min_{x \in I_0} \rho(x) \gtrsim_L \delta L^{1/3}.
\] (1.4)

As we later explain, we obtain the Eq. (1.4) by moving from \( \rho(x) \) to a stochastically dominated curve whose limit (after the same rescaling) is a Brownian excursion, yielding the \( L^{1/3} \) bound. This refines the lower bound in Eq. (1.3) into the estimate \( \max_{x \in I_0} \rho(x) \geq \min_{x \in I_0} \rho(x) \gtrsim_L L^{1/3} \).

Note though that one cannot replace the \( L^{1/3+o(1)} \) in the upper bound of Eq. (1.2) by \( O(L^{1/3}) \), as it addresses \( \max \rho(x) \) over all \( I = [\epsilon_L L, (1 - \epsilon_3)L] \). Our comparison to a Brownian excursion implies that \( \rho(x_0) \geq aL^{1/3} \) with probability at least \( C \exp(-a^2/2) \) for any \( x_0 \in I \) and \( a = a(L) \), as well as that \( \max_{x \in I} \rho(x) \geq cL^{1/3} \sqrt{\log L} \) w.h.p., vs. its predicted order \( L^{1/3}(\log L)^{2/3} \) (see Question 1.3).
To derive the Brownian excursion law, we rely on the powerful Ornstein–Zernike framework as developed by Campanino, Ioffe and Velenik [28, 6, 7, 8], that allows one to couple the interface in hand to a directed random walk. This machinery was the key to several recent advances in the understanding 2D Ising interfaces (e.g., [29] mentioned earlier) and Potts interfaces (e.g., [37, 30]). In fact, the work [30], due to Ioffe, Ott, Velenik and Wachtel, is of particular interest in our setting: there it was shown that the interface of the 2D Potts model in a box with Dobrushin’s boundary conditions has the scaling limit of a Brownian excursion for all \( \beta > \beta_c \). As in our case, one of the main obstacles is the interaction of the interface with the boundary, and in particular, ruling out the scenario whereby the interface is pinned to the wall. This was achieved for the Potts interface in [30]) (and later used as an ingredient in [29]) via a direct analysis of its random cluster interface in [30] (and later used as an ingredient in [29]).

Here we instead appeal to the framework of Ioffe, Shlosman and Toninelli [31] to rule out pinning. That approach, while valid only for large enough \( \beta \) (whereas the analysis in [29] holds for all \( \beta > \beta_c \)), is fairly generic, and applicable to SOS contours as part of the following family of Ising polymers as defined by [31] in the positive half-plane \( \beta > \beta_c \) for \( \sigma > 0 \) such that, if \( \tau(x) = \max \{ y : (x, y) \in \gamma \} \), then \( \tau([xN])/(\sigma \sqrt{N}) \) converges weakly to a standard Brownian excursion on \([0, 1]\), and the same holds for \( \tau(x) = \min \{ y : (x, y) \in \gamma \} \).

In particular, Part (b) applies to the SOS model \( \tau^{0,1,1,1} \) with no floor on a box \( \Lambda \) of side length \( N \), for \( \beta > \beta_0 \) and boundary conditions 0 on the bottom side and 1 elsewhere: namely, the height-1 level line that connects the bottom corners of the box \( \Lambda \) has a scaling limit of a Brownian excursion.
We end this section with a few questions regarding large deviations of $\rho(x_0)$, the vertical distance of $L_0$ to the bottom boundary point $(x_0,0)$. It is plausible that for all $x_0 \in I = [\epsilon_\beta L, (1 - \epsilon_\beta)L]$, the upper tails of $\rho(x_0)$ would feature a rate function as in the Tracy–Widom distribution, i.e.,

$$x_{\lambda L}^0(\rho(x_0) > aL^{1/3}) = e^{-\theta(a^{3/2})}.$$ 

See [27], where an estimate of this type was obtained for the 2D Ising interface in critical prewetting. (Our bounds here give a lower bound of $e^{-ca^2}$, as $L^{-1/3}\rho(x_0)$ stochastically dominates a Gaussian.)

We further note that a routine monotonicity argument allows one to take $\epsilon L^{2/3}$ disjoint sub-intervals of $I$ of length $L^{1/3}$ and treat them as independent for the sake of a lower bound on $\max_{x \in I} \rho(x)$. The above behavior for the upper tail of $\rho(x_0)$ would then yield $\max_{x \in I} \rho(x) \asymp \rho \sim L^{1/3} (\log L)^{2/3}$, as in the local roughness of droplet boundaries in the random cluster model, shown by Alexander [1].

**Question 1.3.** Let $x_0 \in [\epsilon_\beta L, (1 - \epsilon_\beta)L]$. What is the rate function $a \mapsto -\log \pi_{\lambda L}^0(\rho(x_0) > aL^{1/3})$?

As for lower tails, to our knowledge these are still open for the 2D Ising under critical prewetting, where one expects the Ising interface to reach the bottom in $O_p(1)$ locations along $[\epsilon_\beta L, (1 - \epsilon_\beta)L]$. One should stress though that it is unclear that the SOS large deviations would take after the behavior of the 2D Ising interface under prewetting—particularly for the lower tails, where in SOS there are $\Theta(\log L)$ level lines below $L_0$, all of which must cooperate with a downward deviation.

**Question 1.4.** Let $x_0 \in [\epsilon_\beta L, (1 - \epsilon_\beta)L]$. What is the order of $\pi_{\lambda L}^0(\rho(x_0) = 0)$?

The paper is organized as follows. In Section 2, we formalize the setting of Ising polymers, as well as the inputs we need from Ornstein–Zernike theory. We also establish that the SOS model satisfies the required hypotheses of Ising polymers, and thus Theorem 1.2 is applicable to it. Section 3 proves Theorem 1.1, addressing the SOS measure $\pi$, using a monotonicity argument and the conclusion of Theorem 1.2 on $\hat{\pi}$, the SOS measure without a floor. In Section 4, we introduce a random walk in $\mathbb{H}$ that is closely related to Ising polymers, and state the key limit theorems for this random walk. In turn, Section 5 provides the proof of Theorem 1.2 modulo these random walks results that are deferred to Section 6.

## 2. Cluster expansion, Ising polymers, and Ornstein–Zernike theory

In this section, we review the tools needed for our proofs—notably, cluster expansion, prior work on Ising polymers, and Ornstein–Zernike theory. In several cases, we will need variants of existing results, which are not covered by the results proved in the literature. In those cases, we provide proofs of these analogues (either in the main text or in the appendix).

Throughout the paper, we say that an event holds with high probability (w.h.p.) if its probability tends to 1 as the system size (typically, $L$ or $N$) tends to $\infty$. For two functions $f : \mathbb{N} \rightarrow (0, \infty)$ and $g : \mathbb{N} \rightarrow (0, \infty)$, write $f \sim g$ to denote that $\lim_{N \rightarrow \infty} f(N)/g(N) = 1$; write $f \lesssim g$ when there exists a constant $K > 0$ such that $f(N) \leq K g(N)$ for all $N \in \mathbb{N}$; and write $f \asymp g$ when $f \lesssim g$ and $g \lesssim f$.

### 2.1. Contours and cluster expansion

A contour $\gamma$ is a collection of bonds $(e_i)_{i=1}^m$ in the dual lattice $(\mathbb{Z}^2)^*$, where all bonds are distinct except possibly $e_1$ and $e_m$ may coincide, every two consecutive edges share a vertex, and the path formed is simple except in accordance with a splitting rule: if the pair $e_i, e_{i+1}$ and $e_j, e_{j+1}$ all intersect at a vertex $x$, then the two other end-points of $e_i, e_{i+1}$ are on the same side of the line through $x$ with slope 1 (from southwest to northeast), and similarly for $e_j, e_{j+1}$ (this is the northeast splitting rule). We call $\gamma$ an open contour if $e_1 \neq e_m$.

In the context of the SOS model in a finite, connected $\Lambda \subseteq \mathbb{Z}^2$ under 0 boundary conditions, for any $h$, the $h$-level lines (recall that for any configuration $\varphi$ and integer $h$, these are the bonds dual to $x \sim y$ with $\varphi_x < h$ and $\varphi_y \geq h$) give rise to a collection of disjoint loops after applying the global splitting rule. In the presence of a boundary condition $\xi \in \{0,1\}^{\partial \Lambda}$ consisting of a connected stretch of 0’s (and 1’s elsewhere), this gives rise to a unique open contour among the height-1 level
lines (accompanied by a collection of closed contours). We refer to this path as the open 1-contour. Let \( \Delta_\gamma^+ \) and \( \Delta_\gamma^- \) denote the set of sites (of \( \mathbb{Z}^2 \)) immediately above and below \( \gamma \), respectively, and define

\[ \Delta_\gamma := \Delta_\gamma^+ \cup \Delta_\gamma^- . \]

Note that the sites in \( \Delta_\gamma^+ \) have height \( \geq 1 \), while the sites in \( \Delta_\gamma^- \) have height \( \leq 0 \). Each \( \gamma \) divides \( \Lambda \) into two regions, \( \Lambda_\gamma^+ \) and \( \Lambda_\gamma^- \), where we write \( \Lambda_\gamma^+ \) to denote the region that contains \( \Delta_\gamma^+ \) as part of its inner boundary. The next proposition addresses the law of this unique SOS open contour \( \gamma \).

**Proposition 2.1** ([13, Lem. A.2]). Consider the SOS model \( \tilde{\pi}_\Lambda^\xi \) on any finite, connected \( \Lambda \subset \mathbb{Z}^2 \) and under any boundary condition \( \xi \in \{0, 1\} \) that induces a unique open 1-contour \( \gamma \). Then there exists a constant \( \beta_0 > 0 \) such that for all \( \beta \geq \beta_0 \),

\[ \tilde{\pi}_\Lambda^\xi(\gamma) \propto \exp \left( -\beta |\gamma| + \sum_{C \cap \Delta_\gamma \neq \emptyset} \phi(C; \gamma) 1_{\{C \subset \Lambda\}} \right), \]

for some “decoration functions” \( \{\phi(C, \gamma)\}_{C \subset \mathbb{Z}^2} \) satisfying the following properties:

(i) If \( C \) is not connected, then \( \phi(C, \gamma) = 0 \).

(ii) The decoration function \( \phi(C, \gamma) \) depends on \( \gamma \) only through \( C \cap \Delta_\gamma \).

(iii) For all \( v \in \mathbb{Z}^2 \), \( \phi(C; \gamma) = \phi(C + v; \gamma + v) \).

(iv) Letting \( d(C) \) denote the cardinality of the smallest connected set of bonds of \( \mathbb{Z}^2 \) containing all boundary bonds of \( C \) (i.e., bonds connecting \( C \) to \( C^c \)), we have the decay bound

\[ \sup_{\gamma} |\phi(C; \gamma)| \leq \exp \left( - (\beta - \beta_0)d(C) \right). \]

Furthermore, defining the interface partition function

\[ G_\Lambda^\xi := \sum_{\gamma} e^{-\beta |\gamma| + \sum_{C \subset \Lambda} \phi(C; \gamma)} , \]

we have

\[ G_\Lambda^\xi = \tilde{Z}_\Lambda^\xi / \tilde{Z}_\Lambda^0. \]

The decoration functions come from *cluster expansion* applied to the partition functions in \( \Lambda_\gamma^+ \) and \( \Lambda_\gamma^- \). In Appendix A, we recall cluster expansion for the SOS model and provide the proof of Proposition 2.1, as the expression for the decoration function \( \phi \) is needed to verify that it meets the criteria of modified Ising polymers (Definition 2.6). In light of Property (i), from now on we will write \( C \) to denote a connected subset (or *cluster*) of \( \mathbb{Z}^2 \).

Next, we recall the notion of surface tension for the SOS model.

**Definition 2.2** (Dobrushin boundary conditions, surface tension). Fix \( \bar{u} \in \mathbb{S}^1 \) with \( \theta_{\bar{u}} \in [0, \pi/2) \), where \( \theta_{\bar{u}} \) is the angle \( \bar{u} \) makes with the positive horizontal axis. Set \( \Lambda_{N,M} := [1, N] \times [-M, M] \), and let \( \xi(\bar{u}) \) denote the boundary condition defined by \( \xi(\bar{u})_v = 0 \) for all \( v \in \mathbb{Z}^2 \) lying on or below \( \text{span}(\bar{u}) \), and \( \xi(\bar{u})_v = 1 \) otherwise. The set of boundary conditions relevant to us is \( \xi(e_1) \) and we will denote it by \( 0, 1, 1, 1 \) (after the values induced by \( \xi(e_1) \) on the four sides of the box \( \Lambda_{N,M} \)). Define \( d_{N,\bar{u}} := N/\cos(\theta_{\bar{u}}) \). Then the surface tension \( \tau_{\beta,\text{SOS}}(\bar{u}) \) is defined by

\[ \tau_{\beta,\text{SOS}}(\bar{u}) := \lim_{N \to \infty} \lim_{M \to \infty} -\frac{1}{d_{N,\bar{u}}} \log \left( G_{\Lambda_{N,M}}^{\xi(\bar{u})} \right). \]

Using the symmetry of the SOS model, \( \tau_{\beta,\text{SOS}} \) trivially extends to an even function on \( \mathbb{S}^1 \) possessing all lattice symmetries. Finally, \( \tau_{\beta,\text{SOS}} \) extends to a function on all of \( \mathbb{R}^2 \) via homogeneity:

\[ \tau_{\beta,\text{SOS}}(x) := \|x\| \tau_{\beta,\text{SOS}} \left( \frac{x}{\|x\|} \right), \text{ for all } x \in \mathbb{R}^2. \]

The proof of the existence of \( \tau_{\beta,\text{SOS}}(x) \) as well as many of its properties can be found in [20, §1–2].
2.2. **The free Ising polymer model.** In this section, we define the class of Ising polymer models, as given by [31]. The reader is also referred to [20] for many useful results on such polymer models. It will be shown that the SOS open contour from Eq. (2.1) falls in this class.

Recall that we always write $C$ to denote a finite, connected subset of $\mathbb{Z}^2$. For every contour $\gamma$, consider any decoration function $\Phi(C; \gamma)$ satisfying the following four properties:

(P1) **Locality:** $\Phi(C; \cdot)$ depends on $\gamma$ only through $C \cap \Delta_\gamma$.

(P2) **Decay:** There exists some $\chi > 1/2$ such that, for all $\beta > 0$ sufficiently large,

$$\sup_{\gamma} |\Phi(C, \gamma)| \leq \exp(\chi \beta (d(C) + 1)),$$

where $d(C)$ is defined as in Property (iv) of Proposition 2.1.

(P3) **Translational symmetry:** for all $v \in \mathbb{Z}^2$, $\Phi(C; \gamma) = \Phi(C + v; \gamma + v)$.

(P4) **Symmetry of the surface tension:** the surface tension $\tau_\beta(x)$ defined below in Eq. (2.8) possesses all discrete symmetries of $\mathbb{Z}^2$ (rotations by $\pi/4$ and reflections w.r.t. axes and the diagonals $y = \pm x$).

Towards specifying the probability that the model assigns to each polymer—as well as the surface tension $\tau_\beta$ mentioned in Property (P4)—define the free (polymer) weight via

$$q(\gamma) = \exp \left( -\beta |\gamma| + \sum_{C \cap \Delta_\gamma \neq \emptyset} \Phi(C; \gamma) \right),$$

where, here and throughout the article, sums over $C$ are assumed to only go over connected subsets $C \subset \mathbb{Z}^2$. Next, for any $x \in (\mathbb{Z}^2)^*$, consider the partition function going over all contours $\gamma$ with start-point given by the dual origin $\sigma^* := (1/2, 1/2)$ and end-point $x$:

$$G(x) := \sum_{\gamma: \sigma^* \to x} q(\gamma).$$

For any set of contours $E$, consider also the partition function going over all contours in $E$ with end-points $0$ and $x$:

$$G(x \mid E) := \sum_{\gamma: \sigma^* \to x, \gamma \in E} q(\gamma) 1_{\{\gamma \in E\}}.$$

Lastly, define the Ising polymer surface tension $\tau_\beta(\cdot)$ via

$$\tau_\beta(\bar{n}) := -\lim_{N \to \infty} \frac{1}{N} \log G(N\bar{n}) \quad \text{for } \bar{n} \in S^1,$$

where the limit is taken over $N$ such that $N\bar{n}$ is in $(\mathbb{Z}^2)^*$. By homogeneity, $\tau_\beta$ extends to all of $\mathbb{R}^2$.

**Definition 2.3** (Free Ising polymer). The free Ising polymer model in a subset $D \subset \mathbb{R}^2$ is given by the probability measure over contours $\gamma : \sigma^* \to x$ contained in $D$:

$$P^x(\cdot \mid \gamma \subset D) := \frac{G(x \mid \gamma \subset D, \gamma \in \cdot)}{G(x \mid \gamma \subset D)},$$

for a partition function $G$ as above with a decoration function $\Phi$ satisfying Properties (P1) to (P4).

Below, we list some needed properties of $\tau_\beta$, proven in [20].

---

3The Ising polymer model in [31] had a weaker decay condition, taking $d(C)$ in Property (P2) to be the $L^\infty$-diameter of $C$, whereas our arguments require $d(C)$ to be the minimum size of a connected set containing its boundary.

4It is common to define $\tau_\beta$ with the $1/\beta$ pre-factor (such was the case in [20] as well as [14]). We do not include the pre-factor here since related Ornstein–Zernike works (e.g., [33, 29]) do not include this pre-factor, and this keeps various definitions (e.g., $W, W^k, P^k$ defined below) consistent with those works. In [31], $\tau_\beta$ does have a $1/\beta$ pre-factor, though it seems to be a typo, as their inputs from the Ornstein–Zernike theory come from the aforementioned [33] and as such their calculations are consistent with the above definition of $\tau_\beta$. We will prove the Ornstein–Zernike facts we require here, to make the proof more self-contained and avoid potential consistency issues.
Proposition 2.4 (Surface tension properties, [20]). There exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$:
(i) the formula in Eq. (2.8) converges uniformly,
(ii) the surface tension $\tau_\beta : \mathbb{R}^2 \to \mathbb{R}^2$ is analytic,
(iii) (Strong triangle inequality) For any two non-collinear vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{Z}^2$, we have
\[ \tau_\beta(\vec{u}) + \tau_\beta(\vec{v}) > \tau_\beta(\vec{u} + \vec{v}). \]

Proof. Properties (i) and (ii) follow from the combination of Theorem 4.8 and Proposition 4.12 of [20].\footnote{These results are stated in [20] for the Ising polymer model $G(x)$. The analogue of $G$ is defined in [20, Eq. 4.3.11] (where for us, $h = 0$). Though that formula involves decoration functions $\Phi$ coming from the cluster expansion of the Ising model, the results in [20, Section 4] that follow this formula only use Properties (iii) and (iv) of our Proposition 2.1. This is stated in the final paragraph of [20, Section 4.3].} Lastly, Property (iii) appeared in [20, Proposition 1, Section 4.21]. \hfill \Box

For our main application of Theorem 1.2, the SOS model, the following result relates the surface tension $\tau_\beta^{\text{sos}}$ defined in Eq. (2.4) to the Ising polymer surface tension $\tau_\beta$ defined above. We suspect it is known, though could not find an exact reference for it, and as it follows from our other arguments in Section 5, we include its proof in Section 5.3 for completeness.

Proposition 2.5. Fix $\beta \geq \beta_0$. Let $q(\gamma)$ denote the free polymer weight as per Eq. (2.6) with $\Phi(C; \gamma)$ taken to be $\phi(C; \gamma)$ from Proposition 2.1. Consider the Ising polymer surface tension $\tau_\beta$ corresponding to the free polymer weights $q(\gamma)$ as per Eq. (2.8). For all $u \in \mathbb{R}^2$,
\[ \tau_\beta^{\text{sos}}(u) = \tau_\beta(u). \]

Observe that the decoration function $\phi(C; \gamma)$ appearing in the law of the unique open 1-contour under $\tilde{\pi}^\xi_\Lambda$ satisfies Properties (P1) to (P4). Indeed, Proposition 2.1 states that the decoration function $\phi(C; \gamma)$ satisfies Properties (P1) to (P3). Property (P4) holds for $\tau_\beta^{\text{sos}}$ because of the symmetry of the SOS model, and thus for $\tau_\beta$ thanks to Proposition 2.5.

However, the open 1-contour in the SOS model is not a free Ising polymer, due to the $1 \{ C \subseteq \Lambda \}$ term appearing in (2.1), which introduces an interaction of $\gamma$ with the boundary via the decoration function $\phi$. This served as one of the motivations of [31] to study modified Ising polymers—the generalization of the free Ising polymers described above to allow domain-induced modifications on the “free” weights (the SOS open contour does belong to that family of models: see Observation 2.7).

2.3. The modified Ising polymer model. For any $D \subset \mathbb{R}^2$ and for any decoration function $\Phi(C; \gamma)$ satisfying Properties (P1) to (P4), consider any function $\Phi_D(C; \gamma)$ satisfying
(M1) $\Phi_D(C; \gamma) = \Phi(C; \gamma)$ for any $C \subset D$, and
(M2) $\Phi_D(C; \gamma)$ satisfies the same decay bound in Eq. (2.5) for all $C$.

Call $\Phi_D(C; \gamma)$ a modified decoration function (with modifications outside of $D$), and define the modified polymer weight
\[ q_D(\gamma) = \exp \left( -\beta |\gamma| + \sum_{C \cap \Delta_\gamma \neq \emptyset} \Phi_D(C; \gamma) \right) \]
as well as the partition function
\[ G_D(x) := \sum_{\gamma : o^* \rightarrow x, \gamma \subset D} q_D(\gamma) \quad \text{and} \quad G_D(x \mid E) := \sum_{\gamma : o^* \rightarrow x, \gamma \subset D \cap E} q_D(\gamma) \mathbb{1}_{\{ \gamma \in E \}} \]
for a set of contours $E$.

Definition 2.6 (Modified Ising polymer). The modified Ising polymer in a subset $D \subset \mathbb{R}^2$ is given by the probability measure on contours $\gamma : o^* \to x$ contained in $D$:
\[ P_D^\gamma(\cdot) := \frac{G_D(x \mid \gamma \in \cdot)}{G_D(x)}, \quad (2.10) \]
where the partition function $G_D$ is defined for a decoration function $\Phi$ satisfying Properties (P1) to (P4), and a modified decoration function $\Phi_D$ satisfying Properties (M1) and (M2).
Note that the SOS open contour with law $\hat{\pi}_A^\xi$ given by Eq. (2.1) is of the form above, with $\Phi(C; \gamma) = \phi(C; \gamma)$, $D = \Lambda$, and $\Phi_A(C; \gamma) = \phi(C; \gamma)1_{\{C \subset \Lambda\}}$ for $\phi(C; \gamma)$ from Proposition 2.1 (in which this choice of $\Phi_D(C; \gamma)$ clearly satisfies Properties (M1) and (M2)). Namely, the following holds:

**Observation 2.7** (SOS is a modified Ising polymer). Fix $\beta \geq \beta_0$, and consider the SOS model $\hat{\pi}_A^\xi$ in a finite, connected subset $\Lambda \subset \mathbb{Z}^2$ and boundary condition $\zeta \in \{0,1\}$ that induces a unique open 1-contour $\gamma$. Let $\hat{\Lambda}$ denote the region in $\mathbb{R}^2$ enclosed by $\partial \Lambda$ (i.e., the region enclosed by the $\mathbb{Z}^2$-edges connecting the boundary vertices of $\Lambda$). Assume for convenience that the start-point of $\gamma$ is $0^*$, and denote its end-point by $x$. Then $\gamma$ has the Ising polymer law $P_A^x$ defined with decoration weights $\hat{\Phi}(C; \gamma) := \phi(C; \gamma)$ and $\hat{\Phi}_A(C; \gamma) := \phi(C; \gamma)1_{\{C \subset \Lambda\}}$.

It will be very convenient to view Ising polymers as connected paths in the lattice $\mathbb{Z}^2$ rather than the dual lattice $(\mathbb{Z}^2)^*$ via the translation map $\iota : 0^* \to 0$, and this is the convention we will follow for the remainder of the article, excluding Section 3.

Let us now re-state Theorem 1.2 in the above language, which is the form in which we will prove it (Section 5.7). For $N \in \mathbb{N}$, define $Q : = [0,N]^2$.

**Theorem 2.8.** Fix $\beta > 0$ large, take $D$ to be either $\mathbb{H}$ or $Q$, and set $x : = (N,0)$. Consider an Ising polymer $\gamma \sim P_D^x(\cdot)$. There exists $\sigma > 0$ such that, if $\gamma(x) = \max\{y : (x,y) \in \gamma\}$, then $\gamma([xN])/\sigma \sqrt{N}$ converges weakly to a standard Brownian excursion in the Skorokhod space $(D[0,1], \| \cdot \|_\infty)$ and the same holds for $\gamma(x) = \min\{y : (x,y) \in \gamma\}$.

**Remark 2.9.** The variance $\sigma^2$ with $\sigma$ as in Theorem 2.8 (and by extension, the one in Theorem 1.2) is given above Theorem 4.3. It is related to the curvature of the Wulff shape (defined in Section 2.6) associated to the Ising polymer. See [29, Appendix B] for details.

### 2.4. Non-negative decoration functions and a product structure.

Following [31, Section 3.1], we employ a construction going back to [21] that allows us to consider decoration functions which are non-negative. In this discussion, we allow for the case $D = \mathbb{R}^2$, in which case $\Phi_D(C; \gamma) = \Phi(C; \gamma)$ and $q_D(\gamma) = q(\gamma)$.

For a contour $\gamma$, we consider the set of (not necessarily distinct) bonds in $\mathbb{Z}^2$ (recall we have applied the translation map $\iota : 0 \to 0^*$):

$$\nabla_\gamma : = \bigcup_{b = (y,y+e_i) \in \gamma} \{b, b + e_i, b - e_i\}.$$

To be clear, $b$ is a bond in $\gamma$, $e_i$ denotes a standard basis vector, and $b \pm e_i$ denotes the bond obtained by translating $b$ by $\pm e_i$. Thus, $\nabla_\gamma$ contains three bonds for each bond of $\gamma$. Define

$$\Phi_D'(C; \gamma) : = |C \cap \nabla_\gamma| e^{-\chi\beta(d(C)+1)} + \Phi_D(C; \gamma),$$

where $|C \cap \nabla_\gamma|$ is equal to the number of bonds, counted with multiplicity, in $\nabla_\gamma$ that $C$ intersects:

$$|C \cap \nabla_\gamma| = \sum_{b = (y,y+e_i) \in \gamma} \left(1_{\{b \in C \neq \emptyset\}} + 1_{\{b + e_i \cap C \neq \emptyset\}} + 1_{\{b - e_i \cap C \neq \emptyset\}}\right).$$

Observe that $\Phi_D' \geq 0$ by Eq. (2.5) and Property (M2). For any fixed bond $b \in \mathbb{Z}^2$, let $c(\beta)$ denote the value of

$$c(\beta) : = \sum_{C \subset \mathbb{Z}^2, C \ni b \neq \emptyset} e^{-\chi\beta(d(C)+1)}.$$

Note that

$$c(\beta) = \sum_{m \geq 1} e^{-\chi\beta(m+1)} \left|\{ C \subset \mathbb{Z}^2 : \gamma \ni b, d(C) = m\}\right| \leq e^{-\chi\beta}, \quad (2.11)$$

where we used that $\left|\{ C \subset (\mathbb{Z}^2)^* : \gamma \ni b, d(C) = m\}\right| \leq e^{cm}$ for some $c > 0$. (To see this, replace the marked $b \in C$ by a marked boundary edge $e_0 \in \partial C$ with the same $y$-coordinate (say) at the cost
of a factor of $m$; then, when enumerating the smallest connected set of edges of $\mathbb{Z}^2$ containing the marked $e_0$ and specified edges $\partial C$, regard $\partial C$ as the vertices of a 6-regular graph (whose vertices are the $\mathbb{Z}^2$ bonds and two are adjacent if they share an end-point; that is, the line graph of $\mathbb{Z}^2$), and recall that the number of $m$-vertex connected subgraphs of a graph with maximum degree $\Delta$ is at most $(e\Delta)^m$.) Using the fact that $C \cap \Delta_\gamma \neq \emptyset$ implies $C \cap \nabla_\gamma \neq \emptyset$, we have

$$
\sum_{C \cap \Delta_\gamma \neq \emptyset} \Phi_D(C; \gamma) = -3c(\beta)|\gamma| + \sum_{C \cap \nabla_\gamma \neq \emptyset} \Phi'_D(C; \gamma),
$$

Thus, we have

$$
q_D(\gamma) = \exp \left( - (\beta + 3c(\beta))|\gamma| + \sum_{C \cap \nabla_\gamma \neq \emptyset} \Phi'_D(C; \gamma) \right).
$$

Since $f(\beta) := \beta + 3c(\beta)$ is strictly increasing for all $\beta$ large enough, we will henceforth redefine $\beta = f(\beta)$ so that we may drop the laborious $3c(\beta)$ from our weights:

$$
q_D(\gamma) = \exp \left( - \beta|\gamma| + \sum_{C \cap \nabla_\gamma \neq \emptyset} \Phi'_D(C; \gamma) \right). \tag{2.12}
$$

A useful comparison to record at this stage is that, for any $D \subset \mathbb{R}^2$,

$$
\left| \log \frac{q_D(\gamma)}{q(\gamma)} \right| \leq 6e^{-\chi \beta}|\gamma|. \tag{2.13}
$$

Remark 2.10. For any domain $D \subset \mathbb{R}^2$, the non-negative decoration functions $\Phi'(C; \gamma)$ still satisfy Properties (P1) to (P4) above, and the modified non-negative decoration functions $\Phi'_D(C; \gamma)$ still satisfy Properties (M1) and (M2).

We next uncover the product structure of $q_D$. Defining $\Psi_D(C, \gamma) := (\exp(\Phi'_D(C; \gamma)) - 1)\mathbb{1}_{C \cap \nabla_\gamma \neq \emptyset}$, we may write

$$
\exp \left( \sum_{C \cap \nabla_\gamma \neq \emptyset} \Phi'_D(C; \gamma) \right) = \prod_{C \cap \nabla_\gamma \neq \emptyset} \left( (e^{\Phi'_D(C; \gamma)} - 1) + 1 \right) = \sum_{\mathcal{C} = \{C\}} \prod_i \Psi_D(C_i; \gamma),
$$

where the sum goes over all possible finite collections $\mathcal{C}$ of clusters. Given this, for any contour $\gamma$ and any collection $\mathcal{C}$ of clusters, we define the animal weight $q_D(\Gamma)$ of the animal $\Gamma = [\gamma, \mathcal{C}]$ by

$$
q_D(\Gamma) = q_D([\gamma, \mathcal{C}]) := e^{-\beta|\gamma|} \prod_{C \in \mathcal{C}} \Psi_D(C; \gamma), \tag{2.14}
$$

and observe

$$
q_D(\gamma) = \sum_{\Gamma = [\gamma, \mathcal{C}]} q_D(\Gamma).
$$

The above allows us to consider the free and modified Ising polymer measures as probability measures on animals:

$$
P^x(\cdot \mid \gamma \subset D) = \frac{G(x \mid \gamma \subset D, \Gamma \in \cdot)}{G(x \mid \gamma \subset D)} \quad \text{and} \quad P^x_D(\cdot) := \frac{G_D(x \mid \Gamma \in \cdot)}{G_D(x)}. \tag{2.15}
$$

Due to the product structure of $q_D(\Gamma)$, it is often convenient to consider animals rather than contours; indeed, we will see in Section 4.1 that the product structure begets a connection with a random walk, which is crucial to our analysis.

When $D$ is taken to be $\mathbb{R}^2$, we will omit $D$ from the notation laid out above.
2.5. Notation for Ising polymers and animals. Below, we set notation that will be used throughout the article in the context of Ising polymers and animals. Recall that we Ising polymers as connected paths in the lattice $\mathbb{Z}^2$ rather than the dual lattice $(\mathbb{Z}^2)^*$ via the translation map $\iota : \sigma^* \mapsto 0$, and this is the convention we will follow for the remainder of the article, excluding Section 3.

- We write $Q := [0, N]^2$.
- For a point $u \in \mathbb{Z}^2$, we will write $u_1$ and $u_2$ to denote the $x$-coordinate and $y$-coordinate of $u$ respectively.
- For a subset $E \subset \mathbb{Z}^2$, we’ll write $u + E$ to denote the translation of $E$ by the vector defined by $u$.
- For a contour $\gamma$, we write $|\gamma|$ to denote the number of bonds in $\gamma$. We write $(\gamma(0), \ldots, \gamma(|\gamma|))$ to denote the ordered vertices of $\gamma$.
- For an animal $\Gamma := [\gamma, \mathcal{C}]$, we write $|\Gamma| := |\gamma|$, and $X(\Gamma) := X(\gamma)$ to denote the displacement of $\gamma$; that is, the vector in $\mathbb{Z}^2$ given by the end-point of $\gamma$ minus the start-point of $\gamma$.
- For a pair of contours $\gamma := (\gamma(0), \ldots, \gamma(|\gamma|))$ and $\gamma' := (\gamma'(0), \ldots, \gamma'(|\gamma'|))$, we define their concatenation to be
  \[ \gamma \circ \gamma' := (\gamma(0), \ldots, \gamma(|\gamma|), X(\Gamma) + \gamma'(0), \ldots, X(\Gamma) + \gamma'(|\gamma'|)) \,.
\]
  Similarly, for a pair of animals $\Gamma := [\gamma, \mathcal{C}]$ and $\Gamma' := [\gamma', \mathcal{C}']$, we define their concatenation to be
  \[ \Gamma \circ \Gamma' := [\gamma \circ \gamma', \mathcal{C} \cup \mathcal{C}'] \,.
\]
- For $u, v \in \mathbb{Z}^2$, we write $\gamma : u \to v$ to indicate that $\gamma(0) = u$ and $\gamma(|\gamma|) = v$. We write $\Gamma : u \to v$ to indicate that $\Gamma = [\gamma, \mathcal{C}]$ for some contour $\gamma : u \to v$.
- For a set $D \subset \mathbb{R}$, we say $\Gamma = [\gamma, \mathcal{C}] \subset D$ if $\gamma$ as well as all clusters $\mathcal{C} \in \mathcal{C}$ are contained in $D$.
- For any $D \subset \mathbb{R}^2$, we define the set of contours
  \[ \mathcal{P}_D(u, v) := \{ \gamma : \gamma \subset D, \gamma : u \to v \} \,.
\]
  When $u = 0$, we will simply write $\mathcal{P}_D(v)$. We’ll write $\Gamma \in \mathcal{P}_D(u, v)$ for animals $\Gamma$ to mean $\Gamma = [\gamma, \mathcal{C}]$ for some contour $\gamma \in \mathcal{P}_D(u, v)$.

2.6. The Wulff Shape. Now, define the Wulff shape
  \[ \mathcal{W} := \bigcap_{y \in \mathbb{R}^2} \{ h \in \mathbb{R}^2 : h \cdot y \leq \tau_\beta(y) \} \,\]
  which is clearly closed and convex (as it is the intersection of half-spaces). Observe that
  \[ \mathcal{W} = \left\{ h \in \mathbb{R}^2 : \sum_{y \in \mathbb{Z}^2} e^{h \cdot y} g(y) < \infty \right\} \,.
\]
  Indeed, from Eq. (2.8), we have
  \[ \log g(y) = -\tau_\beta(y)(1 + o_{\|y\|_1}(1)) \quad (2.16) \,.
\]
  It follows that the sum in the second expression for $\mathcal{W}$ converges if and only if $h \cdot y < \tau_\beta(y)$ for all $y \in \mathbb{Z}^2$ large, which by homogeneity and continuity of $\tau_\beta$ is equivalent to the same holding for all $y \in \mathbb{R}^2$. Including the equality case $h \cdot y = \tau_\beta(y)$ in the first expression for $\mathcal{W}$ is equivalent to taking the closure of the set in the second expression.
2.7. Cone-points, the irreducible decomposition of animals, and weight factorization. Let us define the forward cone $\gamma^\triangleright := \{(x,y) \in \mathbb{Z}^2 : |y| \leq x\}$ and the backward cone $\gamma^\lhd := -\gamma^\triangleright$. We will also need $\gamma^\triangleright_\delta := \{(x,y) \in \mathbb{Z}^2 : |y| \leq \delta x\}$ for $\delta > 0$. Given a contour $\gamma$ and an animal $\Gamma = [\gamma, C]$, we say that $u \in \gamma$ is a cone-point for $\gamma$ if
\[ \gamma \subset u + \gamma^\triangleright \cup u + \gamma^\lhd, \]
and we say $u \in \gamma$ is a cone-point for $\Gamma$ if
\[ \Gamma \subset u + \gamma^\triangleright \cup u + \gamma^\lhd. \]
Recall from Section 2.5 that the previous display means that the forward and backward cones emanating from $u$ fully contain $\gamma$ as well as all clusters $C$ in $\mathcal{C}$. Of course if $u \in \gamma$ is a cone-point for $\Gamma$, then $u$ is a cone-point for $\gamma$ as well. Note that for any $u,v \in \mathbb{Z}^2$, $u \in v + \gamma^\triangleright$ if and only if $v \in u + \gamma^\triangleright$.

If $\Gamma$ has two cone-points $u$ and $v$, the contour part of $\Gamma$ between $u$ and $v$ as well as all associated clusters are entirely contained within the “diamond” $u + \gamma^\triangleright \cap v + \gamma^\lhd$. An animal is called left-irreducible if it contains no cone-points and if it is entirely contained in the backwards cone emanating from its end-point. Similarly, an animal is called right-irreducible if it contains no cone-points and it is contained in the forward cone emanating from its start-point. An animal is called irreducible if it is both left- and right-irreducible. We let $A_L, A_R$ and $A$ be the sets of left-irreducible, right-irreducible and irreducible animals, respectively, with start-point at the origin.

Consider now an animal $\Gamma$ with at least two cone-points, and decompose it into (left-/right-) irreducible animals as follows:
\[ \Gamma = \Gamma^{(L)} \circ \Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \circ \Gamma^{(R)}, \]
for some $n \geq 1$, $\Gamma^{(L)} \in A_L$, $\Gamma^{(R)} \in A_R$, and $\Gamma^{(i)} \in A$ for $i = 1, \ldots, n$. For $i \in \{L, 1, \ldots, n, R\}$, we will write
\[ \Gamma^{(i)} := [\gamma^{(i)}, C_i]. \]
From the definition of a cone-point, any cluster $C \in \mathcal{C}$ will be such that $C \cap \nabla_{\gamma^{(j)}} = \emptyset$ for $j \neq i$. Informally, this means that any cluster of $\Gamma^{(i)}$ will not be a cluster of $\Gamma^{(j)}$. Thus, the weights $q_D(\Gamma)$ “factorize” into a product of the weights of the irreducible pieces:
\[ q_D(\Gamma) = \prod_{i \in \{L, 1, \ldots, n, R\}} e^{-\beta \gamma^{(i)}} \prod_{C \in \mathcal{C}_i} \Psi_D(C; \gamma) = q_D(\Gamma^{(L)}) q_D(\Gamma^{(R)}) \prod_{i=1}^n q_D(\Gamma^{(i)}) \quad (2.18) \]
When $\gamma \subset Q$, note that the shape of the forward and backward cones necessitates
\[ \Gamma^{(1)} \circ \cdots \Gamma^{(n)} \subset [1, N - 1] \times (-\infty, N], \]
so that Eq. (2.18) becomes
\[ q_Q(\Gamma) = q_Q(\Gamma^{(L)}) q_Q(\Gamma^{(R)}) \prod_{i=1}^n q_{\Xi_i}(\Gamma^{(i)}). \quad (2.19) \]
We write $\text{Cpts}(\gamma)$ and $\text{Cpts}(\Gamma)$ to denote the set of cone-points of $\gamma$ and the set of cone-points of $\Gamma$, respectively. Let $n := n(\Gamma) = |\text{Cpts}(\Gamma)| - 1$ as above, so that $n$ is equal to the number of irreducible pieces of $\Gamma$. We label the cone-points of $\Gamma$ by $\zeta^{(1)}, \ldots, \zeta^{(n+1)}$, where the ordering is strictly increasing in the $x$-coordinates, i.e.,
\[ \zeta^{(i)}_{\lfloor} < \zeta^{(i+1)}_{\lfloor}. \]
The set $\text{Cpts}(\Gamma)$ therefore takes on a natural meaning as an ordered $n(\Gamma)$--tuple, and so we will abuse notation and write
\[ \text{Cpts}(\Gamma) = (\zeta^{(i)})_{i=1}^{n+1} = (X(\Gamma^{(L)}), X(\Gamma^{(L)}), X(\Gamma^{(1)}), \ldots, X(\Gamma^{(L)}) + \cdots + X(\Gamma^{(R)})). \quad (2.20) \]
The following result of [31] states that the free weight of contours in $\mathbb{Z}^2$ that have large length and contain a sub-linear number of cone-points compared to the distance of the end-point is exponentially small compared to $G$.

**Lemma 2.11** ([31, Eq. (4.5)]6. Fix any $\epsilon, \delta \in (0, 1)$. There exist $\beta_0 \in (0, \infty)$, $\nu_0 > 0$, $\delta_0 > 0$, and $c > 0$ such that the following bounds hold uniformly over $\beta \geq \beta_0$, $y \in Y_\delta^* \setminus \{0\}$, and $r \geq 1 + \epsilon$:

\[
G(y \mid |\gamma| \geq r \|y\|_1) \leq ce^{-r_0 \beta r \|y\|_1}G(y) \tag{2.21}
\]

\[
G(y \mid \text{Cpts}(\Gamma) < \delta_0 \|y\|_1) \leq ce^{-\nu \beta \|y\|_1}G(y) \tag{2.22}
\]

In [31]7, it is claimed without proof that the analogue of Lemma 2.11 also holds for animals. For completeness, we supply the result as well as a proof below.

**Proposition 2.12.** For any $\delta \in (0, 1)$, there exist $\nu > 0$ and $c > 0$ such that the following bounds hold uniformly over $\beta \geq \beta_0$, and $y \in Y_\delta^* \setminus \{0\}$:

\[
G(y \mid \text{Cpts}(\Gamma) < \delta_0 \|y\|_1) \leq ce^{-\nu \|y\|_1}G(y),
\]

where $\delta_0$ is as in Lemma 2.11.

**Proof.** The idea is to show that, for a typical $\Gamma = [\gamma, \mathcal{C}]$, many of the cone points for $\gamma$ are also cone points for $\Gamma$. We do this by showing that typically, $\mathcal{C}$ does not contain many clusters (more precisely, the total $d(\cdot)$-size is not too big). The result then follows from Eq. (2.22).

Our starting point is the following, which comes from Property (P2) and Remark 2.10 and states that for some constant $c_1 > 0$, for all $\beta > 0$ sufficiently large, and for any animal $[\gamma, \mathcal{C}]$,

\[
q([\gamma, \mathcal{C}]) \leq e^{-\beta |\gamma|} \exp \left(-c_1 \beta \sum_{\mathcal{C} \in \mathcal{C}} d(\mathcal{C}) \right). \tag{2.23}
\]

Now, define the events

\[
A = \{ \Gamma = [\gamma, \mathcal{C}] \mid \text{Cpts}(\Gamma) < \delta_0 \|y\|_1 \} \\
B = \{ \Gamma = [\gamma, \mathcal{C}] \mid \text{Cpts}(\gamma) \geq 2 \delta_0 \|y\|_1, |\gamma| \leq 1.1 \|y\|_1 \}
\]

Then, from Eqs. (2.21) and (2.22), we have

\[
\frac{G(y \mid A)}{G(y)} = \frac{G(y \mid A, B)}{G(y)} + O(e^{-\nu_0 \beta \|y\|_1}).
\]

Now, consider a contour $\gamma$ with at least $2 \delta_0 \|y\|_1$ cone-points and an animal $\Gamma$ such that $\Gamma = [\gamma, \mathcal{C}]$ for some set of clusters $\mathcal{C}$. Note that if $\Gamma$ has less than $\delta_0 \|y\|_1$ cone-points, then the clusters of $\mathcal{C}$ must intersect $\gamma$ in such a way that at least $\delta_0 \|y\|_1$ cone-points of $\gamma$ are not cone-points of $\Gamma$. This necessitates that the sum of $d(\mathcal{C})$ over $\mathcal{C} \in \mathcal{C}$ exceeds $\delta_0 \|y\|_1$. Hence, we can write

\[
G(y \mid A, B) \leq \sum_{\gamma; 0 < |\gamma| \leq 1.1 \|y\|_1} \sum_{m \geq \delta_0 \|y\|_1} \sum_{n=1}^{m} \binom{3 |\gamma|}{k} \sum_{n_1 + \cdots + n_k = n} \prod_{i=1}^{k} (G_i, \mathcal{C}_m) e^{-\beta |\gamma|} c_1 \beta^m,
\]

where: the second sum accounts for the possible values of $m = \sum_{\mathcal{C} \in \mathcal{C}} d(\mathcal{C})$; the third sum accounts for the possible values of $n = |\mathcal{C}|$ given $m$; the fourth sum accounts for the number of possible bonds of $\nabla_{\gamma}$ that the clusters of $\mathcal{C}$ intersect; the fifth sum accounts for the number of clusters that intersect a particular bond of $\nabla_{\gamma}$ (given $k$ and an arbitrary labeling of the bonds $b_1, \ldots, b_k$); the sixth sum accounts for the possible $d(\cdot)$-value of each of the $n$ clusters given

6In [31], $1 + \epsilon$ is replaced by an unspecified constant $r_0$; however, it is trivial to see that $r_0$ may be taken to be arbitrarily close to 1 in [31, Lemma 4], which is the same $r_0$ as in [31, Eq. 4.5]. Additionally, our condition $y \in Y_\delta^* \setminus \{0\}$ appears as $x \in \mathcal{Q}_+$ in [31].

7See the paragraph containing equation (4.9) in [31].
that their total \(d(\cdot)-\)sum is \(m\), and where \(m = (m_1, \ldots, m_n)\) is ordered such that \(m_1, \ldots, m_n\) represent the \(d(\cdot)-\)values of the \(n_1\) clusters intersecting \(b_1\), \(m_{n_1+1}, \ldots, m_{n_1+n_2}\) represent the \(d(\cdot)-\)values of the \(n_2\) clusters intersecting \(b_2\) and so on; each \(G_{i,n}\) denotes the set of all possible collections of the \(n_i\) clusters intersecting \(b_i\) given the aforementioned \(n_i\) and \(m_i\); and lastly, the exponential is the cumulative upper bound on \(q(\gamma, C)\) given by Eq. (2.23) (given that the cumulative \(d(\cdot)\) is \(m\)).

Now, similar to Eq. (2.11), the number of clusters \(C\) with \(d(C) = r\) and intersecting some fixed bond \(b\) is bounded from above by \(e^{c_2r}\), and therefore

\[
\prod_{i=1}^k |G_{i,n}| \leq e^{c_2(m_1 + \cdots + m_n)} = e^{c_2m}.
\]

Hence, \(G(y \mid A, B)\) is upper-bounded by

\[
\sum_{|\gamma| \leq 1.1 |y|} e^{-\beta|\gamma|} \sum_{m \geq 0} \sum_{n=1}^m \sum_{k=1} \left( \frac{3|\gamma|}{k} \right) \sum_{m=(m_1, \ldots, m_n) \atop n_1 + \cdots + n_k = n, m_{n_1} + \cdots + m_n = m} e^{-c_3m} e^{c_2m} \leq \sum_{|\gamma| \leq 1.1 |y|} e^{-\beta|\gamma|} \sum_{m \geq 0} \sum_{n=1}^m \left( \frac{3|\gamma|}{k} \right) \left( \frac{n-1}{k-1} \right) \left( \frac{m-1}{n-1} \right) \leq e^{-c_4\beta|y|_1} \sum_{|\gamma| \leq 1.1 |y|} e^{-\beta|\gamma|}.
\]

Finally, the non-negativity of the decorations \(\Phi'\) implies

\[
G(y) = \sum_{\Gamma \in \gamma, C} q(\gamma, C) = \sum_{\gamma, \beta \rightarrow y} \exp \left( -\beta|\gamma| + \sum_{\substack{C \subset A \\cap \Delta \neq \emptyset \\cap \Delta \neq \emptyset}} \Phi'(C; \gamma) \right) \geq \sum_{\gamma, \beta \rightarrow y} e^{-\beta|\gamma|},
\]

and thus

\[
\frac{G(y \mid A, B)}{G(y)} \leq e^{-c_5\beta|y|_1},
\]

concluding the proof.

**Remark 2.13.** Note that the proof of Proposition 2.12 is such that, for any Ising polymer model \(G_D\) or \(G(\cdot \mid \gamma \subset D)\) satisfying the analogous bounds in Lemma 2.11, Proposition 2.12 follows with \(G\) replaced by \(G_D\) or \(G(\cdot \mid \gamma \subset D)\). This is used when we prove the existence of cone points for \(\Gamma\) in these models in Lemma 5.7.

### 2.8. Ornstein–Zernike theory and its applications.

For \(h \in \mathbb{R}^2\), we define \(W^h(\cdot)\) by

\[
W^h(\Gamma) := e^{hx(\Gamma)} q(\Gamma)
\]

for any animal \(\Gamma\) (not necessarily irreducible). For \(y \neq 0\), define the dual parameter \(h_y\) by

\[
h_y = \nabla \tau_y(\gamma).
\]

Convexity and homogeneity of \(\tau_y(\gamma)\) imply that \(h_y \in \partial W\) (see the first definition of \(W\)).

Proposition 2.14 below, which forms the main result of the Ornstein–Zernike theory for the Ising polymer models, was stated in [31, Thm. 5] in a slightly different setting (namely, the cones are defined differently here), pointing to [33] as its relevant input. Indeed, one may infer this result via an Ornstein–Zernike analysis as in [33, Sec 3.3 and 3.4], but since (a) the models considered in that work do not include Ising polymers, and (b) our setting differs somewhat from that of [31], we include a full proof of this proposition in Appendix B.
Proposition 2.14. For any $\delta \in (0,1)$, there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ and for any $y \in \mathcal{Y}_\delta \setminus \{0\}$, the collection of weights $W^h_y$ defines a probability distribution on the set of irreducible animals. To emphasize that $W^h_y$ defines a probability distribution (on irreducible animals), and for consistency with [31], we use the notation

$$P^h_y(\Gamma) := e^{h_y X(\Gamma)} q(\Gamma) = W^h_y(\Gamma).$$

Let $E^h_y$ denote expectation under $P^h_y$. Then

$$E^h_y[X(\Gamma)] = \alpha y,$$

for some constant $\alpha := \alpha(\beta, y) > 0$—in particular, the expectation of $X(\Gamma)$ under $P^h_y$ is collinear to $y$. Finally, there exists a “mass-gap” constant $\nu_y > 0$ such that for all $\beta > 0$ large, $y \in \mathcal{Y}_\delta \setminus \{0\}$, and $k \geq 1$,

$$\sum_{\Gamma \in A_L \cup A_R} P^h_y(\Gamma) 1_{\{|\Gamma| \geq k\}} \leq C e^{-\nu_y \beta^k},$$

where $C := C(\beta) > 0$ is a positive constant.

Note that, since $h_y \cdot y = \tau_\beta(y)$, we have the following from Eq. (2.18)

$$q(\Gamma) = e^{-\tau_\beta(y) P^h_y(\Gamma^{(L)}) P^h_y(\Gamma^{(R)}) \prod_{i=1}^n P^h_y(\Gamma^{(i)})},$$

for any $\Gamma$ with at least two cone-points. For $u, v \in \mathbb{Z}^2$ and a set of animals $E$, introduce

$$\mathcal{A}(u, v; E) = \sum_{n \geq 1} \sum_{\Gamma^{(1)}, \ldots, \Gamma^{(n)} \subseteq A} \prod_{i=1}^n P^h_y(\Gamma^{(i)}) 1_{\{u+\gamma^{(1)}_0 \cdots \gamma^{(n)}_0 \in\mathcal{P}_y(u, v)\}} 1_{\{u+\Gamma^{(1)}_0 \cdots \Gamma^{(n)}_0 \in E\}}.$$ (2.28)

When $E$ is taken to be the set of all possible animals, we’ll simply write $\mathcal{A}(u, v; E) = \mathcal{A}(u, v)$. Then, for $D = Q$ or $\mathbb{H}$, Eq. (2.27) allows us to write

$$G(y \mid \gamma \subset D, |\text{Cpts}(\Gamma)| \geq 2) = e^{-\tau_\beta(y)} \sum_{\Gamma^{(L)} \subseteq A_L, \gamma^{(L)} \subset D} P^h_y(\Gamma^{(L)}) \sum_{\Gamma^{(R)} \subseteq A_R, \gamma^{(R)} \subset D} P^h_y(\Gamma^{(R)}) \mathcal{A}(X(\Gamma^{(L)}), y - X(\Gamma^{(R)})).$$ (2.29)

Note that the sum over $n$ in (2.28) above closely resembles the probability of a random walk event. This product structure of probability weights $P^h_y$ naturally leads to considering the random walk model related to the law of $\text{Cpts}(\Gamma)$ described in Section 4.1.

3. Proof of Theorem 1.1 modulo Theorem 1.2

In this section, we infer Theorem 1.1 from Theorem 1.2(a). Throughout the section, let $H_L = \lfloor \frac{1}{47} \log L \rfloor$. Recall that, as per Eq. (1.4), we wish to give a lower bound on the event

$$E := \left\{ \min_{x \in I_0} \rho(x) \geq \delta L^{1/3} \right\}.$$ 

The fact that the definition of $\rho(x)$ pertains to the height-$h^*$ (top) level line, where $h^*$ is random in $\{H_{L-1}, H_L\}$ (as mentioned in the Introduction, the results of [14] determine it w.h.p. for side lengths $L$ outside of a critical subsequence) make this event delicate to work with. We will derive the required lower bound on $E$ via more tractable events, tailored to the analysis of a local rectangle of side-length $\asymp L^{2/3}$ located at the center of the bottom side of $\Lambda_L$.

Let

$$R_0 := \left[ \frac{L}{2} - 3L^{2/3}, \frac{L}{2} + 3L^{2/3} \right] \times \left[ 0, 2L^{2/3} \right],$$

and
and let $R_1 \subset \Lambda_L$ be the set of $x \in \Lambda_L$ at distance at most $(\log L)^2 + 1$ from $R_0$. For fixed $\delta > 0$, define the horizontal line $\mathcal{L} := I_0 \times \{[\delta L^{1/3}]\}$. Theorem 1.1 will follow from the next proposition:

**Proposition 3.1.** Fix $\beta \geq \beta_0$. Let $B_n$ be the event that there exists a simple path in $R_0$ of length $\geq (\log L)^2$ that intersects the line $\mathcal{L}$, and whose sites have height $\geq H_L - n$. Then, for every fixed $n \geq 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\pi^0_{\Lambda_L}(B_n \cap C_n) < \epsilon$ for all $L$ large.

**Proof of Proposition 3.1.** Observe that

$$\pi^0_{\Lambda_L}(E^c, \mathcal{L}) = \pi^0_{\Lambda_L}(B_0 \cap C_0) + o(1).$$

Indeed, $E^c$ and [14, Theorem 2a] imply that the height-$h^*$ level loop $\mathcal{L}^*$ must encompass a path crossing $\mathcal{L}$ as in $B_0$, while $C_0$ holds w.h.p. (also by [14, Theorem 2a]). Similarly,

$$\pi^0_{\Lambda_L}(E^c, \mathcal{L}) = \pi^0_{\Lambda_L}(B_1 \cap C_1) + o(1),$$

again using that the height-$h^*$ level loop $\mathcal{L}^*$ must encompass a path crossing $\mathcal{L}$ as in $B_1$ and that $C_1$ holds w.h.p. when $h^* = H_L - 1$, by [14, Theorem 2a]. The conclusion now follows from Proposition 3.1. \hfill \Box

**Proof of Theorem 1.1.** Fix $n \geq 0$ and $\epsilon > 0$. The following procedure will reveal the “outermost” chain $\mathcal{C}$ of $\mathbb{Z}^2$-connected sites of height at most $H_L - n$ that encloses, together with the southern boundary of $\Lambda_L$, the box $R_0$. For each site $x$ in the north, east, and west boundaries of $R_1$, reveal its $\mathbb{Z}^2$-connected component of sites $y \in R_1$ for which $\varphi_y > H_L - n$. Let $U$ denote the collection of all revealed vertices, and let $\mathcal{C}$ be the exterior boundary of $U$ contained in $R_1$. By definition, $\varphi_y \leq H_L - n$ for all $y \in \mathcal{C}$. Now, let $W$ denote the collection of sites whose exterior boundary is formed by $\mathcal{C}$ and the southern boundary of $\Lambda_L$. Since we are on the event $C_n$, each connected component comprising $U$ has diameter strictly less than $(\log L)^2$. In particular, $R_0 \subset W$.

Now, condition on $\mathcal{C}$, and let $\xi(\mathcal{C})$ denote the boundary condition of $W$ given by the heights on $\mathcal{C}$ and the sites of height 0 on $\partial \Lambda$. From the domain Markov property, we may deduce our desired bound $\pi^0_{\Lambda_L}(B_n \cap C_n) < \epsilon$ if we can show

$$\pi^\xi_{\mathcal{C}}(B_n) < \epsilon,$$

uniformly over $\mathcal{C}$. Define $x_L := \lfloor \frac{L}{2} - 2L^{2/3} \rfloor$ and $x_R := x_L + [4L^{2/3}]$. Using that all heights on $\mathcal{C}$ are bounded by $H_L - n$, that all heights on $\partial \Lambda$ are 0 (in particular, bounded by $H_L - n - 1$), and that $B_n$ is an increasing event, we have

$$\pi^\xi_{\mathcal{C}}(B_n) \leq \pi^{\text{legs}}_{\mathcal{C}}(B_n),$$

where the boundary conditions $\text{legs}$ are $H_L - n = 1$ on the horizontal line of sites $[x_L, x_R] \times \{-1\}$ and $H_L - n$ elsewhere (see Fig. 2, noting the “legs” of sites of height $H_L - n$ protruding inwards from the bottom boundary of $W$). Thus, it suffices to show the right-hand is bounded by $\epsilon$.

Moving to $\pi^{\text{legs}}_{\mathcal{C}}$ has two main advantages. The first is that there exists a unique open $H_L - n$-contour $\gamma$ under this measure, and in particular, the sites of height $\geq H_L - n$ along $\gamma$ contain a simple path of length larger than $(\log L)^2$ (typically of order $L^{2/3}$). Further, a classical Peierls argument shows that, w.h.p. under $\pi^{\text{legs}}_{\mathcal{C}}$, there are no closed contours of length larger than $\log L$. In particular, defining

$$\rho^\gamma(x) := \min\{y \geq 0 : (x, y) \in \gamma\} \text{ for } x \in \mathbb{R},$$

we have

$$\pi^{\text{legs}}_{\mathcal{C}}(B_0) = \pi^{\text{legs}}_{\mathcal{C}}(\min_{x \in I_0} \rho^\gamma(x) < \delta L^{1/3}) + o(1).$$

(3.2)
The second advantage of moving to $\pi^\text{legs}_W$ is that we may couple $\gamma$ under the no-floor measure $\pi^\text{legs}_W$ with a modified Ising polymer law in the half-space $\mathbf{P}_H^\times$ (see Definition 2.6), for which we have Theorem 1.2. After exhibiting this in Claim 3.2, we compare $\pi^\text{legs}_W$ and $\pi^\text{legs}_W$ via Eq. (A.8).

We begin with the argument relating $\pi^\text{legs}_W$ to $\mathbf{P}_H^\times$. From now on, we shift $W$ to the left so that $x_L = 0$ and $x_R = \lfloor 4L^2/3 \rfloor$. Note that $\gamma$ is a contour in $W$ with start-point $o^* \in (\mathbb{Z}^2)^*$ and end-point $x := x_R - (1/2, 1/2)$. We will couple $\pi^\text{legs}_W$ with the Ising polymer law $\mathbf{P}_H^x$, defined using the SOS decoration weights $\Phi(C, \gamma) := \phi(C, \gamma)$ and $\Phi_H(C, \gamma) := \phi(C, \gamma)I_{\{C \subseteq \mathbb{H}\}}$ from Proposition 2.1. This coupling is natural in light of the very far distance of $C$ from both $o^*$ and $x$. Define $N := \lfloor 4L^2/3 \rfloor$. Recall the set $\mathcal{P}_D(u, v)$ from Section 2.5. Let $W$ denote the region in $\mathbb{R}^2$ enclosed by $W$, i.e., the region enclosed by the $\mathbb{Z}^2$-edges connecting the boundary vertices of $W$. Then $\gamma \in \mathcal{P}_W(o^*, x)$. Lower the boundary conditions of $\pi^\text{legs}_W$ to 0, 1, 1, 1 using the shift-invariance of the no-floor model. Then Observation 2.7 states that $\gamma \sim \pi^\text{legs}_W$ has the Ising polymer law $\mathbf{P}_W^x$, and the decoration weights are defined via $\Phi(C, \gamma) := \phi(C, \gamma)$ and $\Phi_H(C, \gamma) := \phi(C, \gamma)I_{\{C \subseteq \mathbb{H}\}}$.

**Claim 3.2.** Extend $\pi^\text{legs}_W$ to a probability measure on all contours in $\mathcal{P}_H(o^*, x)$ via $\pi^\text{legs}_W(\gamma) = 0$ for all $\gamma \notin \mathcal{P}_W(o^*, x)$. For some constant $C > 0$ and for all $\beta > 0$ large enough, we have

$$
\|\mathbf{P}_H^\times - \pi^\text{legs}_W\|_{TV} \leq C e^{-\beta C^{-1}N}.
$$

**Proof of Claim 3.2.** We use the identification of the law of the contour induced by $\pi^\text{legs}_W$ with $\mathbf{P}_W^x$ throughout this proof.

The main input for the proof is the following: for any $\delta \in (0, 1)$ and for all $\beta, N > 0$ large,

$$
\max \left( \frac{\pi^\text{legs}_W(|\gamma| \geq (1 + \delta)N)}{\mathbf{P}_W^x(|\gamma| \geq (1 + \delta)N)} \right) \leq e^{-\beta \delta N/2}. \tag{3.3}
$$

We prove Eq. (3.3) now. Using the decay of $\phi(C, \gamma)$ (Proposition 2.1(iv)) as in Eq. (2.11), we have the following for any $\gamma : o^* \to x$:

$$
\sum_{C \subseteq \Delta, \gamma \neq \emptyset} |\phi(C, \gamma)| \leq \sum_{y \in \Delta} \sum_{C \ni y} \sup_{\gamma} |\phi(C, \gamma)| \leq 3|\gamma| e^{-\beta_0 - \beta \gamma}. \tag{3.4}
$$

Since $|\gamma|$ is always larger than $N$, we have

$$
\min \left( G_H(x), G_W(x) \right) \geq \exp \left( - (\beta + 3e^{-(\beta_0 + \beta)}))N \right). \tag{3.5}
$$

**Figure 2.** The rectangles $R_0$ (purple) and $R_1$ (gray), and in between the region $W$ (blue) whose boundary conditions are raised via monotonicity to $H_L - n$ except on the segment between $x_L$ and $x_R$, where they are set to $H_L - n - 1$. 

\[ \text{Figure 2. The rectangles } R_0 \text{ (purple) and } R_1 \text{ (gray), and in between the region } W \text{ (blue) whose boundary conditions are raised via monotonicity to } H_L - n \text{ except on the segment between } x_L \text{ and } x_R, \text{ where they are set to } H_L - n - 1. \]
On the other hand, Eq. (3.4) implies
\[
\max \left( \mathcal{G}_H(x \mid |\gamma| \geq (1 + \delta)N), \mathcal{G}_W(x \mid |\gamma| \geq (1 + \delta)N) \right) \\
\leq \sum_{\gamma \in \mathcal{H}} \exp \left( - (\beta - 3e^{-(\beta - \beta_0)})|\gamma| \right) \leq e^{-(\beta - c)(1 + \delta)N},
\]
where \( c > 0 \) is a constant independent of \( \beta \) and \( N \). The above two bounds yield Eq. (3.3). Note that a consequence of (3.3) is
\[
\hat{\pi}^{\text{legs}}_W (d_{L \rightarrow \infty} (\gamma, \mathcal{C}) \leq (1 - 2\delta)N) \leq e^{-\beta \delta N/2}.
\]
(3.6)

Now, fix any \( \delta \in (0, 1/2) \). Observe that
\[
q_H(\gamma)/q_W(\gamma) = \exp \left( \sum_{\mathcal{C} \cap \Delta_x \neq \emptyset} \phi(\mathcal{C}; \gamma) 1_{\{\mathcal{C} \subset \mathcal{H} \}} \right)
\]
For any contour \( \gamma \) such that \( d_{L \rightarrow \infty} (\gamma, \mathcal{C}) > (1 - 2\delta)N \), we have the following for all \( \beta > 0 \) large:
\[
\sum_{\mathcal{C} \cap \Delta_x \neq \emptyset} |\phi(\mathcal{C}; \gamma)| 1_{\{\mathcal{C} \subset \mathcal{H} \}} \leq \sum_{\gamma \in \Delta, \mathcal{C} \subset \mathcal{H}} \sup_{\gamma} |\phi(\mathcal{C}; \gamma)| \leq |\gamma| e^{-c\beta N},
\]
where in the last inequality, \( c > 0 \) is a constant independent of \( N \) and \( \beta \) large, and the bound follows similarly to Eq. (3.4). For any \( |\gamma| < (1 + \delta)N \), Eq. (3.4) yields an upper bound on \( \mathcal{G}_H(x) \) of \( \exp(- (\beta - 3(1 + \delta)e^{-(\beta - \beta_0)})N) \). This upper bound, the lower-bound on \( \mathcal{G}_W(x) \) in (3.5), and the previous display imply the following for any \( |\gamma| < (1 + \delta)N \):
\[
|\hat{\pi}^{\text{legs}}_W (\gamma) - P^x_W (\gamma) | \leq e^{-c' \beta N},
\]
where \( c' > 0 \) is a constant independent of \( \beta \) and \( N \). Since the number of connected paths in \((\mathbb{Z}^2)^*\) rooted at \( \mathbf{o}^* \) of length at most \((1 + \delta)N\) is trivially bounded \( 4^{(1 + \delta)N} \), the previous display along with Eq. (3.3) imply that the total variation distance between \( \hat{\pi}^{\text{legs}}_W \) viewed as a probability measure on \( \mathcal{P}_H(\mathbf{o}^*, x) \) and \( P^x_H \) is bounded by \( e^{-c' \beta N} \).

Now, from Eq. (A.8), we have
\[
\hat{\pi}^{\text{legs}}_{R_0} (\gamma) \propto \hat{\pi}^{\text{legs}}_{R_0} (\gamma) \exp \left( - \frac{\lambda (n)}{L} A(\gamma) + o(1) \right),
\]
(3.7)
where \( A(\gamma) \) denotes the area under \( \gamma \) in \( W \). From Eq. (3.1) and Eq. (3.2), the claim will be proven if we can show that for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\pi^{\text{legs}}_W (\min_{x \in I_0} \rho^\gamma (x) < \delta L^{1/3}) < \epsilon
\]
(3.8)
for \( L \) sufficiently large. Theorem 1.1 and Claim 3.2 imply that, under horizontal rescaling by \( |x|_1 = 4L^{2/3} \) and vertical rescaling by \( 4L^{2/3} \) \( 1/2 \), \( \gamma \) under \( \hat{\pi}^{\text{legs}}_W \) converges weakly to a Brownian excursion as \( L \rightarrow \infty \). In particular, \( \hat{\pi}^{\text{legs}}_W (\min_{x \in I_0} \rho^\gamma (x) < \delta L^{1/3}) \) can be made arbitrarily close to 0 by taking \( \delta \) small enough. Letting \( \bar{E} \) denote expectation with respect to \( \hat{\pi}^{\text{legs}}_W \), we have
\[
\pi^{\text{legs}}_W (\min_{x \in I_0} \rho^\gamma (x) < \delta L^{1/3}) = \frac{\bar{E} \left[ 1_{\{\min_{x \in I_0} \rho^\gamma (x) < \delta L^{1/3} \}} e^{\frac{\lambda (n)}{L} A(\gamma) + o(1)} \right]}{\bar{E} \left[ e^{\frac{\lambda (n)}{L} A(\gamma) + o(1)} \right]}. \]
The denominator is bounded away from 0 because \( \lambda (n) \) is bounded for fixed \( \beta \) and the Brownian excursion limit implies that \( A(\gamma) \) is unlikely to be much bigger than \( L^{2/3} \cdot L^{1/3} = L \) (making the
expectation strictly positive). For the numerator, we apply Cauchy–Schwarz to get
\[ \mathbb{E}
\left[ e^{-\frac{A(\gamma) + o(1)}{L}} \mathbb{1}_{E^c} \right] \leq \sqrt{\mathbb{E}^\text{legs} \left( \min_{x \in I_0} \rho^\gamma(x) < \delta L^{1/3} \right)} \sqrt{\mathbb{E}
\left[ e^{-2\frac{A(H_L - h^{\gamma})}{L} A(\gamma) + o(1)} \right]}. \]
The first term can be made arbitrarily small by taking \( \delta \) small enough, as discussed above, while the second term may be bounded as \( 1 + o(1) \). This implies Eq. (3.8), thereby concluding the proof. \( \square \)

4. The effective random walk model, free Ising polymers, and limit theorems

The factorization of \( q(\Gamma) \) from (2.27) and the input from Ornstein-Zernike theory (Proposition 2.14) naturally leads one to consider the two-dimensional effective random walk defined in Section 4.1. In that subsection, we further expose the connection between this effective random walk and the cone-points of the free Ising polymer model in \( \mathbb{H} \). A key result there is Proposition 4.1, which enables a quantitative comparison between the partition function \( \mathcal{G}(x \mid \gamma \subset \mathbb{H}) \) and the probability that the random walk stays in \( \mathbb{H} \). In particular, we will be able to rule out bad events for the cone-points of the free Ising polymer in \( \mathbb{H} \) using random walk estimates. In the next Section, we develop comparison results between free and modified Ising polymers such that bad events for the cone-points of modified Ising polymers can also be ruled out using random walk estimates. Eventually, this will lead to a coupling between the cone-points of the modified Ising polymer and our random walk.

In the rest of this Section, we develop various limit theorems for our random walk that will be used in the analysis of (modified) Ising polymers in Section 5, culminating in the proof of Theorem 2.8 (and thus Theorem 1.2).

4.1. The random walk model and free Ising polymers.

Consider the random walk \( (S(i))_{i \in \mathbb{Z}_{\geq 0}} \), whose law will be denoted by \( \mathbb{P} \), with i.i.d. increments \( \{X_i\}_{i \geq 1} \) of step distribution \( X \), where for any \( \vec{v} \in \mathbb{Z}^2 \),
\[ \mathbb{P}(X = \vec{v}) = \sum_{\gamma \in A} \mathbb{P}^h(\Gamma) \mathbb{1} \{ \chi(\Gamma) = \vec{v} \}. \quad (4.1) \]
From Eq. (2.25), we have \( \mathbb{E}[X] = \alpha \), and from Eq. (2.26), we inherit exponential tail decay: for all \( \beta \) sufficiently large, there exists a constant \( C' := C'(\beta) > 0 \) such that for all \( k \geq 1 \), we have
\[ \sum_{\vec{v} \in \mathbb{Z}^2 : ||\vec{v}||_1 \geq k} \mathbb{P}(X = \vec{v}) \leq C' e^{-\nu \beta k}. \quad (4.2) \]
We remark that the step-distribution of \( X \) need not be symmetric in the \( y \)-coordinate \( X_2 \) due to the northeast splitting rule (this is the key difference between our random walk and the random walk considered in [30], and why we cannot use their random walk results here). Also note that \( \mathbb{P}(X \in \mathbb{Y}_x) = 1 \), and thus the \( x \)-coordinate \( X_1 > 0 \) a.s.

We will write \( H^A_u \) to denote the first hitting time of a set \( E \subset \mathbb{R}^2 \) by a stochastic process \( A \), omitting the \( A \) from the notation when the stochastic process is clear. For singleton sets, we will simply write \( H^A_u \) rather than \( H^A_u \). Let \( \mathbb{H} := \mathbb{R} \times \mathbb{R}_{<0} \). In light of the factorization (2.27), for fixed \( \Gamma^{(L)} \) and \( \Gamma^{(R)} \), we will be interested in our random walk \( S(\cdot) \) started from \( \Gamma^{(L)} \) and conditioned on the event
\[ \{H_{x\times X(\Gamma^{(R)})} < H_{\mathbb{H}^+} \}. \]
It will therefore be helpful to obtain an expression for \( \mathbb{P}_u(H_v < H_{\mathbb{H}^+}) \) in terms of animal weights, for any \( u, v \in \mathbb{Z}^2 \). Towards this end, define the sets
\[ V_{u,v}^+ := \{(\vec{v}_1, \ldots, \vec{v}_n) \in (\mathbb{Z}^2)^n : n \geq 1, u + \vec{v}_1 + \cdots + \vec{v}_m = v, u + \vec{v}_1 + \cdots + \vec{v}_l \in \mathbb{H} \text{ for all } 1 \leq i \leq n\} \]
\[ A_{u,v}^+ := \{(\Gamma^{(1)}, \ldots, \Gamma^{(n)}) \in A^n : n \geq 1, (X(\Gamma^{(1)}), \ldots, X(\Gamma^{(n)})) \in V_{u,v}^+\}, \quad (4.3) \]
and, for a set of animals $E$, introduce
\[
\mathcal{A}^+(u, v; E) = \sum_{n \geq 1} \sum_{(\Gamma^{(1)}, \ldots, \Gamma^{(n)}) \in \mathcal{A}^+_V} \prod_{i=1}^n \mathbb{P}^h(\Gamma^{(i)}_i) \mathbbm{1}_{\{\Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)} \in E\}}.
\]
When $E$ is taken to be the set of all possible animals, we’ll simply write $\mathcal{A}^+(u, v; E) = \mathcal{A}^+(u, v)$. Observe that
\[
\mathbb{P}_u(H_v < H_{\mathbb{H}_-}) = \sum_{n \geq 1} \sum_{(\vec{v}_1, \ldots, \vec{v}_n) \in \mathcal{V}^+_V} \prod_{i=1}^n \mathbb{P}(X = \vec{v}_i) = \mathcal{A}^+(u, v),
\]
(4.4)
In words, the above equation states that the probability of the random walk hitting $v$ before leaving $\mathbb{H}_-$ is equal to a sum over all possible products of weights of irreducible animals whose concatenation has all cone-points in $\mathbb{H}$.

Recall $\mathcal{A}(\cdot, \cdot ; \cdot)$ from (2.28), and note that for any $u, v \in \mathbb{Z}^2$ and any set of animals $E$,
\[
\mathcal{A}(u, v; E) \leq \mathcal{A}^+(u, v; E),
\]
(4.5)
since the (only) difference between $\mathcal{A}^+(u, v; E)$ and $\mathcal{A}(u, v; E)$ is that the sum defining $\mathcal{A}(u, v; E)$ restricts to tuples of irreducible animals such that the entire concatenation $u + \Gamma^{(1)} \circ \cdots \circ \Gamma^{(n)}$ is contained in $\mathbb{H}$, while the sum defining $\mathcal{A}^+(u, v; E)$ requires only the cone-points to stay in $\mathbb{H}$. In particular, $\text{Cpts}(\Gamma)$ (viewed as an ordered tuple, see Eq. (2.20)) under the conditioned law $\mathbb{P}^x(\cdot \mid \Gamma^{(L)}, \Gamma^{(R)}, \gamma \subset \mathbb{H})$ and the trajectory of the random walk $\mathbb{P}_{X(\Gamma^{(L)})}(\cdot \mid \mathbb{H}_{x - X(\Gamma^{(R)})} < \mathbb{H}_{\mathbb{H}_-})$ only differ due to these requirements.

Ultimately, we will achieve a coupling between $\text{Cpts}(\Gamma)$ with the trajectory of the random walk because of an entropic repulsion result (Proposition 5.11): the cone-points of $\Gamma$ in the “bulk” of the strip $[0, N] \times [0, \infty)$ stay far away from the boundary of $\mathbb{H}$ with high probability, so that the aforementioned difference between $\mathcal{A}^+$ and $\mathcal{A}$ becomes negligible. A major step towards showing entropic repulsion is the following crucial result from [31].

**Proposition 4.1** ([31, Theorem 7]). Recall $\nu_g$ from Proposition 2.14. There exists $\delta \in (0, \nu_g/4)$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$, there exists a constant $C := C(\beta) > 0$ such that
\[
\sup_{u, x, v \in \mathbb{H}} e^{-\beta(||u||_1 + ||x - v||_1)} \mathbb{P}_u(H_v < H_{\mathbb{H}_-}) \leq Ce^{-\beta} \mathcal{G}(x \mid \gamma \subset \mathbb{H}).
\]

Proposition 4.1 combined with the exponential tail decay of $\Gamma^{(L)}$ and $\Gamma^{(R)}$ (2.26) (note the significance of $\delta < \nu_g/4$) will allow us to eliminate bad events for $\text{Cpts}(\Gamma)$ under the free Ising polymer law via estimates for random walks in a half-space, which are significantly easier to obtain compared to directly analyzing the polymer law.

4.2. Limit theorems for our random walk, confined to the half-space. We now state two important limit theorems, Theorems 4.2 and 4.3, for our random walk $S(\cdot)$ confined to positive half-space, though our results apply to a more general class of two-dimensional random walks. Our random walk $S(\cdot)$ is on $\mathbb{Z}^2$ and has i.i.d. increments $X$ with mean $(\mu, 0)$ for some $\mu > 0$, exponential tails, and satisfies $X_1 > 0$ a.s. Let us also write $\text{Var}(X) = (\sigma_1^2, \sigma_2^2)$.

Let $V_1$ denote the unique positive harmonic function for the one-dimensional random walk $S_2$ killed upon leaving $(0, \infty)$ satisfying
\[
\lim_{a \to \infty} \frac{V_1(a)}{a} = 1.
\]
The uniqueness of $V_1$ was established by Doney in [22]. Similarly, let $V_1'$ denote the analogous harmonic function for the random walk $-S_2$.\footnote{The notation in [31] differs from ours: note that their quantity $\mathbb{P}^h(u, v)$ (defined in [31, Eq. 5.10]) is exactly our $\mathcal{A}(u, v)$. We have also used Eqs. (4.4) and (4.5) to replace $\mathcal{A}(u, v)$ by $\mathbb{P}_u(H_v < H_{\mathbb{H}_-})$.}
Theorem 4.2. Fix any $A > 0$ and any $\delta \in (0, 1/2)$. Uniformly over $k \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}]$ and $u, v \in (0, N^{1/2-\delta}] \cap \mathbb{N}$, we have the following results.

1. There exist constants $C > 0$ and $\kappa := \kappa(X) > 0$ such that
   \[
   \mathbb{P}_{(0,u)}(S(k) = (N, v), H_{\mathbb{H}_-} > k) \sim C\kappa \frac{V'_1(u)V'_2(v)}{k^2} \exp\left(-\frac{(N - k\mu)^2}{2k\sigma_1^2}\right). \tag{4.6}
   \]
   Furthermore, if
   \[
   p_{N,A} := \mathbb{P}_{(0,u)}(H_{(N,v)} \in \left[\frac{N}{\mu} - A\sqrt{N}, \frac{N}{\mu} + A\sqrt{N}\right] \mid H_{(N,v)} < H_{\mathbb{H}_-})
   \]
   then
   \[
   \lim_{A \to \infty} \lim_{N \to \infty} p_{N,A} = 1. \tag{4.7}
   \]

2. Let $\tilde{S}_2(\cdot)$ denote the denote the linear interpolation of the points $(n, S_2(n))_{n \in \mathbb{N}}$, viewed as an element of $C[0, \infty)$. The family of conditional laws
   \[
   Q_{u,v}^k(\cdot) := \mathbb{P}_{(0,u)}\left(\frac{\tilde{S}_2(tk)}{\sigma_2 \sqrt{k}} \in \cdot \mid S(k) = (N, v), H_{\mathbb{H}_-} > k\right)
   \]
   converges as $k \to \infty$ to the law of the standard Brownian excursion in $C[0, 1]$.\]

Next, we describe the diffusive limit of our random walk. For $\ell, n \in \mathbb{N}$ and any ordered $\ell$-tuple of points $S$ in $\mathbb{R}^2$, enumerated as
   \[S = ((s_1(1), s_2(1)), (s_1(2), s_2(2)), \ldots, (s_1(\ell), s_2(\ell)))\]
and satisfying $s_1(i) < s_1(i + 1)$ for each $i \in [1, \ell - 1]$, we define $\tilde{S}_n(S)$ to be the linear interpolation through the diffusively-rescaled points
   \[
   \tilde{S}_n(S) := \left(\frac{1}{n} s_1(i), \frac{1}{\sigma n} s_2(i)\right)_{i=1}^\ell \tag{4.8}
   \]
where $\sigma := \sigma_2^2/\mu$ (c.f. Remark 2.9). Consider now the linear interpolation corresponding to our random walk $\tilde{S}(\cdot)$ up to time $H_{(N,v)}$:
   \[\tilde{c}_{N,v}^S := \tilde{S}_N ((S(i))_{i=0}^{H_{(N,v)}}),\]
Viewing $\tilde{c}_{N,v}^S$ as a random element of $C[0, 1]$, Theorem 4.3 below gives a convergence result to the Brownian excursion on $[0, 1]$.

Theorem 4.3. Fix any $\delta \in (0, 1/2)$. Uniformly over $u, v \in (0, N^{1/2-\delta}] \cap \mathbb{N}$, we have the following. The family of conditional laws
   \[
   Q_{u,v}^N(\cdot) := \mathbb{P}_{(0,u)}\left(\left(\tilde{c}_{N,v}^S(t)\right)_{t \in [0,1]} \in \cdot \mid H_{(N,v)} < H_{\mathbb{H}_-}\right),
   \]
converges weakly as $N \to \infty$ to the law of the standard Brownian excursion in $(C[0, 1], \| \cdot \|_{\infty})$.

Theorems 4.2 and 4.3 are proved in Section 6.3. These results are modifications of a ballot-type theorem from [17] and an invariance principle from [23], respectively. Those important results were proved for a much broader class of random walks in very general cones; however, they were only proved for fixed start-points and end-points. The two results below have been modified to hold uniformly in appropriate ranges of start-points and end-points, which is crucial for our application. We remark that [30, Section 5] states and proves results that similarly modify the results of [17] and [23]. However, first, the random walk they consider is symmetric in the $y$-coordinate, which simplifies their analysis; and second, their proofs do not always make explicit the aforementioned uniformity, in particular for the $x$-coordinate of the end-point of their random walk. In Section 6, where our random walk estimates are proved, we take care to describe explicitly how we modify the proofs in [17] and [23] to handle uniformity in a broad range of start- and end-points.
5. Proof of Theorem 1.2 via random walk coupling

Throughout this Section, set \( x := x(N) = (N, 0) \in \mathbb{Z}^2 \). Recall the notation set forth in Section 2.5.

In this section we prove Theorem 2.8, thereby proving Theorem 1.2, via a certain coupling between \( \text{Cpts}(\Gamma) \) under the modified Ising polymer law \( P^D_Q \) and the effective random walk, where \( D = Q \) or \( \mathbb{H} \). This is accomplished as follows.

In Section 4.1, we described the particular role played by the free Ising polymer model in the half-space due to its direct connection with an effective random walk. It is then crucial for our analysis to be able to compare the modified Ising polymers we are interested in with free Ising polymers. In Section 5.1, we state our needed comparison results, Theorem 5.1 (the main theorem of [31]) and Proposition 5.2, the latter being an extension of the former that allows us to handle the \( D = Q \) case. The key consequence of these comparisons is that contour events of small probability in one model are still small in the other model (Corollary 5.3).

In Section 5.2, we prove Proposition 5.2, showing along the way that \( \text{Cpts}(\Gamma) \) and Proposition 5.1 in a similar way to Proposition 5.2.

In Section 5.3, we prove Proposition 2.5 in a similar way to Proposition 5.2. Comparing Ising polymers with modified and unmodified weights.

In Section 5.4, we show that each (left/right-)irreducible piece \( \Gamma^{(i)} \) has size bounded by \((\log N)^2\), \( i \in \{1, \ldots, n, R\} \), w.h.p. under \( P^D_Q \).

In Section 5.5, we show the key result enabling a coupling between \( \text{Cpts}(\Gamma) \) and the effective random walk, Proposition 5.11. This result, which is an expression of entropic repulsion, states that, with high probability, each cone point of \( \Gamma \) stays above height \( N^{\delta} \) in the large interval \([N^{\delta}, N - N^{\delta}]\), for any \( \delta \in (0, 1/4) \). This is achieved via Corollary 5.3, which reduces the result in the free polymer model; and Proposition 4.1, which will allow us to reduce the result to an entropic repulsion result for the random walk conditioned to stay in \( \mathbb{H} \).

The entropic repulsion, the shape of the cones \( \mathcal{Y}^\bullet \) and \( \mathcal{Y}^\circ \), and the bound \(|\Gamma^{(i)}| \leq (\log N)^2\) implies that the entire animal stays bounded away from \( \partial \mathbb{H} \), the boundary of the half-plane, in the strip \([N^{\delta}, N - N^{\delta}] \times [0, \infty)\). The first consequence of this is that the weight modifications play no role in this strip; that is, for each cluster \( C \) of \( \Gamma \) in this strip, we have \( \Phi_D(C; \gamma) = \Phi(C; \gamma) \). Thus, the portion of \( \Gamma \) contained in this strip behaves like a free Ising polymer, which can be related to the random walk. The second consequence is that the requirement that \( \gamma \) stays in \( D \) in this strip becomes trivial. In particular, following the discussion after Eq. (4.5), we will be able to couple the cone points of \( \Gamma \) in this strip with an effective random walk. This is achieved in Section 5.6.

Theorem 2.8 is finally proved in Section 5.7.

5.1. Comparing Ising polymers with modified and unmodified weights. We begin with the main theorem of [31], which states that the partition functions of the Ising polymers in \( \mathbb{H} \), with and without modifications, are equivalent up to a \( \beta \)-dependent constant.

**Theorem 5.1 ([31, Theorem 1]).** For all \( \beta > 0 \) large enough, we have

\[
\tau_\beta(x) = -\lim_{N \to \infty} \frac{1}{\|x\|} \log G_{\mathbb{H}}(x) .
\]  

In fact, there exist constants \( C_1 := C_1(\beta), C_2 := C_2(\beta) > 0 \) such that for all \( \beta \) large enough,

\[
C_1 G(x \mid \gamma \subset \mathbb{H}) \leq G_{\mathbb{H}}(x) \leq C_2 G(x \mid \gamma \subset \mathbb{H}) .
\]  

Though we only consider \( \mathbb{H} \) in this article, [31, Theorem 1] is stated more generally for half-spaces whose interior normal has argument lying in \([-\pi/4, 3\pi/4]\).

The next proposition can be seen as an extension of Theorem 5.1 to address polymers in \( Q \).
Proposition 5.2. For all $\beta > 0$ sufficiently large, there exists a constant $C := C(\beta) > 0$ such that for any $N \in \mathbb{N}$,
\[
C^{-1} \mathcal{G}(x \mid \gamma \subset \mathbb{H}) \leq \mathcal{G}(x \mid \gamma \subset Q) \leq C \mathcal{G}(x \mid \gamma \subset \mathbb{H}),
\]
and
\[
C^{-1} \mathcal{G}(x \mid \gamma \subset Q) \leq \mathcal{G}_Q(x) \leq C \mathcal{G}(x \mid \gamma \subset Q).
\]

The proof of Proposition 5.2 is given in Section 5.2.

One can readily imply from Eqs. (5.2) and (5.4) the following, relating $\mathbf{P}^A(\cdot \mid \gamma \subset D)$ and $\mathbf{P}^D_A$, exactly as [31] derived its analogue from Eq. (5.2) (see the discussion below Eq. (2.7) in that paper):

Corollary 5.3. Let $D = \mathbb{H}$ or $Q$. There exists a constant $C := C(\beta) > 0$ such that for any set of contours $A$ contained in $D$,
\[
\mathbf{P}^A_D(A) \leq C \sqrt{\mathbf{P}^A(\cdot \mid \gamma \subset D)} \quad \text{and} \quad \mathbf{P}^A(\cdot \mid \gamma \subset D) \leq C \sqrt{\mathbf{P}^D_A(A)}.
\]

For instance, if $Y_\gamma := \exp(\sum_{C \cap \nabla \neq \emptyset} \Phi'_D(C; \gamma) - \Phi'(C; \gamma))$ and $\mathbf{E}[\cdot]$ denotes expectation under $\mathbf{P}^\gamma(\cdot \mid \gamma \subset D)$, then $\mathbf{E}[e^{Y_\gamma}]$ is just $\tilde{\mathcal{G}}_D(x)/\mathcal{G}(x \mid \gamma \subset D)$, and therefore $\mathbf{P}^A_D(A) = \mathbf{E}[\mathbf{1}_A e^{Y_\gamma}]/\mathbf{E}[e^{Y_\gamma}]$. Notice $C \leq \mathbf{E}[e^{Y_\gamma}] \leq C$ by Eq. (5.2) (for $D = \mathbb{H}$) and Eq. (5.4) (for $D = Q$). By Cauchy–Schwarz, $\mathbf{P}^A_D(A) \leq C \sqrt{\mathbf{P}^A(\cdot \mid \gamma \subset D)} \sqrt{\mathbf{E}[e^{2Y_\gamma}]}$, and the last expectation is uniformly bounded (as $\mathbf{E}[e^{2Y_\gamma}] = \tilde{\mathcal{G}}_D(x)/\mathcal{G}(x \mid \gamma \subset D)$, where $\tilde{\mathcal{G}}_D(x)$ is defined w.r.t. $\mathbf{P}^D_\gamma(D, \gamma)$).

A particular consequence of Corollary 5.3 is that contour events that hold with probability tending to 1 in the unmodified models (which are much easier to study) still hold with probability tending to 1 in the modified models. The most significant application of this fact is in the proof of entropic repulsion under $\mathbf{P}^Q_\gamma(\cdot)$ (Proposition 5.11), which, as explained at the start of this Section, is a crucial step towards the random walk coupling achieved in Section 5.6.

Remark 5.4. In Corollary 5.3, we specified that the event $A$ should be a set of contours. The reason this was emphasized is that in Eq. (2.15) we extended the measure $\mathbf{P}_D$ from contours to animals, but Corollary 5.3 does not hold for general animal events $A$. Indeed, it is not true that small animal events in one model stay small in another model, as the ratio $|\mathcal{G}(\Gamma)/\mathcal{Q}(\Gamma)|$ is not bounded away from 0 nor $\infty$ (as there are no such bounds on $|\Phi'_Q(C; \gamma)/\Phi'(C; \gamma)|$ in the generality of weight modifications that we must consider).

5.2. Proof of Proposition 5.2. We will need two auxiliary lemmas. The first, Lemma 5.5, states that in any model where at least two cone-points exist, the partition function is dominated by animals whose first (left) and last (right) irreducible pieces have size of order 1.

Lemma 5.5. Let $D$ be either $Q$ or $\mathbb{H}$. For any $\epsilon \in (0, 1)$, there exists $K_\epsilon := K_{\epsilon, \beta} > 0$ such that
\[
\mathcal{G}_D(x \mid |\text{Cpts}(\Gamma)| \geq 2, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) > K_\epsilon) \leq \epsilon \mathcal{G}(x)
\]
and
\[
\mathcal{G}(x \mid \gamma \subset D, |\text{Cpts}(\Gamma)| \geq 2, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) > K_\epsilon) \leq \epsilon \mathcal{G}(x \mid \gamma \subset \mathbb{H}).
\]

Remark 5.6. Observe that once Proposition 5.2 is proved, the domain $\mathbb{H}$ in the right-hand side of Eqs. (5.5) and (5.6) may be replaced by $D$, for $D$ either $Q$ or $\mathbb{H}$.

The second auxiliary result, Lemma 5.7, extends Lemma 2.11, and thus Proposition 2.12 thanks to Remark 2.13, to the modified models on $Q$ and $\mathbb{H}$.

Lemma 5.7. Let $D$ be either $Q$ or $\mathbb{H}$. Fix any $\epsilon, \delta \in (0, 1)$. There exist positive constants $\beta_0, \delta_0$, and $\nu_0 > 0$ such that for all $\beta \geq \beta_0$, there exists $C := C(\beta) > 0$ such that the following hold.
In particular, there exists $\nu > 0$ such that for all $\beta \geq \beta_0$, there exists $C := C(\beta) > 0$ such that the following holds uniformly over all $N \in \mathbb{N}$:

$$G_D(x \mid Cpts(\Gamma)) < 2\delta_0 \|x\|_1 \leq Ce^{-\nu \beta \|x\|_1} G_D(x).$$  \hfill (5.9)

We postpone the proof of Lemma 5.5 to the end of the subsection. We prove Proposition 5.2 and Lemma 5.7 in tandem, proceeding as follows:

(a) We begin by proving Lemma 5.7 for $D = \mathbb{H}$.
(b) We then prove the first part of Proposition 5.2 (Eq. (5.3)), as well as its analog for $G_\mathbb{H}$, i.e.,

$$C^{-1}G_\mathbb{H}(x) \leq G_\mathbb{H}(x \mid \gamma \subset Q) \leq C G_\mathbb{H}(x).$$  \hfill (5.10)

(c) These results are then used to prove Lemma 5.7 for $D = Q$.
(d) Finally, we prove Eq. (5.4), thereby completing the proof of Proposition 5.2.

**Proof of Part (a).** Recall Lemma 2.11, which states that the analogous bounds to Eq. (5.7) and Eq. (5.8) hold for $G(x)$. Eq. (5.1) implies that $\log G(x)$ and $\log G(x \mid \gamma \subset \mathbb{H})$ have the same leading order, and thus, for some $\nu_0 \in (0, \nu_0]$, we have

$$G(x \mid \gamma \subset \mathbb{H}, \ |\gamma| \geq r \|x\|_1) \leq G(x \mid |\gamma| \geq r \|x\|_1) \leq Ce^{-\nu_0 \beta r \|x\|_1} G(x \mid \gamma \subset \mathbb{H}),$$  \hfill (5.11)

and

$$G(x \mid \gamma \subset \mathbb{H}, Cpts(\gamma)) < 2\delta_0 \|x\|_1 \leq G(x \mid Cpts(\gamma)) < 2\delta_0 \|x\|_1 \leq Ce^{-\nu_0 \beta \|x\|_1} G(x \mid \gamma \subset \mathbb{H}).$$

Eq. (5.7) and Eq. (5.8) for $D = \mathbb{H}$ now follow from Corollary 5.3, and (5.9) from Remark 2.13. \hfill \Box

**Proof of Part (b).** The upper-bounds in both Eqs. (5.3) and (5.10) hold with $C = 1$, since $Q \subset \mathbb{H}$. We now show the lower bound for Eq. (5.10). The lower bound in Eq. (5.3) is obtained simply by replacing $q_\mathbb{H}$ with $q$ in what follows. By Lemmas 5.5 and 5.7 (for $D = \mathbb{H}$), we have the following: for any $\epsilon \in (0, 1)$, there exists $K := K(\beta, \epsilon) > 0$ such that for $\|x\|_1$ large,

$$(1 - \epsilon)G_\mathbb{H}(x) \leq G_\mathbb{H}(x \mid \{Cpts(\Gamma) \mid \gamma \leq \gamma_0 \|x\|_1 \}, \max(|\Gamma(L)|, |\Gamma(R)|) \leq K),$$  \hfill (5.12)

Now, using the weight factorization of $q_\mathbb{H}(\Gamma)$ in Eq. (2.18), we may express the right-hand-side of the above (for $N = \|x\|_1 \geq 2/\delta_0$) as

$$\sum_{\Gamma(L) \in A_L, \Gamma(R) \in A_R} q_\mathbb{H}(\Gamma(L)) q_\mathbb{H}(\Gamma(R)) \mathbb{I}_{\{\|\Gamma(L)\|, \|\Gamma(R)\| \leq K\}} \times \sum_{n \geq \delta_0 \|x\|_1} \prod_{i=1}^{n} q_\mathbb{H}(\Gamma(i)) \mathbb{I}_{\{\Gamma^{(n)}(1) \cdots \Gamma^{(n)}(n) \in \mathcal{P}_\mathbb{H}(X(\Gamma(L)), x - X(\Gamma(R)))\}}.$$

For any $u \in \mathcal{Y}$, let $\gamma^{(L)}(u)$ denote an arbitrary up-right path from 0 to $u$, and let $\gamma^{(R)}(u)$ denote an arbitrary down-right path from $x - u$ to $x$ (if one exists). Since the backwards cone of $\Gamma^{(L)}$ must contain the origin and the forwards cone of $x - \Gamma^{(R)}$ must contain $x$, it follows that

$$\gamma^{(L)}(X(\Gamma(L))) \circ \gamma_1 \circ \cdots \circ \gamma_m \circ \gamma^{(R)}(X(\Gamma(R))) \subset Q.$$
for any collection \( \{ \Gamma^{(L)}, \Gamma^{(1)}, \ldots, \Gamma^{(m)}, \Gamma^{(R)} \} \) contributing to a nonzero term in the second-to-last display. In particular, for any \( \Gamma^{(L)} \in A_L \) and \( \Gamma^{(R)} \in A_R \) such that \( \{ |\Gamma^{(L)}|, |\Gamma^{(R)}| \leq K \} \), we have

\[
q_{\mathbb{H}}(\gamma^{(L)}(X(\Gamma^{(L)})))q_{\mathbb{H}}(\gamma^{(R)}(X(\Gamma^{(R)}))) \sum_{m \geq 1} \sum_{\Gamma^{(m)} \in A} \prod_{i=1}^{m} q_{\mathbb{H}}(\Gamma^{(i)}) \mathbb{I}_{\{ \Gamma^{(1)} \cap \cdots \cap \Gamma^{(m)} \in \mathcal{P}_{\mathbb{H}}(X(\Gamma^{(L)}), x - X(\Gamma^{(R)})) \}} \leq G_{\mathbb{H}}(x \mid \gamma \subset Q) \quad (5.12)
\]

Since \( q_{\mathbb{H}}(\Gamma^{(L)}) \leq q_{\mathbb{H}}(\gamma^{(L)}(X(\Gamma^{(L)}))) \) and \( q_{\mathbb{H}}(\Gamma^{(R)}) \leq q_{\mathbb{H}}(\gamma^{(R)}(X(\Gamma^{(R)}))) \), it follows from the last three displays that the right-hand side of Eq. (5.11) is bounded above by

\[
\left( \sum_{\Gamma^{(L)} \in A_L} \frac{q_{\mathbb{H}}(\Gamma^{(L)})}{q_{\mathbb{H}}(\gamma^{(L)}(X(\Gamma^{(L)})))} \mathbb{I}_{\{ |\Gamma^{(L)}| \leq K \}} \right) \left( \sum_{\Gamma^{(R)} \in A_R} \frac{q_{\mathbb{H}}(\Gamma^{(R)})}{q_{\mathbb{H}}(\gamma^{(R)}(X(\Gamma^{(R)})))} \mathbb{I}_{\{ |\Gamma^{(R)}| \leq K \}} \right) G_{\mathbb{H}}(x \mid \gamma \subset Q) \leq \left( \sum_{u \in \mathbb{Z}^2 : ||u||_1 \leq K} e^{\beta ||u||_1} G_{\mathbb{H}}(u) \right)^2 G_{\mathbb{H}}(x \mid \gamma \subset Q) .
\]

The pre-factor in the last display is a constant depending only \( K := K(\beta, \epsilon) \), finishing the proof. \( \square \)

**Proof of Part (c).** In Part (a), we proved Eqs. (5.7) and (5.8) for \( D = \mathbb{H} \). Bounding below the left-hand sides of these equations by adding the restriction \( \gamma \subset Q \), and giving an upper bound on the right-hand sides by using Eq. (5.10) to replace \( G_{\mathbb{H}}(x) \) by \( CG_{\mathbb{H}}(x \mid \gamma \subset Q) \) yields the same results for \( G_{\mathbb{H}}(x \mid \gamma \subset Q) \):

\[
G_{\mathbb{H}}(x \mid \gamma \subset Q, |\gamma| \geq 1.1 ||x||_1) \leq Ce^{-\nu_0 ||x||_1} G_{\mathbb{H}}(x \mid \gamma \subset Q) \quad (5.13)
\]

\[
G_{\mathbb{H}}(x \mid \gamma \subset Q, \text{Cpts}(\gamma) < 2\delta_0 ||x||_1) \leq Ce^{-\nu_0 ||x||_1} G_{\mathbb{H}}(x \mid \gamma \subset Q) \quad (5.14)
\]

uniformly in \( \beta \) large enough and \( x \). Similar to Eq. (2.13), we have

\[
\left| \log \frac{q_{\mathbb{H}}(\gamma)}{q_{\mathbb{H}}(\gamma')} \right| \leq 6e^{-\chi |\gamma|} . \quad (5.15)
\]

Eq. (5.7) and Eq. (5.8) for \( D = Q \) are then simple consequences of Eqs. (5.13) to (5.15). Once again, Eq. (5.9) for \( D = Q \) follows by Remark 2.13. \( \square \)

**Proof of Part (d).** Let us start by proving the upper bound in Eq. (5.4). As in Eq. (5.11), Lemmas 5.5 and 5.7 (now for \( D = Q \)) yield the following for any \( \epsilon \in (0, 1) \), some \( K := K(\beta, \epsilon) > 0 \), and all \( ||x||_1 \) large enough:

\[
G_{\mathbb{H}}(x) \leq \epsilon G_{\mathbb{H}}(x) + G_{\mathbb{H}}(x \mid |\text{Cpts}(\Gamma)| \geq \delta_0 ||x||_1 , \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) \leq K) . \quad (5.16)
\]

The first term on the right-hand side, \( \epsilon G_{\mathbb{H}}(x) \), may be bounded by \( C_0(\beta) G(x \mid \gamma \subset Q) \) via Eqs. (5.2) and (5.3). We now follow the proof of Part (b) to bound the second term: using the factorization Eq. (2.19) of \( q_{\mathbb{H}}(\Gamma) \), expanding the right-hand side as in Part (b), and replacing \( \Gamma^{(L)} \) and \( \Gamma^{(R)} \) by cluster-less up-right paths \( \gamma^{(L)}(X(\Gamma^{(L)})) \) and \( \gamma^{(R)}(X(\Gamma^{(R)})) \), we obtain an upper bound by \( C_1 G_{\mathbb{H}}(x \mid \gamma \subset Q) \), which by Eqs. (5.2) and (5.3) is bounded above by \( C_1 G(x \mid \gamma \subset Q) \).

For the lower bound in Eq. (5.4), start instead from \( G(x \mid \gamma \subset Q) \). Bound it from above by \( C_2 G(x \mid \gamma \subset \mathbb{H}) \) using Eq. (5.3), and bound the latter by \( C_2 G_{\mathbb{H}}(x) \) using Eq. (5.2). Now, use the inequality in Eq. (5.11), expand the right-hand side exactly as in the proceeding display, and again replace \( \Gamma^{(L)} \) and \( \Gamma^{(R)} \) by cluster-less, up-right paths. Observe that the bound in Eq. (5.12) holds with the right-hand side replaced with \( G_{\mathbb{H}}(x) \), since all animals on the left-hand side are contained in \( Q \) (and therefore the \( q_{\mathbb{H}} \)-weights of these animals are equal to the \( q_{\mathbb{H}} \)-weights). Following the remainder of the proof of Part (b), we obtain the lower bound in Eq. (5.4). \( \square \)
Proof of Lemma 5.5. For any $u,v \in \mathbb{Z}^2$ and for any set of animals $E$, define the partition functions

$$G_D(u \to v) := \sum_{\gamma : u \to v | \gamma \subset D} q_D(\gamma) \quad \text{and} \quad G_D(u \to v | E) := \sum_{\Gamma \in \mathcal{P}_D(u,v) \cap E} q_D(\Gamma).$$

We begin with three estimates pertaining to $G_D(u \to v)$.

First, we have the analogue of Eq. (2.26) for the modified animal weights: for all $\beta > 0$ sufficiently large, there exists a constant $c' := c'(\beta) > 0$ such that for all $k, N \geq 1$,

$$\sum_{\Gamma \in A_L \cup A_R} e^{h_\epsilon X(\Gamma)} q_D(\Gamma) 1_{\{|\Gamma| \geq k\}} \leq c' e^{-\nu_g \beta k}. \quad (5.17)$$

This is proved at the end of Appendix B.

Next, below Theorem 3 of [31], it is noted that Eq. (5.2) in the current article actually holds if the end-points of the contour are not on the line $\partial \mathbb{H}$; that is, for any $u,v \in \mathbb{H}$, we have

$$C_1(\beta)G(u \to v \mid \gamma \subset \mathbb{H}) \leq G_\mathbb{H}(u \to v) \leq C_2(\beta)G(u \to v \mid \gamma \subset \mathbb{H}). \quad (5.18)$$

Similarly, in Sections 4.1 and 4.2 of [31], it is shown that contours with linear size and many cone-points dominate $G(x \mid \gamma \subset \mathbb{H})$, and the arguments there yield that the same is true when the end-points of the contour are not on $\partial \mathbb{H}$; that is, for any fixed $\epsilon \in (0,1)$, there exist constants $\delta_0, \nu, \epsilon > 0$ such that the following bounds hold uniformly over $\beta > 0$ sufficiently large, over $u, v \in \mathbb{H}$ satisfying $v \in u + \mathcal{Y}_\delta \setminus \{0\}$, and over $r \geq 1 + \epsilon$:

$$G(u \to v \mid \gamma \subset \mathbb{H}, |\gamma| \geq r \|v - u\|_1) \leq c e^{-\nu \|v - u\|_1} G(u \to v \mid \gamma \subset \mathbb{H}).$$

and

$$G(u \to v \mid \gamma \subset \mathbb{H}, |\text{Cpts}(\gamma)| < 2\delta_0 \|v - u\|_1) \leq c e^{-\nu \|v - u\|_1} G(u \to v \mid \gamma \subset \mathbb{H}).$$

As a consequence of Remark 2.13, we have many cone-points for animals $\Gamma$ as well:

$$G(u \to v \mid \gamma \subset \mathbb{H}, |\text{Cpts}(\Gamma)| < \delta_0 \|v - u\|_1) \leq c e^{-\nu \|v - u\|_1} G(u \to v \mid \gamma \subset \mathbb{H}). \quad (5.19)$$

Now, towards Eq. (5.5), we only show $G_D(x \mid |\text{Cpts}(\Gamma)| \geq 2, |\Gamma^{(L)}| > K_\epsilon) \leq \epsilon G_D(x)$, as the analogous bound with $\Gamma^{(R)}$ instead of $\Gamma^{(L)}$ follows via the same argument. We begin by using Eqs. (2.18) and (2.19) to obtain the expansion

$$G_D(x \mid |\text{Cpts}(\Gamma)| \geq 2, |\Gamma^{(L)}| > K_\epsilon)$$

$$= \sum_{u \in \mathcal{Y}} \sum_{x - v \in \mathcal{Y}} q_D(\Gamma^{(L)}) 1_{|\Gamma^{(L)}| > K_\epsilon} \sum_{\Gamma^{(R)} \in \mathcal{A}_L} \sum_{\Gamma^{(R)} \in \mathcal{P}_D(u,x)} \prod_{i=1}^{n} q_D(\Gamma^{(i)}). \quad (5.20)$$

The above double sum over $n$ and irreducible animals is bounded by $G_\mathbb{H}(u \to v)$, which by Eq. (5.18) is bounded by $C_2(\beta)G(u \to v \mid \gamma \subset \mathbb{H})$. It then suffices to show

$$G(u \to v \mid \gamma \subset \mathbb{H}) \leq C e^{h_\epsilon (u + x - v) + \tilde{\delta} \|v\|_1 + \|x - v\|_1} G(x \mid \gamma \subset \mathbb{H}). \quad (5.21)$$

where $C := C(\beta) > 0$ is a constant and $\tilde{\delta}$ is as in Proposition 4.1. Indeed, substituting the bound in Eq. (5.21) into Eq. (5.20), using the exponential tail from Eq. (5.17), and recalling $\tilde{\delta} < \nu_g / 4$ yields

$$G_D(x \mid |\text{Cpts}(\Gamma)| \geq 2, |\Gamma^{(L)}| > K_\epsilon) \leq C' e^{-\frac{\nu_g}{2} \beta K_\epsilon} G(x \mid \gamma \subset \mathbb{H}).$$

By Theorem 5.1, the right-hand side above is bounded by $c G_\mathbb{H}(x)$ for $K_\epsilon := K_{\epsilon, \beta} > 0$ large enough.
To show Eq. (5.21), we use the existence of cone-points Eq. (5.19) (for which we write a $1 + o(1)$, where the $o(1)$ term vanishes as $\|v - u\|_1$ tends to infinity) and the weight factorization in Eq. (2.27):
\[
G(u \to v \mid \gamma \subset \mathbb{H})
\leq (1 + o(1)) \sum_{u' \in u + \mathcal{Y}^d} \sum_{x - v - v' \in \mathcal{Y}^d} q(\Gamma(L)) q(\Gamma(R)) \sum_{n \geq 1} \prod_{i=1}^{n} \prod_{\gamma(\alpha)} \prod_{v \in v' \in \mathcal{A}} \prod_{\gamma(\alpha) \in \mathcal{A}} \prod_{i=1}^{n} q(\Gamma(i)) I_{\Gamma(\alpha)}(u', v')
\]
\[
= (1 + o(1)) e^{-h_x(u - v)} \sum_{u' \in u + \mathcal{Y}^d} \sum_{x - v - v' \in \mathcal{Y}^d} \mathbb{P}_{h_x}(\Gamma(L)) e^{-\beta \|v'\|_1} \mathbb{P}_{h_x}(\Gamma(R)) A(u', v')
\]
\[
\leq (1 + o(1)) e^{-h_x(u + x - v)} G(x \mid \gamma \subset \mathbb{H})
\times \left( \sum_{u' \in u + \mathcal{Y}^d} \sum_{x - v - v' \in \mathcal{Y}^d} \mathbb{P}_{h_x}(\Gamma(L)) e^{-\tilde{h} \|v'\|_1} \mathbb{P}_{h_x}(\Gamma(R)) e^{-\tilde{h} \|x - v\|_1} \right)
\times \left( \sum_{\Gamma(L) \in \mathcal{A}_L} \mathbb{P}_{h_x}(\Gamma(L)) e^{-\tilde{h} \|\gamma(\alpha)\|_1} \right)
\]
\[
\leq (1 + o(1)) e^{-h_x(u + x - v) + \tilde{h} \|\gamma\|_1} G(x \mid \gamma \subset \mathbb{H})
\times \left( \sum_{\Gamma(L) \in \mathcal{A}_L} \mathbb{P}_{h_x}(\Gamma(L)) e^{-\tilde{h} \|\gamma(\alpha)\|_1} \right)
\]
\[
In the second-to-last line we used Eqs. (4.4) and (4.5) and then Proposition 4.1 to bound the $A(u', v')$ term, as well as $h_x \cdot \gamma = \tau_\beta(x)$. In the third-line, we used the triangle inequality. Equation (5.21) then follows from the exponential tails of $\Gamma(L)$ and $\Gamma(R)$ in Eq. (2.26), again recalling $\tau < \tau_G/4$. This yields Eq. (5.5). The proof of Eq. (5.6) is identical: just replace the left-hand side of Eq. (5.20) by $\mathbb{G}(x \mid |\text{Cpts}(\Gamma)| \geq 2, |\Gamma(L)| > K, \gamma \subset D)$, and then replace all modified weights $q_D$ by $q$. 

5.3. Proof of Proposition 2.5. Fix a direction $\vec{u} \in S^{d-1}$, and let $t_N^N(\vec{u}) := (N, y_N) \in \mathbb{Z}^2$ denote the lattice point whose $x$-coordinate is equal to $N$ and is on or below the line span(\vec{u}). Let $\ast := (1/2, 1/2)$ be the origin of the dual lattice $(\mathbb{Z}^2)^*$, and define $x_N(\vec{u}) := t_N^N(\vec{u}) + (-1/2, 1/2) \in (\mathbb{Z}^2)^*$. Then we have the formula
\[
\tau_\beta^{\text{SOS}}(\vec{u}) = \lim_{N \to \infty} \lim_{M \to \infty} -\frac{1}{\beta \|x_N(\vec{u})\|} \log \mathbb{G}_{\mathcal{I}_{N,M}}(x_N(\vec{u}))
\]
where $\mathcal{I}_{N,M}$ denotes the strip $[0, N] \times [-M, M]$. [20, Theorem 4.16] implies that the surface tension $\tau_\beta$ is equal to the surface tension of the model in $\mathcal{I}_{N,M}$ with the same (free) weights $q(\gamma)$, that is:
\[
\tau_\beta(\vec{u}) = \lim_{N \to \infty} \lim_{M \to \infty} -\frac{1}{\beta \|x_N(\vec{u})\|} \log \mathbb{G}(x_N(\vec{u}) \mid \gamma \subset \mathcal{I}_{N,M}).
\]
Thus, the proof will be finished if we can compare to exponential order the partition function with modified weights $\mathbb{G}_{\mathcal{I}_{N,M}}(x_N(\vec{u}))$ and the partition function with free weights $\mathbb{G}(x_N(\vec{u}) \mid \gamma \subset \mathcal{I}_{N,M})$. The proof of this comparison is extremely similar to the proof of Eq. (5.4) above, though much simpler, and so it is omitted for brevity’s sake. 

\[\text{It is well-known that this result of [20] contains a mistake, corrected in the appendix of [31]. However, the mistake has to do with a part of the result that is irrelevant to us, namely, a part that claims an unchanged surface tension even after modification of the decoration functions $\Phi(C; \Gamma)$. In our application of [20, Theorem 4.16], we are not making any modifications to the decoration functions.}\]

\[\text{A crucial input for the proof of Eq. (5.4) was Proposition 2.12, which we proved using Lemma 2.11, cited from the paper of [31]. Note [31] was written for Ising polymers, so in particular, Property (P4) of the surface tension is always assumed in that paper. However, we use Proposition 2.5 in order to prove Property (P4) of $\tau_\beta$, so this might appear to be circular reasoning. However, Lemma 2.11 is proved in [31] (Eq. (4.5) there) without assuming Property (P4), so there is no issue in using it. No other results from [31] are needed for the proof of Proposition 2.5.}\]
5.4. Boundedness of the irreducible pieces. The main result of this subsection is Proposition 5.8, which states that each irreducible piece of $\Gamma$ has size bounded by $(\log N)^2$ with high $\mathbb{P}_D(\cdot)$-probability.

When $\Gamma$ has at least two cone-points, recall from Eq. (2.17) that we write $\Gamma^{(1)}, \ldots, \Gamma^{(n)}$ to denote the irreducible components of $\Gamma$, where $n := |\text{Cpts}(\Gamma)| - 1$.

**Proposition 5.8.** For $D = Q$ or $D = \mathbb{H}$,
\[
\pi_D(\{|\text{Cpts}(\Gamma)| \geq 2\}, \{\exists i \in \{L, 1, \ldots, |\text{Cpts}(\Gamma)| - 1, R\} : |\Gamma^{(i)}| \geq (\log N)^2\}) = o(1).
\]

We will need the following two results on the asymptotic size of the partition functions of interest. Recall the notation for asymptotic relations set out at the start of Section 2.

**Lemma 5.9.** There exists some $C := C(\beta) > 0$
\[
\mathcal{G}(x) \leq Ce^{-\tau_\beta(x)}, \tag{5.22}
\]

**Proof.** Similar to Eq. (2.29), we have
\[
e^{-\beta(x)}\mathcal{G}(x \mid |\text{Cpts}(\Gamma)| \geq 2) = \sum_{\Gamma^{(L)} \in A_L} \sum_{\Gamma^{(R)} \in A_R} \mathbb{P}^h_x(\Gamma^{(L)})\mathbb{P}^h_x(\Gamma^{(R)})\mathbb{P}_{x^{(\Gamma^{(L)})}}(H_{x-x^{(\Gamma^{(R))}}) < \infty}) \leq \left( \sum_{\Gamma^{(L)} \in A_L} \mathbb{P}^h_x(\Gamma^{(L)}) \right) \left( \sum_{\Gamma^{(R)} \in A_R} \mathbb{P}^h_x(\Gamma^{(R)}) \right).
\]

The above summations are finite by Claim B.1. The Lemma then follows from Eq. (2.22). \(\square\)

We remark that a precise first-order asymptotic for $\mathcal{G}(x)$ is given in [20, Equation 4.12.3]. The asymptotic can be proved by showing $\mathbb{P}_{x^{(\Gamma^{(L)})}}(H_{x-x^{(\Gamma^{(R))}}) < \infty})$ is of order $|x|^{-1/2}$ via random walk estimates, similar to the proof of Theorem 4.2.

**Lemma 5.10.** We have
\[
\mathcal{G}(x \mid \gamma \subset \mathbb{H}) \gtrsim e^{-\tau_\beta(x)}N^{-3/2}, \tag{5.23}
\]

where the implied constant depends on $\beta$.

**Proof.** Item (1) of Theorem 4.2 shows that for any positive, fixed $u$ and $v$ in $\mathbb{N}$ (independent of $N$), we have
\[
\mathbb{P}(0,u)(H_{(N,v}) < H_{(1)}) \sim C_k\frac{V_1(u)V_1(v)}{N^{3/2}}. \tag{5.24}
\]

The lower-bound of Eq. (5.23) then follows immediately from Proposition 4.1 and Eq. (5.24). \(\square\)

It is not too hard to show a matching upper-bound, so that $\mathcal{G}(x \mid \gamma \subset \mathbb{H}) \asymp e^{-\tau_\beta(x)}N^{-3/2}$. Again, we do not pursue this here as the lower-bound suffices.

**Proof of Proposition 5.8.** Define the set of animals
\[
\mathcal{P}^{\text{cp,len}}_D(x, K) := \{\Gamma \in \mathcal{P}_D(x) : |\text{Cpts}(\Gamma)| \geq 2, \max(|\Gamma^{(L)}|, |\Gamma^{(R)}|) \leq K, \ |\Gamma| \leq 1.1 \|x\|_1\}. \tag{5.25}
\]

In Lemmas 5.5 and 5.7 and Remark 5.6, it was shown that this set of animals dominates $\mathcal{P}_D$, i.e.,
\[
\lim_{K \to \infty} \lim_{N \to \infty} \mathbb{P}_D(\mathcal{P}^{\text{cp,len}}_D(x, K)) = 1. \tag{5.26}
\]

Therefore, in light of Lemma 5.10, the proof of Proposition 5.8 will be complete upon showing
\[
\mathcal{G}_D(x \mid \mathcal{P}_D^{\text{cp,len}}(x, K), \{\exists i \leq |\text{Cpts}(\Gamma)| - 1 : |\Gamma^{(i)}| \geq (\log N)^2\}) = o(e^{-\tau_\beta(x)}N^{-3/2}) \tag{5.27}
\]

for any fixed $K$.

Define the sets of animals
\[
A_{L,K} := \{\Gamma \in A_L : |\Gamma| < K\} \quad \text{and} \quad A_{R,K} := \{\Gamma \in A_R : |\Gamma| < K\}. \tag{5.27}
\]
Using the factorization of weights (Eq. (2.18), and Eq. (2.19) for \( D = Q \), we have
\[
\mathcal{G}_D(x \mid \mathcal{P}_D^{cpl,\text{len}}(x, K), \{ \exists 1 \leq i \leq |\text{Cpts}(\Gamma)| : |\Gamma(i)| \geq (\log N)^2 \})
\]
\[
= \sum_{\Gamma(L) \in \mathcal{AL}_K, \gamma(L) \subset D} q_D(\Gamma(L)) \sum_{\Gamma(R) \in \mathcal{AR}_K, \gamma(R) \subset D} q_D(\Gamma(R)) \prod_{n \geq 1} g_{\mathbb{H}}(\Gamma(i)) \times 1_{1 \leq i \leq n : |\Gamma(i)| \geq (\log N)^2} \in \mathcal{P}_D(\mathcal{X}(\Gamma(L)), x - \mathcal{X}(\Gamma(R))) \prod_{1 \leq k \leq n : |\Gamma(k)| \geq (\log N)^2} 1 \times \bigcup_{\Gamma(1) \in \mathcal{A}} \bigcup_{\Gamma(n) \in \mathcal{A}} 1_{|\Gamma| \geq (\log N)^2}.
\]

For any \( a, b \in \mathbb{H} \), Eqs. (5.18) and (5.22) along with the trivial bound \( \mathcal{G}(a \rightarrow b \mid \gamma \subset \mathbb{H}) \leq \mathcal{G}(b - a) \) implies \( \mathcal{G}_D(a \rightarrow b) \leq C e^{\tau_\beta(b - a)} \) for some constant \( C := C(\beta) > 0 \). Define the weights \( W_{\mathbb{H}}^H(\Gamma) := e^{\gamma \mathcal{X}(\Gamma)_{\mathbb{H}}(\gamma)} \), under which \( \Gamma \) has exponential tails by Eq. (5.17). Then
\[
\mathcal{G}_D(x \mid \mathcal{P}_D^{cpl,\text{len}}(x, K), \{ \exists 1 \leq i \leq |\text{Cpts}(\Gamma)| : |\Gamma(i)| \geq (\log N)^2 \})
\]
\[
\leq C^2 \sum_{\Gamma(L) \in \mathcal{AL}_K, \gamma(L) \subset D} q_D(\Gamma(L)) e^{\tau_\beta(\mathcal{X}(\Gamma(L)))} \sum_{\Gamma(R) \in \mathcal{AR}_K, \gamma(R) \subset D} q_D(\Gamma(R)) e^{\tau_\beta(\mathcal{X}(\Gamma(R)))}
\]
\[
\times \bigcup_{\Gamma(1) \in \mathcal{A}} \bigcup_{\Gamma(n) \in \mathcal{A}} 1_{|\Gamma| \geq (\log N)^2} \bigcup_{\Gamma(1) \in \mathcal{A}} \bigcup_{\Gamma(n) \in \mathcal{A}} 1_{|\Gamma| \geq (\log N)^2}.
\]

The exponential tails in Eq. (5.17) implies both
\[
\sum_{\Gamma(L) \in \mathcal{AL}_K, \gamma(L) \subset D} q_D(\Gamma(L)) e^{\tau_\beta(\mathcal{X}(\Gamma(L)))}
\]
and
\[
\sum_{\Gamma(R) \in \mathcal{AR}_K, \gamma(R) \subset D} q_D(\Gamma(R)) e^{\tau_\beta(\mathcal{X}(\Gamma(R)))}
\]
are bounded by a constant \( C_K > 0 \), while
\[
\sum_{\Gamma \in \mathcal{A}} W_{\mathbb{H}}^{\nu - u}(\Gamma) 1_{|\Gamma| \geq (\log N)^2} \leq C e^{-\nu_\beta(\log N)^2}.
\]

These bounds, the strong triangle inequality (Proposition 2.4), and the fact that there are at most \(|0 + \mathcal{Y} \cap x + \mathcal{Y} \cap \mathbb{Z}^2| \leq (N + 1)^2 \) possible values for \( u \) and for \( v \) yield
\[
\mathcal{G}_Q(x \mid \{ \exists 1 \leq i \leq |\text{Cpts}(\Gamma)| : |\Gamma(i)| \geq (\log N)^2 \}) \leq C'_K (N + 1) e^{-\nu_\beta(\log N)^2} e^{-\tau_\beta(x)},
\]
for some constant \( C'_K > 0 \). Thus, we have Eq. (5.26), finishing the proof.

5.5. **Entropic repulsion of the animal.** Fix any \( \delta \in (0, 1/4) \), and define the rectangles \( \mathcal{R}_0(\delta) := [N^{4\delta}, N - N^{4\delta}] \times [0, 2N^\delta] \) and \( \mathcal{R}(\delta) := [N^{4\delta}, N - N^{4\delta}] \times [0, N^\delta] \). The main result in this subsection is the entropic repulsion result Proposition 5.11, which states that, under \( \mathcal{P}_D^*(\cdot) \), the cone points of \( \Gamma \) do not intersect \( \mathcal{R}(\delta) \) with probability tending to 1 as \( N \rightarrow \infty \).

**Proposition 5.11.** Let \( D = Q \) or \( D = \mathbb{H} \). For any \( \delta \in (0, 1/4) \),
\[
\lim_{N \rightarrow \infty} \mathcal{P}_D^*(\text{Cpts}(\Gamma) \cap \mathcal{R}(\delta) \neq \emptyset) = 0.
\]
From the shape of $\mathcal{Y}^\bullet$ and the $(\log N)^2$ bound on each $|\Gamma^{(i)}|$ (Proposition 5.8), Proposition 5.11 is enough to deduce that, with high $\mathbb{P}_{\partial\mathbb{H}}$-probability, the entire animal (contour and clusters) stays away from $\partial\mathbb{H}$ in a slightly shorter rectangle (5.30). We will use this fact in the following subsection to couple $\text{Cpts}(\Gamma)$ with the random walk. We begin with an analogous result in the free Ising polymer case, where we can exploit the connection with the random walk and associated estimates, à la Proposition 4.1.

**Lemma 5.12.** Let $D = Q$ or $D = \mathbb{H}$. For any $\delta \in (0,1/8)$,
\[
\lim_{N \to \infty} \mathbb{P}^x(\Gamma \cap \mathcal{R}(\delta) \neq \emptyset \ | \ \gamma \subset D) = 0.
\]

**Proof.** In light of Proposition 5.8 and the shape of $\mathcal{Y}^\bullet$, it suffices to show that the cone-points of $\Gamma$ avoid the larger rectangle $\mathcal{R}_0(\delta)$ with high probability, i.e.,
\[
\lim_{N \to \infty} \mathbb{P}^x(E_\delta \ | \ \gamma \subset D) = 0, \quad \text{where } E_\delta \coloneqq \{\text{Cpts}(\Gamma) \cap \mathcal{R}_0(\delta) \neq \emptyset\}.
\]

Recall $\mathcal{P}_{D,\text{cp},\text{len}}(x,K)$ from (5.25), and observe that Lemmas 5.5 and 5.7 and Remark 5.6 imply
\[
\lim_{K \to \infty} \lim_{N \to \infty} \mathbb{P}^x(\mathcal{P}_{D,\text{cp},\text{len}}(x,K) \ | \ \gamma \subset D) = 1.
\]

From the previous two displays, the proof of the Lemma will be complete upon showing
\[
\lim_{K \to \infty} \lim_{N \to \infty} \mathbb{P}^x(\mathcal{P}_{D,\text{cp},\text{len}}(x,K),E_\delta \ | \ \gamma \subset D) = 0.
\]

Now, fix $K \geq 1$ and $N$ large compared to $K$. Recall the sets $A_{L,K}$ and $A_{R,K}$ from (5.27). Using Eq. (2.29), we have
\[
\mathbb{P}^x(\mathcal{P}_{D,\text{cp},\text{len}}(x,K),E_\delta \ | \ \gamma \subset D)
\leq \frac{1}{g(x \mid \gamma \subset D)} e^{-\gamma}(x) \sum_{\Gamma^{(L)}} \mathbb{P}^h_x(\Gamma^{(L)}) \sum_{\Gamma^{(R)}} \mathbb{P}^h_x(\Gamma^{(R)}) A(X(\Gamma^{(L)}), x - X(\Gamma^{(R)}); E_\delta).
\]

Using Eq. (5.3), Eq. (4.5) and Proposition 4.1, we may bound from above the right-hand side of the previous display
\[
C^{-1} \sum_{\Gamma^{(L)} \in A_{L,K}} \mathbb{P}^h_x(\Gamma^{(L)}) e^{\beta ||X(\Gamma^{(L)})||_1} \sum_{\Gamma^{(R)} \in A_{R,K}} \mathbb{P}^h_x(\Gamma^{(R)}) e^{\beta ||X(\Gamma^{(R)})||_1} \frac{A^+(X(\Gamma^{(L)}), x - X(\Gamma^{(R)}); E_\delta) H_{x - X(\Gamma^{(R)})} > H_{\mathbb{H}_-}}{\mathbb{P}_{X(\Gamma^{(L)})}(H_{x - X(\Gamma^{(R)})} > H_{\mathbb{H}_-})}
\]
\[
\times \mathbb{P}_{X(\Gamma^{(L)})}(\exists i < H_{x - X(\Gamma^{(R)})} : S_1(i) \in [N^{\delta}, N - N^{4\delta}], S_2(i) \in [0, 2N^{\delta}] \mid H_{x - X(\Gamma^{(R)})} < H_{\mathbb{H}_-}).
\]

The ballot-type result Theorem 4.2(1) and standard random walk estimates yield
\[
\lim_{N \to \infty} \mathbb{P}_{X(\Gamma^{(L)})}(\exists i < H_{x - X(\Gamma^{(R)})} : S_1(i) \in [N^{\delta}, N - N^{4\delta}], S_2(i) \in [0, 2N^{\delta}] \mid H_{x - X(\Gamma^{(R)})} < H_{\mathbb{H}_-}) = 0
\]
uniformly over $\Gamma^{(L)} \in A_{L,K}$ and $\Gamma^{(R)} \in A_{R,K}$. On the other hand, the exponential tails in Eq. (2.26) and the relation $4\delta < \nu_g$ yield
\[
C^{-1} \left( \sum_{\Gamma^{(L)} \in A_{L,K}} \mathbb{P}^h_x(\Gamma^{(L)}) e^{\beta ||X(\Gamma^{(L)})||_1} \right) \left( \sum_{\Gamma^{(R)} \in A_{R,K}} \mathbb{P}^h_x(\Gamma^{(R)}) e^{\beta ||X(\Gamma^{(R)})||_1} \right) \leq C_K,
\]
for some constant $C_K = C_{K,\beta} > 0$. Taking the limit as $N$, then $K \to \infty$ in (5.29) yields (5.28). \qed
Proof of Proposition 5.11. Lemma 5.12 and Corollary 5.3 imply that \( \gamma \) under \( P_D^* (\cdot) \) stays above \( \Re (\delta) \). Of course, all cone-points of \( \Gamma \) lie on \( \gamma \), so we are done.

5.6. Coupling with the effective random walk. Let \( D = Q \) or \( D = \mathbb{H} \), and fix \( \delta \in (0, 1/8) \). The entropic repulsion result, Proposition 5.11, sets the stage for a coupling between the cone-points of \( \Gamma \) lying in the strip \( S_{N^{4\delta}, N - N^{4\delta}} := [N^{4\delta}, N - N^{4\delta}] \times [0, \infty) \) and the random walk \( S (\cdot) \) defined in Section 4.1. Before defining this coupling explicitly, we set up some notation.

For \( u, v \in \mathbb{Z}^2 \), define
\[
P_{D, \delta}^*(u, v) := \left\{ \Gamma \in \mathcal{P}_D(u, v) : |\text{Cpts}(\Gamma)| \geq 2, \max_{i \in \{L, 1, \ldots, |\text{Cpts}(\Gamma)| - 1, R\}} |\Gamma^{(i)}| < (\log N)^2, \text{Cpts}(\Gamma) \cap \Re (\delta) = \emptyset \right\},
\]
where, as usual, we write \( P_{D, \delta}^*(x) := P_{D, \delta}^*(0, x) \). Observe that the last two conditions in the definition, along with the shape of \( J^* \), ensure that
\[
\Gamma \cap ([N^{4\delta}, N - N^{4\delta}] \times [0, N^{\delta}/2]) = \emptyset \quad \text{for all } \Gamma \in \mathcal{P}_{D, \delta}^*(u, v),
\]
where the intersection pertains to both contour and clusters of \( \Gamma \). From Lemma 5.7 and Propositions 5.8 and 5.11, we know that
\[
P_D^*(\mathcal{P}_{D, \delta}^*(x)) = 1 + o(1).
\]

Define the measure
\[
P_{D, \delta}^{*, x} (\cdot) := P_D^*(\cdot \mid \mathcal{P}_{D, \delta}^*(x)).
\]
For \( \Gamma \in \mathcal{P}_{D, \delta}^* \), let \( \zeta^{(L)}(\cdot) \) and \( \zeta^{(R)}(\cdot) \) denote the left-most and right-most cone-points of \( \Gamma \) in \( S_{N^{4\delta}, N - N^{4\delta}} \), respectively. Note that, since \( \Gamma \in \mathcal{P}_{D, \delta}^* \), we have
\[
\zeta^{(L)}(\Gamma) \in [N^{4\delta}, N^{4\delta} + (\log N)^2] \times [N^{\delta}, N^{4\delta}(\log N)^2]
\]
and
\[
\zeta^{(R)}(\Gamma) \in [N - N^{4\delta} - (\log N)^2, N - N^{4\delta}] \times (N^{\delta}, N^{4\delta}(\log N)^2).
\]
Let \( \Gamma^{(L)} \in \mathcal{P}_{D, \delta}^*(0, \zeta^{(L)}(\cdot)) \) denote the portion of \( \Gamma \) connecting 0 to \( \zeta^{(L)}(\cdot) \) (including all clusters connected to \( \gamma^{(L)}(\cdot) \)). Similarly, define \( \Gamma^{(R)} \in \mathcal{P}_{D, \delta}^*(\zeta^{(R)}(\cdot), x) \) as the portion of \( \Gamma \) connecting \( \zeta^{(R)}(\cdot) \) to \( x \), and define \( \Gamma^{*} \in \mathcal{P}_{D, \delta}^{*, x} (\zeta^{(L)}(\cdot), \zeta^{(R)}(\cdot)) \) as the portion of \( \Gamma \) connecting \( \zeta^{(L)}(\cdot) \) to \( \zeta^{(R)}(\cdot) \) so that
\[
\Gamma = \Gamma^{(L)} \circ \Gamma^{*} \circ \Gamma^{(R)}.
\]
Note that \( \Gamma^{*} \) is a concatenation of irreducible animals, each of which are completely contained in \( Q \) by (5.30). Therefore, from (2.18), we have
\[
q_D (\Gamma) = q_D (\Gamma^{(L)}(\cdot)) q(\Gamma^{*}(\cdot)) q_D (\Gamma^{(R)}(\cdot)).
\]
Crucially, the \( q_D \)-weight of \( \Gamma^{*} \) is equal to its (free) \( q \)-weight.

For any fixed \( \zeta^{(L)}(\cdot) \) and \( \zeta^{(R)}(\cdot) \) satisfying (5.32) and (5.33) respectively, and for any fixed animals \( \Gamma^{(L)} \in \mathcal{P}_D(0, \zeta^{(L)}(\cdot)) \) and \( \Gamma^{(R)} \in \mathcal{P}_D(\zeta^{(R)}(\cdot), x) \), consider the measure
\[
P_{D, \delta}^{*, x} (\cdot) := P_D^{*, x} (\cdot \mid \Gamma^{(L)} = \Gamma^{(L)}(\cdot), \Gamma^{(R)} = \Gamma^{(R)}(\cdot)),
\]
which describes the conditional law of \( \Gamma^{*} \) and defines a probability measure on the following set of animals:
\[
P_{D, \delta}^{*, x} (\zeta^{(L)}(\cdot), \zeta^{(R)}(\cdot)) := \left\{ \Gamma^{(1)} \circ \ldots \circ \Gamma^{(n)} \in \mathcal{P}_{D, \delta}^{*, x} (\zeta^{(L)}(\cdot), \zeta^{(R)}(\cdot)) : n \in \mathbb{N}, \Gamma^{(i)} \in A \text{ for each } i \in [1, n] \right\}.
\]
Note that, for fixed $\zeta^{(L^*)}$ and $\zeta^{(R^*)}$, the choice of $\Gamma^{(L^*)}$ and $\Gamma^{(R^*)}$ does not change $\mathbf{P}_{\vec{D},\zeta^{(L^*)}, \zeta^{(R^*)}}^{x,*}(\cdot)$. In particular, we have the formula

$$
\mathbf{P}_{\vec{D},\zeta^{(L^*)}, \zeta^{(R^*)}}^{x,*}(E) = \sum_{\Gamma \in \mathcal{E} \cap \mathcal{P}_D^{\ast}(\zeta^{(L^*)}, \zeta^{(R^*)})} q(\Gamma)/ \sum_{\Gamma \in \mathcal{P}_D^{\ast}(\zeta^{(L^*)}, \zeta^{(R^*)})} q(\Gamma). 
$$

(5.34)

Again, note that only the free weight appears in the above formula.

Lastly, for any $c > 0$, define the half-space $\mathbb{H}_c := \{(x,y) \in \mathbb{R}^2 : y \geq c\}$.

**Proposition 5.13.** Let $D = Q$ or $D = \mathbb{H}$. There exists $\nu_2 > 0$ such that for all $N \in \mathbb{N}$, for all $\beta > 0$ sufficiently large, and for all $\zeta^{(L^*)}$ and $\zeta^{(R^*)}$ satisfying (5.32) and (5.33) respectively, if we let $T = H_{\zeta^{(R^*)}}$ and view $\text{Cpts}(\Gamma^*)$ as an ordered tuple (see Eq. (2.20)),

$$
\left\| \mathbf{P}_{\vec{D},\zeta^{(L^*)}, \zeta^{(R^*)}}^{x,*}\left( \text{Cpts}(\Gamma^*) \in \cdot \right) - \mathbf{P}_{\zeta^{(L^*)}}^x\left( (S(i))_{i=0}^T \in \cdot \mid T < H_{\mathbb{H}_c N^2} \right) \right\|_{TV} \leq Ce^{-\nu_2 \beta (\log N)^2}
$$

for some constant $C := C(\beta) > 0$.

**Proof.** Fix $\zeta^{(L^*)}$ and $\zeta^{(R^*)}$ satisfying (5.32) and (5.33) respectively. Recall $V^\delta_{\vec{u}, \vec{v}}$ from (4.3), and let

$$
V^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}} := \{(\vec{v}_1, \ldots, \vec{v}_n) \in V^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}} : n \geq 1, \|\vec{v}_k\|_1 \leq (\log N)^2 \text{ and } \zeta^{(L^*)} + \sum_{i=1}^k \vec{v}_i \geq N^\delta \forall k \in [1,n]\}
$$

and

$$
A^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}} := \{(\vec{\Gamma}^{(1)}, \ldots, \vec{\Gamma}^{(n)}) \in A^n : n \geq 1, (X(\vec{\Gamma}^{(1)}), \ldots, X(\vec{\Gamma}^{(n)})) \in V^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}}\}
$$

Note that the sets $A^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}}$ and $\mathcal{P}_D^{\ast}(\zeta^{(L^*)}, \zeta^{(R^*)})$ are in bijection with one another, since for any $(\vec{\Gamma}^{(1)}, \ldots, \vec{\Gamma}^{(n)}) \in A^\delta_{\zeta^{(L^*)}, \zeta^{(R^*)}}$, the animal $\vec{\Gamma}^{(1)} \circ \cdots \circ \vec{\Gamma}^{(n)}$ is in $\mathcal{P}_D^{\ast}(\zeta^{(L^*)}, \zeta^{(R^*)})$. This observation along with (2.27) yields the following

$$
\sum_{\Gamma \in \mathcal{P}_D^{\ast}(\zeta^{(L^*)}, \zeta^{(R^*)})} q(\Gamma) = e^{h_{\vec{D}}(\vec{\zeta}^{(L^*)} + x - \vec{\zeta}^{(R^*)})} \sum_{n \geq 1} \prod_{i=1}^n \mathbf{P}_{\zeta^{(L^*)}}^x(\vec{\Gamma}^{(i)})
$$

$$
= e^{h_{\vec{D}}(\vec{\zeta}^{(L^*)} + x - \vec{\zeta}^{(R^*)})} \sum_{n \geq 1} \sum_{(\vec{v}_1, \ldots, \vec{v}_n) \in V^d_{\zeta^{(L^*)}, \zeta^{(R^*)}}} \prod_{i=1}^n \left( \sum_{\vec{r}^{(i)} \in A} \mathbf{P}_{\zeta^{(L^*)}}^x(\vec{r}^{(i)}) I_{\{X(\vec{r}^{(i)}) = \vec{v}_i\}} \right)
$$

$$
= e^{h_{\vec{D}}(\vec{\zeta}^{(L^*)} + x - \vec{\zeta}^{(R^*)})} \sum_{n \geq 1} \left(\sum_{(\vec{v}_1, \ldots, \vec{v}_n) \in V^d_{\zeta^{(L^*)}, \zeta^{(R^*)}}} \prod_{i=1}^n \mathbf{P}(S(1) = \vec{v}_i) \right)
$$

$$
= e^{h_{\vec{D}}(\vec{\zeta}^{(L^*)} + x - \vec{\zeta}^{(R^*)})} \mathbf{P}_{\zeta^{(L^*)}}^x(E^{\delta}_x),
$$

(5.35)

where we define the random walk event

$$
E^{\delta}_x := \{H_{\zeta^{(R^*)}} < H_{\mathbb{H}_c N^2} \mid \|S(i + 1) - S(i)\|_1 \leq (\log N)^2, \forall i \in [0,H_{\zeta^{(R^*)}} - 1]\}.
$$

A nearly identical calculation but for the numerator of the right-hand side of (5.34) yields the following: for any $n \geq 1$ and for any $(\vec{v}_1, \ldots, \vec{v}_n) \in (\mathbb{Z}^2)^n$,

$$
\mathbf{P}_{\vec{D},\zeta^{(L^*)}, \zeta^{(R^*)}}^{x,*}\left( \text{Cpts}(\vec{\Gamma}) = \left(\vec{\zeta}^{(L^*)} + \vec{v}_1, \ldots, \vec{\zeta}^{(L^*)} + \sum_{i=1}^n \vec{v}_i, \vec{\zeta}^{(R^*)}\right)\right)
$$

$$
= \mathbf{P}_{\zeta^{(L^*)}}^x(\vec{H}_{\zeta^{(R^*)}} = n + 1, S^{(k)} = \vec{\zeta}^{(L^*)} + \sum_{i=1}^k \vec{v}_i, \forall k \in [1,n] \mid E^{\delta}_x) I_{\{(\vec{v}_1, \ldots, \vec{v}_n) \in V^d_{\zeta^{(L^*)}, \zeta^{(R^*)}}\}}.
$$
Thus, the law of \( Cpts(\Gamma) \) under \( P_{D,\delta}^{x,*}(\cdot) \) is equal to the law of \( (S_i)_{i=0}^{H_{\zeta(\Gamma^*)}} \) under \( P_{\zeta(\Gamma^*)}^x(\cdot) \). Now, observe that Theorem 4.2 along with shift-invariance of the random walk (to shift the walk downwards by \( N^\delta \)) give the following estimate, which holds uniformly over our ranges of \( \zeta^{(L^*)} \) and \( \zeta^{(R^*)} \):

\[
\mathbb{P}_{\zeta^{(L^*)}}(H_{\zeta(\Gamma^*)} < H_{\mathbb{H},N^\delta}) = \mathbb{P}_{\zeta^{(L^*)}}(H_{\zeta(\Gamma^*)} - (0,N^\delta) < H_{\mathbb{H},-}) \gtrsim N^{-3/2}.
\]

Exponential tails of the random walk increments (4.2) and the above estimate then imply that

\[
\mathbb{P}_{\zeta^{(L^*)}}(E_\delta^* \mid H_{\zeta(\Gamma^*)} < H_{\mathbb{H},N^\delta}) \geq 1 - C e^{-\nu\beta(\log N)^2}
\]

for some constant \( \nu > 0 \) independent of \( \beta \), and a constant \( C := C(\beta) > 0 \), finishing the proof.

\[\Box\]

5.7. **Proof of Theorem 2.8.** We see from the shape of the cone \( Y^* \) and the restriction on \( |\Gamma(i)| \) that for all \( \Gamma \in \mathcal{P}_{D,\delta}(x) \),

\[
\frac{1}{\sqrt{N}} \max_{x \in [0,N]} |\overline{\gamma}(x) - \gamma(x)| \leq \frac{2(\log N)^2}{\sqrt{N}}.
\]

In addition to (5.31), this tells us that it suffices to show \( \mathcal{J}_N(Cpts(\Gamma)) \) under \( P_{D,\delta}^{x,*}(\cdot) \) converges weakly to a Brownian excursion. Now, we have the following on \( \mathcal{P}_{D,\delta}^{x,*} \):

\[
\sup \{ y : \exists x \in \mathbb{Z} \text{ s.t. } (x,y) \in \gamma^{(L^*)} \}, \sup \{ y : \exists x \in \mathbb{Z} \text{ s.t. } (x,y) \in \gamma^{(R^*)} \} \leq N^{4\delta}(\log N)^2 \quad (5.36)
\]

as well as the bounds in Eqs. (5.32) and (5.33). These bounds imply that \( \Gamma^{(L^*)} \) and \( \Gamma^{(R^*)} \) do not impact the diffusive scaling limit of \( Cpts(\Gamma) \). Thus, recalling the linear interpolation \( \mathcal{J}_N \) from (4.8), it suffices to show \( \mathcal{J}_N(Cpts(\Gamma^*)) \) under law \( P_{D,\delta}^{x,*}(\cdot) \) converges weakly to a Brownian excursion, uniformly in \( \zeta^{(L^*)} \) and \( \zeta^{(R^*)} \) satisfying (5.32) and (5.33). But this is an immediate consequence of Proposition 5.13, the shift invariance of the random walk, and Theorem 4.3.

This concludes the proof of Theorem 2.8, and hence also Theorem 1.2.

\[\Box\]

6. Random Walks in a Half-space

This section is devoted to the proofs of Theorems 4.2 and 4.3 via the analysis of random walks in a half-space.

As mentioned after Theorem 4.3, we seek to extend many of the results from [17] and [23]. These papers are written for random walks whose coordinates are uncorrelated. Though \( S(\cdot) \) may not have this property, we can obtain such a random walk by rotating— this is done in Section 6.3.

As such, we must consider random walks in general half-spaces. Throughout the rest of this section, fix a unit vector \( \vec{n} \in S^1 \), and consider the half-space through the origin with inward normal \( \vec{n} \), and call this space \( \mathbb{H}_{\vec{n}} \subset \mathbb{R}^2 \). Let \( \vec{n}^\perp \in S^1 \) be a unit vector orthogonal to \( \vec{n} \) (the choice between \( \vec{n} \) and \( -\vec{n} \) does not matter). We will think of our random walk in terms of coordinates with respect to \( \vec{n} \) and \( \vec{n}^\perp \) instead of \( e_1 \) and \( e_2 \). Let \( S(\cdot) \) denote a general 2D random walk with step distribution \( X = (X_1, X_2) \) such that \( \mathbb{E}X = 0, \text{ Cov } X = 1 \), and \( \mathbb{P}(|X| > m) \leq c_1 e^{-c_2m} \) for some constants \( c_1, c_2 > 0 \) and for all \( m \geq 1 \). Additionally, we impose a lattice assumption: assume that \( X \) takes values on a lattice \( \mathcal{L} \) that is a non-degenerate linear transformation of \( \mathbb{Z}^2 \), and that the distribution of \( X \) is strongly aperiodic; that is, for each \( u \in \mathcal{L}, \mathcal{L} \) is the smallest subgroup of \( \mathbb{Z}^2 \) containing \( \{ v : v = u + w \text{ for some } w \text{ such that } \mathbb{P}(X = w) > 0 \} \).

The theory of random walks confined to a half-space is intimately related to the theory of harmonic functions for processes killed upon exiting a half-space, and so we recall the relevant facts from this theory before proceeding.
6.1. Harmonic functions for processes in a half-space. The study of Markov processes conditioned to stay in a domain is intimately related to the study of their corresponding positive harmonic functions. Our domain of interest is a relatively simple one, the half-space $\mathbb{H}_n$, and we are concerned with the harmonic functions of the Brownian motion and of the random walk $S(\cdot)$ killed upon exit from $\mathbb{H}_n$.

The harmonic function of the Brownian motion killed at $\partial \mathbb{H}_n$ is given by the minimal (up to a constant), strictly positive harmonic function on $\mathbb{H}_n$ with zero boundary conditions. For us, such functions take a very explicit form. Consider the rotation that sends $\vec{n} \mapsto e_2$, and thus $\mathbb{H}_n$ to $\mathbb{H}$. This is a conformal mapping that induces a bijection between positive harmonic functions in $\mathbb{H}_n$ that vanish on $\partial \mathbb{H}_n$ to positive harmonic functions in $\mathbb{H}$ that vanish on $\partial \mathbb{H}_n$. Since\footnote{See, e.g., [2, Theorem 7.22].} every such harmonic function in $\mathbb{H}$ takes the form $h(v) = cv \cdot e_2$, for some constant $c > 0$, it follows that every harmonic $h$ function in $\mathbb{H}_n$ takes the form $h(v) = cv \cdot \vec{n}$. In particular, the harmonic function only depends on the projection of $v$ onto $\vec{n}$. In what follows, we take $c = 1$ and define

$$u(v) := v \cdot \vec{n}.$$  

We refer to [15] for a general discussion of the harmonic function of the Brownian motion in cones.

One of the many achievements of [17] is the construction of a positive harmonic function $V$ for a wide class of random walks $S(\cdot)$ killed upon exit from $K$, i.e., $V$ solves the equation\footnote{In [17] and [23], the authors consider random walks $(S)$ from $S_0 = 0$, and study the law of $(v + S)$ for $v \in K$. In particular, they write $\tau_v$ to the hitting time $H_{K_v}^S$ under law $\mathbb{P}_v$.}

$$\mathbb{E} [V(S(1)), H_{K'} > 1] = V(v),$$

where $K$ is an element of a wide class of cones. Many of their limit theorems that we wish to extend were stated in terms of $V$. The function $V$ was constructed as

$$V(v) = \lim_{n \to \infty} \mathbb{E} [h(S(n)), H_{K'} > n],$$

where $h$ is a choice of the harmonic function for Brownian motion killed at $\partial K$. For $K := H_{\mathbb{H}_n}$, $h := u$, and so we see

$$V(v) = V_1(v \cdot \vec{n}),$$

where $V_1 := V_1^\vec{n}$ is the unique positive harmonic function for the one-dimensional random walk $\vec{n} \cdot S(\cdot)$ killed at leaving $(0, \infty)$ satisfying

$$\lim_{a \to \infty} \frac{V_1(a)}{a} = 1.$$  

The uniqueness of $V_1$, as well as an exact formula for $V_1$, was established by Doney in [22].

6.2. Limit theorems for general random walks in a half-space. The following two results, Propositions 6.1 and 6.2, are modifications of a ballot-type theorem from [17] and an invariance principle from [23], respectively. We’ll write $\xi(\cdot)$ to denote the standard Brownian excursion on $[0, 1]$.

**Proposition 6.1** (Modification of [17, Theorem 6]). Fix any $A > 0$ and $\delta \in (0, 1/2)$. Then there exists a constant $C_1 > 0$ such that, uniformly over sequences $a_k, b_k$ and $u_k$ satisfying

$$a_k \in [-A \sqrt{k}, A \sqrt{k}], \quad \frac{a_k}{\sqrt{k}} \to a \in [-A, A], \quad b_k, u_k \in (0, k^{1/2-\delta}], \quad \{u_k \vec{n}, a_k \vec{n} \perp + b_k\} \subset \mathcal{L},$$

we have

$$\mathbb{P}_{u_k \vec{n}} \left( S(k) = a_k \vec{n} \perp + b_k \vec{n}, H_{\mathbb{H}_n}^S > k \right) \sim C_1 k^2 \frac{V_1(u_k) V_1'(b_k)}{k^2} e^{-\frac{a^2}{2}},$$ \hspace{1cm} (6.3)

\footnote{See also [18], where such a harmonic function was constructed for a more general class of cones; as well as [19], which addressed a more general class of random walks.}
where $V'_1$ is the positive harmonic function for $-\bar{n} \cdot S_2(\cdot)$ killed upon leaving $(0, \infty)$.

Proposition 6.2 (Modification of [23, Theorem 6]). Fix any $\delta \in (0, 1/2)$. Uniformly over sequences 
\[ a_k \in [-A\sqrt{k}, A\sqrt{k}] \text{ and } b_k, u_k \in (0, k^{1/2-\delta}] \] and \( \{u_k \bar{n}, a_k \bar{n}^\perp + b_k \} \subset \mathcal{L}, \) \hspace{1cm} (6.4)
the family of conditional laws 
\[ Q_{u_k, a_k, b_k}(\cdot) := P_{u_k \bar{n}} \left( \begin{array}{c} \bar{n} \cdot S(\lfloor k \rfloor) \\ V(\cdot) \end{array} \right) \in [0,1] \in \cdot | S(k) = a_k \bar{n}^\perp + b_k \bar{n}, H_{\mathbb{H}_{\nu=\bar{n}}}^\perp > k \)
converge as $k \to \infty$ to the law of $\varepsilon(\cdot)$ in the Skorokhod space $(D[0,1], \| \cdot \|_{\infty})$.

For the next result, let $S$ denote a two-dimensional random walk with step distribution $X = (X_1, X_2)$ satisfying the same lattice, covariance, and tail decay assumptions as $X$, but with mean $E X = \mu \bar{n}^\perp$ for some $\mu > 0$.

Proposition 6.3. Fix any $\delta \in (0, 1/2)$. There exists a constant $C' := C'(X) > 0$ such that, uniformly over $u, v \in (0, N^{1/2-\delta}]$ and \( \{u \bar{n}, N \bar{n}^\perp + v \bar{n}\} \subset \mathcal{L}, \) we have
\[ P_{u \bar{n}} \left( H_{N \bar{n}^\perp + v \bar{n}}^S < H_{\mathbb{H}_{\nu=\bar{n}}}^S \right) \sim C' \frac{V_1(u)V_1(v)}{N^{3/2}}. \hspace{1cm} (6.5) \]
where $V_1'$ denotes the harmonic function for $-\bar{n} \cdot S_2(\cdot)$ killed upon leaving $(0, \infty)$.

Furthermore, if 
\[ p_{N,A} := P_{u \bar{n}} \left( H_{N \bar{n}^\perp + v \bar{n}}^S \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}] \right) \frac{H_{N \bar{n}^\perp + v \bar{n}}^S < H_{\mathbb{H}_{\nu=\bar{n}}}^S} \]
then 
\[ \lim_{A \to \infty} \liminf_{N \to \infty} p_{N,A} = 1. \hspace{1cm} (6.6) \]

Propositions 6.1 to 6.3 will be proved in Section 6.5. Note that when $\bar{n} = e_2$ (so $\mathbb{H}_{\nu} = \mathbb{H}$), Proposition 6.3 reduces to [30, Theorem 5.1].

6.3. Proof of Theorems 4.2 and 4.3. Recall that the random walk $S(\cdot)$ is a two-dimensional random walk whose step distribution $X = (X_1, X_2)$ satisfies $\mathbb{E}|X| = (\mu, 0)$, for some $\mu > 0$, and exponential tail decay for $|X|$. Recall $\sigma_1^2 := \text{Var}(X_1)$ and $\sigma_2^2 := \text{Var}(X_2)$.

Proof of Theorem 4.2. Theorem 4.2 follows from Propositions 6.1 to 6.3 by linearly transforming the random walk $S(\cdot)$ to satisfy the hypotheses of these results. We demonstrate this explicitly for item (2) of Theorem 4.2; the proof of item (1) follows readily.

Following from [17, Example 2], we derive the aforementioned linear transformation, . Consider the re-centered, normalized random walk 
\[ \mathcal{S}(n) := \begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{pmatrix} \left( S(n) - n \begin{pmatrix} \mu \\ 0 \end{pmatrix} \right). \]
This is a random walk with increments $\tilde{X} := (\tilde{X}_1, \tilde{X}_2)$ of mean 0 such that, for some $\rho \in (0, 1)$, we have $C := \text{Cov} \tilde{X} = \mathbb{E} \tilde{X} \tilde{X}^t = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Let $C = ODO^t$. Since $C$ is a covariance matrix, it is positive semi-definite; however, if $C$ had 0 as an eigenvalue, then that would imply the random walk lives on a line, which we know is not true as $X$ can be $e_1$ or $e_1 + e_2$ with positive probability. So, $C$ is positive-definite. Consider the transformation matrix $M := O\sqrt{D^{-1}}O$. Then 
\[ \text{Cov}(MX) = \mathbb{E}[MXMX^tM^t] = \text{Id}. \]
To be explicit, straight-forward calculations reveal that if $\theta \in [-\pi, \pi]$ solves $\sin 2\theta = \rho$, then we may take 
\[ M = \frac{1}{\sqrt{1-\rho^2}} \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \]
though we will not need this explicit form.

Now, observe that \( \vec{n} := M^{-1}e_2 \) has norm 1 (explicitly, \( \vec{n} = (\sin \theta, \cos \theta) \)). Using the fact that \( M^{-1} \) is symmetric and \( \vec{n} \) is outward normal to \( M\mathbb{H}_- = \mathbb{H}_-\vec{n} \),

\[
\{ H^S_{\mathbb{H}_-} > k \} = \{ e_2^S(i) > 0, \forall i \in [0, k] \} = \{ e_2^S(i) > 0, \forall i \in [0, k] \} = \{ M^S(i) \cdot M^{-1}e_2 > 0, \forall i \in [0, k] \}
\]

\[
= \{ H^{M^S}_{\mathbb{H}_-} > k \}.
\]

In what follows, we write \( M(a, b) \) to denote the matrix applied to the vector \( ae_1 + be_2 \). Then for any measurable set \( B \),

\[
P_{(0,u)} \left( S(\cdot) \in B \mid S(k) = (N, v), H^S_{\mathbb{H}_-} > k \right)
\]

\[
= P_{(0,\sigma_2^{-1}u)} \left( \mathbb{S}(\cdot) \in B - k(\mu,0) \mid \mathbb{S}(k) = \left( \frac{N-k\mu}{\sigma_1}, \frac{v}{\sigma_2} \right), H^S_{\mathbb{H}_-} > k \right)
\]

\[
= P_M(0,u) \left( M\mathbb{S}(\cdot) \in M(B - k(\mu,0)) \mid M\mathbb{S}(k) = M \left( \frac{N-k\mu}{\sigma_1}, \frac{v}{\sigma_2} \right), H^{M\mathbb{S}}_{\mathbb{H}_-} > k \right). \tag{6.7}
\]

Let us now check the hypotheses of Proposition 6.2. Take \( \vec{n}^\perp := Me_1/\|Me_1\| \). Note that \( \vec{n}^\perp \) spans \( \partial\mathbb{H}_-\vec{n} \) and \( \vec{n} \) is the inward normal of \( M\mathbb{H}_- = \mathbb{H}_-\vec{n} \). \( M\mathbb{S}(\cdot) \) is a random walk on the lattice \( \mathcal{L} \) generated by \( \|Me_1\| \vec{n}^\perp \) and \( Me_2 \), the latter of which can be expressed as

\[
Me_2 = (Me_2 \cdot \vec{n}^\perp) \vec{n}^\perp + \vec{n} = \frac{e_1^t M^t Me_2}{\|Me_1\|} \vec{n}^\perp + \vec{n} = \frac{e_1^t C^{-1}e_2}{\|Me_1\|} \vec{n}^\perp + \vec{n}.
\]

Thus, for any \( x, y \in \mathbb{R} \),

\[
M(x,y) = \frac{x + ye_2^t C^{-1}e_2}{\|Me_1\|} \vec{n}^\perp + y\vec{n}.
\]

For any \( k \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}] \), for \( A \) arbitrarily large (but fixed with respect to \( N \)), and for any \( u, v \in [1, N^{1/2-\delta}] \), we see that

\[
u_k \vec{n} := M(0,u) \quad \text{and} \quad a_k \vec{n}^\perp + b_k \vec{n} := M \left( \frac{N-k\mu}{\sigma_1}, \frac{v}{\sigma_2} \right)
\]

satisfies Eq. (6.2). With the hypotheses of Proposition 6.2 satisfied, it follows from Eq. (6.7) and

\[
\sigma_2^{-1}S_2(\cdot) = \mathbb{S}(\cdot) \cdot e_2 = M\mathbb{S}(\cdot) \cdot M^{-1}e_2 = M\mathbb{S}(\cdot) \cdot \vec{n}
\]

that item (2) of Theorem 4.2 is an immediate consequence of Proposition 6.2.

We now use Theorem 4.2 to prove the Brownian excursion limit.

**Proof of Theorem 4.3.** Due to Eq. (4.7), it suffices to show that, for each fixed \( k \), uniformly in the number of steps \( k \in [N/\mu - A\sqrt{N}, N/\mu + A\sqrt{N}] \), the family of conditional laws

\[
Q_{u,v}^{N,k} (\cdot) := P_{(0,u)} \left( (e^{S,v}(t))_{t \in [0,1]} \in \cdot \mid S(k) = (N,v), k < H^S_{\mathbb{H}_-} \right)
\]

converges as \( k \to \infty \) to the law of the standard Brownian excursion on \( [0,1] \). We begin with Claim 6.4, which localizes the \( x \)-coordinate \( S_1(j) \) to an interval of size \( o(N) \), for \( j \in [0, k] \), and allows us to compare the linear interpolation \( e^{S,v} \) with the linear interpolation in item (2) of Theorem 4.2.

**Claim 6.4.** For any \( \eta > 0 \), uniformly over \( u \) and \( v \) as in the theorem statement, we have

\[
\lim_{N \to \infty} P_{(0,u)} \left( \max_{j \in [0,k]} | S_1(j) - \mu j | > N^{1/2+\eta}, k < H^S_{\mathbb{H}_-} \right) = 1. \tag{6.8}
\]

**Proof.** In light of the bound on \( P_{(0,u)}(S(k) = (N,v), k < H^S_{\mathbb{H}_-}) \) provided by Eq. (4.6), it suffices to show

\[
P_0 \left( \max_{j \in [0,N]} | S_1(j) - \mu j | > N^{1/2+\eta} \right) = o(N^{-3/2}). \tag{6.9}
\]
This follows from a union bound over \( j \), the exponential tail bound on the steps of \( S_1(\cdot) \) from Eq. (4.2), and Hoeffding’s inequality:

\[
\mathbb{P}_0 \left( \max_{j \in [0,N]} |S_1(j) - \mu j| > N^{1/2+\eta} \right) \leq N \max_{j \in [0,N]} \mathbb{P}_0 \left( |S_1(j) - \mu j| > N^{1/2+\eta} \right) \\
\leq N \max_{j \in [0,N]} \mathbb{P}_0 \left( |S_1(j) - \mu j| > N^{1/2+\eta} \right), \max_{i \in [0,j-1]} |S(i) - S(i+1)| \leq (\log N)^2 + c N^2 e^{-\nu g(\log N)^2} \\
\leq 2Ne^{-2N^{2\eta} / (\log N)^2} + c N^2 e^{-\nu g(\log N)^2} = o(N^{-3/2}). \tag{6.10} \]

Note that Theorem 4.2(2) and Claim 6.4 are the equivalents of Equations (76) and (77) of [30] for our random walk. We therefore finish the proof of our Theorem 4.3 exactly as in the proof of [30, Theorem 5.3].

6.4. Uniform estimates for the random walk in a half-space. Before proving Propositions 6.1 to 6.3, we need to modify several key estimates of [17] to address the range of parameters in Eqs. (6.2) and (6.4). In what follows, we will repeatedly refer back to [17], explaining how their arguments can be adapted to give the desired uniformity. When stating or citing the results of [17] and [23], we will for the most part use their notation, pointing out any discrepancies explicitly.

6.4.1. Needed inputs. The results of [17] are often stated in terms of the dimension \( d \) and a positive constant \( p \), where \( p \) is related to the asymptotic behavior of the relevant harmonic function in the cone. For us, \( d = 2 \) and \( p = 1 \).

Our first input is an extension of [17, Equation 7], which gives a tail estimate on the hitting time of \( \mathbb{H}_{-\tilde{n}} \).

**Proposition 6.5** (Modification of [17, Equation 7]). Fix any \( \delta \in (0,1/2) \). The following estimate holds uniformly over all \( a_k \in \mathbb{R} \) and \( b_k \in (0,k^{1/2-\delta}] \) such that \( a_k \tilde{n}^+ + b_k \tilde{n} \in \mathcal{L} \):

\[
\mathbb{P}_{a_k \tilde{n}^+ + b_k \tilde{n}} (H_{\mathbb{H}_{-\tilde{n}}} > k) \sim \kappa V_1(b_k) k^{-1/2}. \tag{6.10} \]

**Proof.** Eq. (6.10), for \( a_k := a \) and \( b_k := b \) fixed, is proved in [17] as an immediate consequence of the lemmas of [17, Sections 3 and 4]. Of these, only [17, Lemma 21] is insufficient for the uniformity that we require. \footnote{It may appear that the constant \( C(x) \) in [17, Lemma 16] also poses an issue for uniformity, but here \( C(x) \) can be taken to be \( C(\epsilon) \) for \( \epsilon \) in that lemma, as a consequence of [17, Lemma 14] in our special case of \( u(\nu) := \nu \cdot \tilde{n} \).} That is, we must show that for all \( \epsilon > 0 \) sufficiently small,

\[
\mathbb{E}_{a_k \tilde{n}^+ + b_k \tilde{n}} \left[ u(S(\nu_k)) ; H^S_{\mathbb{H}_{-\tilde{n}}} > \nu_k , \nu_k \leq k^{1-\epsilon} \right] = V(a_k \tilde{n}^+ + b_k \tilde{n})(1 + o(1)),
\]

where \( \nu_k \) is defined as the first hitting time of \( k^{1/2-\epsilon} \tilde{n} + \mathbb{H}_{\tilde{n}} \) and \( o(1) \to 0 \) as \( k \to \infty \) uniformly over the ranges of interest for \( a_k \) and \( b_k \). Uniformity in \( a_k \) is trivial. Uniformity of \( b_k \) was proven in [30, Section 5.6] (see their equation 60, and recall that \( u(S(\nu_k)) = S(\nu_k) \cdot \tilde{n} \) and \( V(a_k \tilde{n}^+ + b_k \tilde{n}) = V_1(b_k) \)). Thus, the proof of [17, Equation 7] given at the end of [17, Section 4] extends to prove our proposition.

**Proposition 6.6** (Modification of [17, Theorem 3]). Uniformly over sequences \( a_k \) and \( b_k \) satisfying Eq. (6.2), the family of measures

\[
\mathbb{P}_{a_k \tilde{n}^+ + b_k \tilde{n}} \left( \frac{S(\lfloor tk \rfloor)}{\sqrt{k}} \in \cdot ; H^S_{\mathbb{H}_{-\tilde{n}}} > k \right)
\]

converges weakly as \( k \to \infty \) to the probability measure on \( \mathbb{H}_{\tilde{n}} \) with density \( H_0(y \cdot \tilde{n}) e^{-\|a \tilde{n}^+ - y\|^2/2} dy \), where \( \overline{H}_0 > 0 \) is the normalizing constant.
Proof. Let \( B \) denote a measurable subset of \( K \). Theorem 3 of [17] gives us
\[
\lim_{k \to \infty} P\left( \frac{S(|tk|)}{\sqrt{k}} \in B - a\bar{n}^+ \mid H_{\mathcal{H}_{\alpha}}^S > k \right) = H_0 \int_{B - a\bar{n}^+} (y \cdot \bar{n}) e^{-|y|^2/2} dy = H_0 \int_B (y \cdot \bar{n}) e^{-|\bar{n}^+ - y|^2/2} dy.
\]
A simple continuity argument then yields
\[
\lim_{k \to \infty} P\left( \frac{S(|tk|)}{\sqrt{k}} + a_k\bar{n}^+ + b_k\bar{n} \in B \mid H_{\mathcal{H}_{\alpha}}^S > k \right) = H_0 \int_B (y \cdot \bar{n}) e^{-|\bar{n}^+ - y|^2/2} dy.
\]
as well. This concludes the proof. \( \square \)

**Proposition 6.7** (Modification of [17, Theorem 5]). Recall \( H_0 \) from Proposition 6.6. Then uniformly over sequences \( a_k \) and \( b_k \) satisfying Eq. (6.2),
\[
\lim \sup_{k \to \infty} \sup_{y \in \mathbb{H}_{\alpha}} \left| \frac{k^{3/2}}{V_1(b_k)} \mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S(k) = y, H_{\mathcal{H}_{\alpha}}^S > k) - \kappa H_0 \frac{\sqrt{k}}{\sqrt{\kappa}} e^{-\|a\bar{n}^+ - y\|^2/2k} \right| = 0. \tag{6.11}
\]

**Proof of Proposition 6.7.** Below, we adapt the proof of Theorem 5 given in [17, Section 6.2], beginning as in [17] by splitting \( \mathbb{H}_{\alpha} \) into three parts:
\[
K^{(1)} := \{ y \in \mathbb{H}_{\alpha} : \|y\| > R\sqrt{k} \}
\]
\[
K^{(2)} := \{ y \in \mathbb{H}_{\alpha} : \|y\| \leq R\sqrt{\kappa}, y \cdot \bar{n} \leq 2\epsilon\sqrt{\kappa} \}
\]
\[
K^{(3)} := \{ y \in \mathbb{H}_{\alpha} : \|y\| \leq R\sqrt{k}, y \cdot \bar{n} > 2\epsilon\sqrt{k} \},
\]
for some \( R > 0 \) and \( \epsilon > 0 \). Below, we let \( C > 0 \) denote a constant independent of \( k, y, R, \) and \( \epsilon \) that may change from line to line. Since
\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} \sup_{y \in K^{(1)} \cup K^{(2)}} \frac{\sqrt{\kappa}}{\sqrt{\kappa}} e^{-\|a\bar{n}^+ - y\|^2/2k} = 0,
\]
Proposition 6.7 will be proved if we can show
\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} \lim_{k \to \infty} \sup_{y \in K^{(1)} \cup K^{(2)}} \frac{k^{3/2}}{V_1(b_k)} \mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S(k) = y, H_{\mathcal{H}_{\alpha}}^S > k) = 0 \tag{6.12}
\]
and
\[
\lim_{\epsilon \to 0} \lim_{k \to \infty} \sup_{y \in K^{(3)}} \left| \frac{k^{3/2}}{V_1(b_k)} \mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S(k) = y, H_{\mathcal{H}_{\alpha}}^S > k) - \kappa H_0 \frac{\sqrt{k}}{\sqrt{\kappa}} e^{-\|a\bar{n}^+ - y\|^2/2k} \right| = 0. \tag{6.13}
\]
We begin with the more complicated Eq. (6.13).

Set \( m := \lceil e^3k \rceil \). Our starting point is [17, Equation 82], reproduced below:
\[
\mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S(k) = y, H_{\mathcal{H}_{\alpha}}^S > k) = \sum_{z \in \mathbb{H}_{\alpha}} \mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S_{k-m} = z, H_{\mathcal{H}_{\alpha}}^S > k-m) \mathbb{P}_z(S_m = y, H_{\mathcal{H}_{\alpha}}^S > m). \tag{6.14}
\]
Let \( \mathbb{H}^{(1)}_{\alpha}(y) := \{ z \in \mathbb{H}_{\alpha} : \|z - y\| < \epsilon\sqrt{k} \} \). Then we may follow the computations in [17, Equations 83, 84] exactly, yielding a constant \( a > 0 \) such that the following inequalities hold uniformly in \( a_k, b_k \), and \( y \) such that \( y \cdot \bar{n} > 2\epsilon\sqrt{k} \):
\[
\frac{k^{3/2}}{V_1(b_k)} \sum_{z \in \mathbb{H}_{\alpha} \setminus \mathbb{H}^{(1)}_{\alpha}(y)} \mathbb{P}_{a_k\bar{n}^+ + b_k\bar{n}}(S_{k-m} = z, H_{\mathcal{H}_{\alpha}}^S > k-m) \mathbb{P}_z(S_m = y, H_{\mathcal{H}_{\alpha}}^S > m) \leq C e^{-3} e^{-\frac{a}{2}} \tag{6.15}
\]
and
\[
\frac{k^{3/2}}{V_1(b_k)} \sum_{z \in \mathbb{Z}^{(1)}(y)} \mathbb{P}_{a_k \vec{n}^\perp + b_k \vec{n}}(S_{k-m} = z, H_{S_{k-m}}^S > k-m) \mathbb{P}_z(S_m = y, H_{S_{k-m}}^S < k) \leq C e^{-\frac{\epsilon y}{2\pi}}. \tag{6.16}
\]

Both right-hand sides go to 0 as \( \epsilon \to 0 \), and so we turn our attention to the following expression:
\[
\Sigma(y) = \sum_{z \in \mathbb{Z}^{(1)}(y)} \mathbb{P}_{a_k \vec{n}^\perp + b_k \vec{n}}(S_{k-m} = z, H_{S_{k-m}}^S > k-m) \mathbb{P}_z(S_m = y).
\]

Equation 85 of [17] also holds uniformly over \( a_k, b_k \), and \( y \cdot \vec{n} > 2\epsilon \sqrt{k} \), except that there should be a \( V(x) \) factor in the big-Oh expression of Equation 85, where for us \( x := a_k \vec{n}^\perp + b_k \vec{n} \) and so \( V(x) = V_1(b_k) \).\(^{15}\) Altogether, we find
\[
\Sigma(y) = (2\pi k^3)^{-1} \mathbb{P}_{a_k \vec{n}^\perp + b_k \vec{n}}(H_{S_{k-m}}^S > k-m) \Sigma_1(y) + O\left(\frac{V_1(b_k)}{k^{3/2}} e^{-\frac{3}{4} \frac{a}{\epsilon}}\right), \tag{6.17}
\]
where
\[
\Sigma_1(y) := \sum_{z \in \mathbb{Z}^{(1)}(y)} \mathbb{P}_{a_k \vec{n}^\perp + b_k \vec{n}}(S_{k-m} = z \mid H_{S_{k-m}}^S > k-m) e^{-2\epsilon y \cdot \vec{n}^\perp}.
\]

From Proposition 6.6 and a compactness argument, we have
\[
\limsup_{k \to \infty} \sup_{y \in K(3)} \left| \Sigma_1(y) - H_0 \int_{\| (1-\epsilon^3)^{1/2} - y / \sqrt{k} \| < \epsilon} (r \cdot \vec{n}) e^{-\| a \vec{n}^\perp - r \|^2 / 2} e^{-\| (1-\epsilon^3)^{1/2} - y / \sqrt{k} \|^2 / 2} dr \right| = 0.
\]

We can follow the steps up to the display before [17, Equation 86], appealing this time to the uniform continuity of \( (r \cdot \vec{n}) e^{-\| a \vec{n}^\perp - r \|^2 / 2} \) (instead of the function \( u(r) e^{-\| r \|^2 / 2} \) as in [17]), to obtain
\[
\limsup_{k \to \infty} \sup_{y \in K(3)} \left| \Sigma_1(y) - H_0 \frac{y \cdot \vec{n}}{\sqrt{k}} e^{-\frac{\| a \vec{n}^\perp - y \|^2}{2k}} \right| = o(\epsilon^3).
\]

Combining the above with Eq. (6.17) and applying Proposition 6.5, we find
\[
\limsup_{\epsilon \to 0} \sup_{k \to \infty} \sup_{y \in K(3)} \left| \frac{k^{3/2}}{V_1(b_k)} (\Sigma(y) - \kappa H_0 \frac{y \cdot \vec{n}}{\sqrt{k}} e^{-\frac{\| a \vec{n}^\perp - y \|^2}{2k}}) \right| = 0.
\]

Combining this with Eqs. (6.15) and (6.16) yields Eq. (6.13).

The modifications of [17, Section 6.2] required to obtain Eq. (6.12) are much simpler than those needed to obtain Eq. (6.13). Equation (6.12) follows very similarly to the proofs of Equations 77 and 81 in [17], and so we just highlight the main differences. The main technical modification comes from the following, which adapts their Lemmas 27 and 28:
\[
\mathbb{P}_{a \vec{n}}(S(k) = a \vec{n}^\perp + b \vec{n}, H_{S_{k-m}}^S > k) \leq C(1 + V_1(u)) m^{-3/2} \land C(1 + V_1(u))(1 + V_1(b)) n^{-2}, \tag{6.18}
\]
for all \( u, a, b \geq 0 \). Indeed, note that Proposition 6.5 gives the bound
\[
\mathbb{P}_{a_k \vec{n}^\perp + b_k \vec{n}}(H_{S_{k-m}}^S > k) \leq C(1 + V_1(b_k)) k^{-1/2}.
\]

Then the proofs of Lemmas 27 and 28 of [17] can be repeated to yield Eq. (6.18) (in particular, the \( C(x) \) in Lemma 27 can be expressed as \( C V_1(x \cdot \vec{n}) \), and the \( C(x, y) \) in Lemma 28 can be expressed as \( CV_1(x \cdot \vec{n}) V_1(y \cdot \vec{n}) \)). So all \( C(x) \) terms from the proofs of [17, Equations 77, 81] should be replaced by \( V_1(b_k) \), after which one finds that their work up to [17, Equation 81] yields Eq. (6.12).\(^{15}\)

\(^{15}\)The \( V_1(b_k) \) factor comes from an application of Proposition 6.5 in the second line of [17, Equation 85]. This term was dropped in [17] because they consider \( x \) fixed, and so \( V_1(b_k) \) is order 1.
6.5. Proofs of Propositions 6.1 to 6.3.

Proof of Proposition 6.1. In what follows, all estimates will be uniform over \(a_k, b_k,\) and \(u_k\) satisfying Eq. (6.2).

We begin by following the proof given in [17, Section 6.3]. In particular, we also set \(m = \lfloor (1-t)k \rfloor\) for some \(t \in (0, 1)\) and write the decomposition
\[
\mathbb{P}_{u_k \tilde{n}} \left( S(k) = a_k \tilde{n} + b_k \tilde{n}, H_{\mathbb{H}_{-\rho}}^S > k \right) = \sum_{z \in \mathbb{H}_{\rho}} \mathbb{P}_{u_k \tilde{n}} \left( S(k - m) = z, H_{\mathbb{H}_{-\rho}}^S > k - m \right) \mathbb{P}_{a_k \tilde{n} + b_k \tilde{n}} \left( S'(m) = z, H_{\mathbb{H}_{-\rho}}^{S'} > m \right),
\]
where \(S'\) is distributed as \(-S\). Letting \(x := u_k \tilde{n}\) and \(y := a_k \tilde{n} + b_k \tilde{n}\), and recalling Eq. (6.18) as the needed modification of Lemmas 27 and 28 of [17], we can follow the proof given in [17, Section 6.3] up to their equation (89) to find
\[
\lim_{R \to \infty} \lim_{k \to \infty} \frac{\left( V_1(u_k)V'(b_k) \right)}{k^2} = 0,
\]
where
\[
\Sigma_1(R, k) := \sum_{z \in \mathbb{H}_{\rho}; \|z\| > R \sqrt{k}} \mathbb{P}_{u_k \tilde{n}} \left( S(k - m) = z, H_{\mathbb{H}_{-\rho}}^S > k - m \right) \mathbb{P}_{a_k \tilde{n} + b_k \tilde{n}} \left( S'(m) = z, H_{\mathbb{H}_{-\rho}}^{S'} > m \right).
\]
We are now in a position to apply Proposition 6.7 to the remainder term
\[
\Sigma_2(R, k) := \sum_{z \in \mathbb{H}_{\rho}; \|z\| \leq R \sqrt{k}} \mathbb{P}_{u_k \tilde{n}} \left( S(k - m) = z, H_{\mathbb{H}_{-\rho}}^S > k - m \right) \mathbb{P}_{a_k \tilde{n} + b_k \tilde{n}} \left( S'(m) = z, H_{\mathbb{H}_{-\rho}}^{S'} > m \right)
\]
\[
= \frac{H_0^2 \sqrt{k^3} V_1(u_k)V_1(b_k)(t(1-t))^{3/2}k^3}{(t(1-t))^{3/2}k^3} \sum_{z \in \mathbb{H}_{\rho}; \|z\| \leq R \sqrt{k}} \left( \frac{z \cdot \tilde{n}}{\sqrt{tk}} \left( \frac{z \cdot \tilde{n}}{\sqrt{(1-t)k}} \right) \exp \left( \frac{|z|^2}{2tk} - \frac{a \sqrt{k} \tilde{n}^2}{2(1-t)k} \right) + o(V_1(u_k)V'(b_k)k^{-2}) \right.
\]
\[
= \frac{H_0^2 \sqrt{k^3} V_1(u_k)V_1(b_k)(t(1-t))^{3/2}k^3}{(t(1-t))^{3/2}k^3} \sum_{z \in \mathbb{H}_{\rho}; \|z\| \leq R \sqrt{k}} \left( \frac{z \cdot \tilde{n}}{\sqrt{tk}} \left( \frac{z \cdot \tilde{n}}{\sqrt{(1-t)k}} \right) \exp \left( - \frac{(z \cdot \tilde{n})^2}{2(tk)} - \frac{(z \cdot \tilde{n}^2 - ta \sqrt{k})^2}{2(t(1-t)k)} \right) + o(V_1(u_k)V'(b_k)k^{-2}) \right). \]
Note that, compared to the first display after equation (89) of [17], the only different terms are those involving \(a\) and the little-oh terms: this is a consequence of the modifications in our Proposition 6.7 compared to their Theorem 5. From here, we can follow their proof step-by-step until the end, yielding Eq. (6.3).

Proof of Proposition 6.2. We begin by showing convergence of the finite-dimensional distributions. For this, it is enough to consider sequences \(a_k \in [-A \sqrt{k}, A \sqrt{k}]\) such that \(a_k/\sqrt{k} \to a\), and show that for any \(a \in [-A, A]\), the finite-dimensional distributions converge to the same limit (as then every subsequence has a further subsequence converging to the same distribution).

We proceed by following the arguments as in [23, Section 4], making adaptations as necessary. We begin with [23, Equation 40], which states that for any \(t \in [0, 1)\) and \(B \in \sigma(\{S^{(2)}_i, i \leq kt\})\), we have
\[
\mathbb{P}_{u_k \tilde{n}}(B \mid S(k) = a_k \tilde{n} + b_k \tilde{n}, H_{\mathbb{H}_{-\rho}}^S > k) = \mathbb{E} \left[ h_{u_k,a_k,b_k}(t, X_{k,t}) 1_B \mid H_{\mathbb{H}_{-\rho}}^S > kt \right], \quad (6.20)
\]
where \( X_{k,t} := S([tk]) / \sqrt{k} \) and
\[
h^{(k)}_{u_k,a_k,b_k}(t,w) = \frac{\mathbb{P}_{u_k\bar{n}}(H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}} > kt) \mathbb{P}_{w\sqrt{k}}(S(1-t)k = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > (1-t)k)}{\mathbb{P}_{u_k\bar{n}}(S(k) = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > k)}.\]

From Propositions 6.1 and 6.5, we have
\[
\frac{\mathbb{P}_{u_k\bar{n}}(H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}} > kt)}{\mathbb{P}_{u_k\bar{n}}(S(k) = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > k)} \sim t^{-1/2} e^{a^2/2} \frac{C_1}{C_1} V^t(a_k(b_k))^3/2.
\]

Now, let \( S'(\cdot) \) denote the random walk whose increments are independent copies of \(-X\). Considering the walk \( S(\cdot) \) in reversed time, we have
\[
\mathbb{P}_{w\sqrt{k}}(S((1-t)k) = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > (1-t)k) = \mathbb{P}_{a_k\bar{n}^\perp + b_k\bar{n}}(S'((1-t)k) = w\sqrt{k}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > (1-t)k).
\]

Applying Proposition 6.7 yields
\[
\limsup_{k \to \infty} \sup_{w \in \mathbb{H}_{\bar{n}}} \left| \frac{(1-t)^{3/2}k^{3/2}}{V^t_1(b_k)} \mathbb{P}_{w\sqrt{k}}(S(1-t)k = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > (1-t)k) \right|
\]
uniformly over \( a_k \) and \( b_k \). Altogether, we find
\[
h^{(k)}_{u_k,a_k,b_k}(t,w) = (1 + o(1)) h(a,t,w),
\]
uniformly over \( w \in \mathbb{H}_{\bar{n}} \).

Recall \( D[0,t] \) the space of càdlàg functions from \([0,t]\) to \( \mathbb{R} \). For any bounded and continuous functional \( g_t : D[0,t] \to \mathbb{R} \), Eq. (6.20) gives us
\[
\mathbb{E}_{u_k\bar{n}}[g_t(X_{k,.} \cdot \bar{n}) \mid S(k) = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > k] = (1 + o(1)) \mathbb{E}_{u_k\bar{n}}[g_t(X_{k,.} \cdot \bar{n})h(a,t,X_{k,t}) \mid H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > kt],
\]
where we have used that for fixed \( a \) and \( t \), \( h(a,t,w) \) is uniformly bounded in \( w \). Applying the convergence result [23, Theorem 2] (see also Remark 1 in [23]), we find
\[
\lim_{k \to \infty} \mathbb{E}_{u_k\bar{n}}[g_t(X_{k,.} \cdot \bar{n}) \mid S(k) = a_k\bar{n}^\perp + b_k\bar{n}, H_{\mathbb{H}_{\bar{n}} ^{\mathbb{R}}}^S > k] = \mathbb{E}[g_t(t^{1/2} \mathcal{M}_{\mathbb{H}_{\bar{n}}} \cdot \bar{n})h(a,t,t^{1/2} \mathcal{M}_{\mathbb{H}_{\bar{n}}}^{(1)}(1))],
\]
where \( (\mathcal{M}_{\mathbb{H}_{\bar{n}}}^{(s)}(s))_{s \in [0,1]} \) denotes the Brownian meander in \( \mathbb{H}_{\bar{n}} \) started from the origin. Recall that \( \mathcal{M}_{\mathbb{H}_{\bar{n}}}^{(s)} = W_s\bar{n}^\perp + M_s\bar{n} \), where \( (W_s)_{s \in [0,1]} \) is a standard one-dimensional Brownian motion, \( (M_s)_{s \in [0,1]} \) is a standard one-dimensional Brownian meander, and \( W \) and \( M \) are independent processes. Then the expectation over \( W \) factors out of the right-hand side of Eq. (6.22) as follows:
\[
\mathbb{E}\left[ \frac{H_0}{C_1} (1-t)^{-\frac{3}{2}} M_{t^{1/2} \mathcal{M}_{\mathbb{H}_{\bar{n}}}^{(1)}}(1) \right] \mathbb{E}\left[ \exp\left( -\frac{tN^2}{2(1-t)} + \frac{a\sqrt{t}}{1-t} N - \frac{ta^2}{2(1-t)} \right) \right],
\]
where $\mathcal{N} \sim N(0,1)$. That second expectation evaluates to $\sqrt{1-t}$ (in particular, there is no dependence on $a$ in the above expression). So, letting $C_2 := H_0/C_1$, we’ve found

$$
\lim_{k \to \infty} E_{uk}\left[gt(X_k, \cdot, \hat{n}) \mid S(k) = a_k\hat{n} + b_k\hat{n}, H_{E,\hat{n}}^k > k\right] = E\left[C_2(1-t)^{-\frac{3}{2}}M_1e^{-\frac{c(M_1)^2}{4+\hat{n}^2}}g_t(t^{1/2}M_1)\right].
$$

Thus, for every fixed $t < 1$, we have shown convergence in distribution on $D[0,t]$ for every fixed $t < 1$ to the same limit for every $a$. This in particular implies convergence of all finite dimensional distributions to the same limit for every $a$, as well as tightness on $[0,1-\delta]$, for any $\delta \in (0,1)$. Tightness on $[1-\delta,1]$ follows by applying the exact same arguments for the random walk reversed in time, just as in the end of [23, Section 4].

Thus, we have the convergence of the family of laws $Q_{u_k,a_k,b_k}^k$. The limit may be identified as that of the standard Brownian excursion by taking $S(i) := \hat{S}_1(i)\hat{n} + \hat{S}_2(i)\hat{n}$, where $\hat{S}_1(\cdot)$ and $\hat{S}_2(\cdot)$ are independent simple symmetric random walks on $\mathbb{Z}$.

**Proof of Proposition 6.3.** Following the proof of Theorem 5.1 in [30] until their Equation 59 yields equation Eq. (6.6). Equation (6.5) is then an immediate consequence of Proposition 6.1. \hfill \Box

**Appendix A. Cluster expansion and open contours in the SOS model**

**A.1. Cluster expansion.** Consider the SOS model in a region $\Lambda \subset \mathbb{Z}^2$ without floor, with boundary condition $j \in \mathbb{Z}$. Fix any subset $U$ of the inner boundary of $\Lambda$, and condition on $\{\varphi|U \geq j\}$. Let $\hat{Z}_{\Lambda,U} := \hat{Z}^+_\Lambda U$ denote the partition function of this model. Note that the partition function has no dependence on $j$ and is also unchanged by a replacement of the conditioning with $\{\varphi|U \leq j\}$. [13] proves that there exists a constant $\beta_0 > 0$ such that for all $\beta > \beta_0$, one has

$$
\log \hat{Z}_{\Lambda,U} = \sum_{V \subset \Lambda} f_U(V) \tag{A.1}
$$

for some function $f_U$. Equation (A.1) was proven using the main theorem from [35], which can actually be used to show (with the same proof as in [13]) that the formula holds simultaneously for all subsets $\Lambda' \subset \Lambda$ (with the same $f_U$):

$$
\log \hat{Z}_{\Lambda',U} = \sum_{V \subset \Lambda'} f_U(V) \tag{A.2}
$$

(here, $Z_{\Lambda',U}$ is defined as before, with the understanding that the condition on $U$ only needs to be satisfied on $\Lambda' \cap U$, or in other words $Z_{\Lambda',U} := Z_{\Lambda',U \cap \Lambda'}$). The latter observation leads, using Möbius inversion, to the following formula for $f_U$:

$$
f_U(V) = \sum_{W \subset V} (-1)^{|W|-|V|} \log \hat{Z}_{W,U}. \tag{A.3}
$$

From Eq. (A.3), we can read off some properties of the function $f_U$:

1. $f_U(V)$ only depends on $U$ only through $U \cap V$, and in particular, if $U \cap V = \emptyset$, $f_U(V) = f_0(V)$ for some universal function $f_0$. Moreover, $f_0(V)$ only depends on the “shape” and size of the volume $V$, i.e., $f_0(\cdot)$ is invariant with respect to lattice symmetries.

2. If $V$ is not connected, $f_U(V) = 0$. Indeed, if $V = V_1 \sqcup V_2$, each log-partition function on the right-hand side of Eq. (A.2) splits into the sum of two terms, both of which appear in the global sum with the same frequency of plus and minus signs.

3. There exists a constant $\beta_0 > 0$ such that

$$
\sup_U |f_U(\mathcal{C})| \leq \exp ((\beta - \beta_0)d(\mathcal{C})), \tag{A.4}
$$

where $d(\mathcal{C})$ denotes the cardinality of the smallest connected set of bonds of $\mathbb{Z}^2$ containing all boundary bonds of $\mathcal{C}$ (i.e., bonds connecting $\mathcal{C}$ to $\mathcal{C}^c$), see [20, Proposition 3.9] for details.
A.2. Proof of Proposition 2.1. Recall $\hat{Z}^{j+}_{\Lambda;U}$ from above Eq. (A.1), and define $\hat{Z}^{j-}_{\Lambda;U}$ similarly but with the conditioning $\{\varphi|_U \leq j\}$ (as noted above Eq. (A.1), $\hat{Z}^{j+}_{\Lambda;U} = \hat{Z}^{j-}_{\Lambda;U}$. The different notation is mostly for clarity in the calculations that follow). Recall also $\Lambda^+_\gamma, \Lambda^-_\gamma, \Delta^+_\gamma$, and $\Delta^-_\gamma$ from above Proposition 2.1.

Observe that

$$\hat{Z}^\xi_{\Lambda} = \sum_{\gamma} e^{-\beta|\gamma|} \hat{Z}^{j+}_{\Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma}}, \quad \text{(A.5)}$$

This is a simple consequence of the following identity: for any $u, v \in \mathbb{Z}^2$ such that $\varphi_u \geq 1$ and $\varphi_v \leq 0$, we have

$$|\varphi_u - \varphi_v| = |(\varphi_u - 1) - (\varphi_v - 0) + (1 - 0)| = (\varphi_u - 1) + (0 - \varphi_v) + 1.$$  

Now, cluster-expanding each of the partition functions on the right-hand side of Eq. (A.5) yields

$$\hat{Z}^{1}_{\Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma}} = \exp \left( \sum_{C \subset \Lambda^+_{\gamma}} f_0(C) + \sum_{C \subset \Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma}} f^{\Delta^+_{\gamma}}(C) + \sum_{C \subset \Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma}} f^{\Delta^+_{\gamma}}(C) \right)$$

$$= e^{\sum_{C \subset \Lambda} f_0(C)} \exp \left( \Psi_{\Lambda}(\gamma) \right) = \hat{Z}^0_{\Lambda} \exp \left( \Psi_{\Lambda}(\gamma) \right), \quad \text{(A.6)}$$

where

$$\Psi_{\Lambda}(\gamma) := - \sum_{C \subset \Lambda, \Delta^+_{\gamma}, \Delta^-_{\gamma} \neq \emptyset} f_0(C) + \sum_{C \subset \Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma} \neq \emptyset} f^{\Delta^+_{\gamma}}(C) + \sum_{C \subset \Lambda^+_{\gamma}, \Delta^+_{\gamma}, \Delta^-_{\gamma} \neq \emptyset} f^{\Delta^+_{\gamma}}(C).$$

Defining

$$\phi(C; \gamma) := -f_0(C) + f^{\Delta^+_{\gamma}}(C) \mathbb{1}_{\{C \cap \Delta^+_{\gamma} = \emptyset\}} + f^{\Delta^+_{\gamma}}(C) \mathbb{1}_{\{C \cap \Delta^-_{\gamma} = \emptyset\}}, \quad \text{(A.7)}$$

it follows from Eqs. (A.5) and (A.6) that

$$\hat{Z}^\xi_{\Lambda} = \hat{Z}^0_{\Lambda} \sum_{\gamma} \exp \left( - \beta|\gamma| + \sum_{C \subset \Lambda, \Delta^+_{\gamma}, \Delta^-_{\gamma} \neq \emptyset} \phi(C; \gamma) \right).$$

Equations (2.1) and (2.3) follow immediately. From the definition of $\phi$ in Eq. (A.7) and the properties of $f_U$ from Appendix A.1, we obtain Properties (i) to (iv) of the proposition.

A.3. Law of an open contour in the SOS model with floor. Consider the SOS model $\pi^\xi_R$ in a domain $R$ with floor at 0 and boundary condition $\xi \in \{h-1, h\}$ inducing a unique open $h$-contour $\gamma$. Consider $h = \left[\frac{1}{12} \log L \right] - n$, for $n \in \mathbb{N}$ fixed relative to $L$, and assume that the domain $R$ satisfies $|R| \leq L^{4+2\epsilon}$ and $|\partial R| \leq L^{4+2\epsilon}$. In this subsection, we derive an asymptotic expression for the law of the random contour $\gamma$ in terms of the no-floor law $\tilde{\pi}^\xi_R$.

For a subset $U$ of the inner boundary of $R$, recall $\hat{Z}^{j+}_{R;U}$ as in Appendix A.2, and define $Z^{j+}_{R;U}$ similarly but with a floor at 0. As in Eq. (A.5), we can write the law of $\gamma$ in terms of the partition functions above and below it:

$$\tilde{\pi}^\xi_R(\gamma) = e^{-\beta|\gamma|} \hat{Z}^{h+}_{R^+_\gamma, \Delta^+_{\gamma}, \Delta^-_{\gamma}} \hat{Z}^{h-1,-}_{R^-_{\gamma}, \Delta^-_{\gamma}}.$$

The partition functions $Z^{j+}_{R;U}$ can be related to the corresponding partition function $\hat{Z}^{j+}_{R;U}$ of the model without floor via the following (taken from [14, Prop. A.1]):

$$Z^{j+}_{R;U} = \hat{Z}^{j+}_{R;U} \exp(-\tilde{\pi}(\varphi_o > j)|R| + o(1)),$$
where $\hat{\pi}$ is the infinite volume measure obtained as the thermodynamic limit of $\pi_A^n$, see [3]. It follows that
\[
\pi^\xi_R(\gamma) \propto e^{-\beta|\gamma|} \hat{\pi}^h \exp \left( - \left( \hat{\pi}(\varphi_o > h - 1) - \hat{\pi}(\varphi_o > h) \right) |R^-_n| + o(1) \right),
\]
where we used $|R^-_n| = |R| - |R^-|$ and dropped the term $|R|\hat{\pi}(\varphi_o > h)$, which is independent of $\gamma$.

It was proved in [14, Lemma 2.4] that there exists a constant $c_\infty := c_\infty(\beta)$ such that, for $j \geq 1$,
\[
\hat{\pi}(\varphi_o \geq j) = c_\infty e^{-4\beta j} + O(e^{-6\beta j}).
\]
Define $a(L) := \frac{1}{L} \log L - \frac{1}{L^2} \log L$, $\lambda := \lambda(L) = c_\infty e^{4\alpha(L)}(1 - e^{-4\beta})$ and $\lambda(n) := \lambda e^{4\beta n}$. We have
\[
\hat{\pi}(\varphi_o > h - 1) - \hat{\pi}(\varphi_o > h) = c_\infty e^{-4\beta h}(1 - e^{-4\beta}) + O(e^{-6\beta h})
\]
\[
= c_\infty e^{4\beta n}L^{-\alpha(L)}\log L + 4\beta n(1 - e^{-4\beta}) + O(e^{-6\beta h}) = \lambda(n)/L + O(e^{-6\beta h}).
\]

Putting everything together, we find
\[
\pi^\xi_R(\gamma) \propto \pi^\xi_R(\gamma) \exp \left( - \frac{\lambda(n)}{L} \right) A(\gamma) + o(1),
\]
where $A(\gamma) = |R^-|$ is the area under $\gamma$ in $R$.

**Appendix B. Proof of Proposition 2.14 and Eq. (5.17)**

**Proof of Proposition 2.14.** Fix $\delta \in (0, 1)$. We begin by showing that for any $y \in Y^\bullet_\delta \setminus \{0\}$, we have $f(hy) = 1$, where
\[
f(h) := \sum_{\Gamma \in A} W^h(\Gamma).
\]
Let
\[
W_{in}^\bullet := \{sh_y : y \in Y^\bullet_\delta \setminus \{0\}, s \in [0, 1]\}
\]
and
\[
W_{out}^\bullet := \{sh_y : y \in Y^\bullet_\delta \setminus \{0\}, s \in [1, 1 + \beta \nu \|h_y\|^{-1}]\},
\]
where $\nu$ is the same as in Proposition 2.12. In words, $W_{in}^\bullet$ is the sector of the Wulff shape $W$ where the sector boundary (part of $\partial W$) is $\{h_y : y \in Y^\bullet_\delta\}$, while $W_{out}^\bullet$ lies right outside of $W$ along the continuation of such sector (note that $W_{in}^\bullet \subset W$ comes from $0 \in W$ and the convexity of $W$).

In what follows, we let $C := C(\beta) > 0$ denote a constant that may depend on $\beta$ and may change from line-to-line. We begin with two claims.

**Claim B.1.** For all $h \in W_{in}^\bullet \cup W_{out}^\bullet$, we have
\[
\sum_{\Gamma \in A \cup A_R} W^h(\Gamma) < \infty,
\]
and the series converges uniformly over $h$. In particular, $f(h)$ is continuous on $W_{in}^\bullet \cup W_{out}^\bullet$.

**Proof of Claim B.1.** We will use the following bound many times: from Eq. (2.16), we have that for any $\epsilon \in (0, 1)$ and for all $\beta > 0$ sufficiently large, there exists a constant $C := C(\epsilon, \beta) > 0$ such that for all $y \in \mathbb{Z}^2$
\[
G(y) \leq C e^{-\tau(\beta)(y) + \epsilon \|y\|_1}.
\]

For $h \in W_{in}^\bullet \cup W_{out}^\bullet$, we have $h \cdot y - \tau(\beta)(y) \leq \frac{\beta \nu}{2} \|y\|_2 \leq \frac{\beta \nu}{2} \|y\|_1$ for all $y \in \mathbb{Z}^2 \setminus \{0\}$. Then Proposition 2.12 and Eq. (B.1) yield
\[
\sum_{\Gamma \in A \cup A_R \atop \Gamma \ni y} W^h(\Gamma) \leq e^{hy}G(y \mid \text{Cpts}(\Gamma) = 0) \leq C \exp \left( - \left( \frac{\beta \nu}{2} - \epsilon \right) \|y\|_1 \right).
\]
Take $\varepsilon < \nu \beta / 2$. Then since $A \subset A_L$, the sequence of continuous functions

$$f_N(h) := \sum_{y \in \mathbb{Z}^2 \setminus \{0\}} \sum_{\Gamma \in A} W^h(\Gamma)$$

converges uniformly to $f(h)$, and so the claim follows.

**Claim B.2.** For $h \in \mathcal{W}_{\text{in}}$, $f(h) \leq 1$. For $h \in \mathcal{W}_{\text{out}}$, $f(h) \geq 1$.

**Proof of Claim B.2.** Similar to Eq. (B.2), we have the following for any $h \in \mathcal{W}_{\text{in}} \cup \mathcal{W}_{\text{out}}$:

$$\sum_{y \in \mathcal{Y}^\bullet} \sum_{\Gamma = [\chi, \eta]} \sum_{\gamma \in \gamma} W^h(\Gamma) 1_{\{\text{pts}(\Gamma) < 2\}} \leq C + C \sum_{y \in \mathcal{Y}^\bullet} e^{h \cdot y - \beta y \|y\|} G(y) < \infty. \quad (B.3)$$

Using the factorization of $q(\Gamma)$ (2.18) and (B.3), we have

$$\sum_{y \in \mathcal{Y}^\bullet} e^{h \cdot y} G(y) \leq C + \sum_{y \in \mathcal{Y}^\bullet} \sum_{\Gamma^{(m)} \in A_L^{(m)} \Gamma^{(1)} \in A^{(1)} \cdots \Gamma^{(n)} \in A_L^{(n)}} \sum_{\gamma \in \gamma} W^h(\Gamma^{(1)}) \cdots W^h(\Gamma^{(n)}) 1_{X(\Gamma) = y}$$

$$= C + \left( \sum_{\Gamma^{(1)} \in A_L} W^h(\Gamma^{(1)}) \right) \left( \sum_{\Gamma^{(n)} \in A_L} W^h(\Gamma^{(n)}) \right) \sum_{m \geq 1} f(h)^m$$

$$=: C + B_L(h) B_R(h) \sum_{m \geq 1} f(h)^m,$$

where in the inequality we replaced $\mathcal{Y}^\bullet$ by the full cone $\mathcal{Y}$ and in the next line we exchanged the sums and used the fact that an animal with at least one cone-point must necessarily satisfy $X(\Gamma) \in \mathcal{Y}$. Hence, we have the chain of inequalities

$$\sum_{y \in \mathcal{Y}^\bullet} e^{h \cdot y} G(y) \leq C + B_L(h) B_R(h) \sum_{m \geq 1} f(h)^m \leq C + \sum_{y \in \mathbb{Z}^2} e^{h \cdot y} G(y). \quad (B.4)$$

Since $B_L(h), B_R(h) < \infty$ by Claim B.1, finiteness of the middle term in (B.4) depends only on the convergence of $\sum_{m \geq 1} f(h)^m$.

Now, if $h$ is in the interior of $\mathcal{W}_{\text{in}}$ (and thus in the interior of $\mathcal{W}$), the rightmost term in (B.4) converges due to the second definition of $\mathcal{W}$, and thus $\sum_{m \geq 1} f(h)^m < \infty$, and thus $f(h) < 1$. By continuity of $f$, we get $f(h) \leq 1$ on $\mathcal{W}_{\text{in}}$. On the other hand, if $h$ is in the interior of $\mathcal{W}_{\text{out}}$, the leftmost term in (B.4) diverges (by the same argument that showed the equivalence of the two definitions of $\mathcal{W}$ in subsection 2.6), and thus $\sum_{m \geq 1} f(h)^m = \infty$, and thus $f(h) \geq 1$. □

By continuity of $f$, it follows that $f(h) = 1$ on $\mathcal{W}_{\text{in}} \cap \mathcal{W}_{\text{out}}$, i.e., $f(h_y) = 1$ for $y \in \mathcal{Y}^\bullet$. This proves that $\mathbb{P}^h$ defines a probability distribution on $A$ for $y \in \mathcal{Y}^\bullet$.

Next, we show (2.25). For any $y \in \mathcal{Y}^\bullet \setminus \{0\}$, define

$$F(y) := f(h_y) = \sum_{\Gamma \in A} \mathbb{P}^h(\Gamma).$$

From above, we know that $F(y) \equiv 1$. Since $\tau_\beta$ is analytic outside of the origin (Proposition 2.4), it is in particular twice-differentiable, and thus so is $\mathbb{P}^h(\Gamma)$ with $\nabla \mathbb{P}^h(\Gamma) = J_{h_y} X(\Gamma) \mathbb{P}^h$, where $J_{h_y}$ is the Jacobian matrix of $y \mapsto h_y$. Using the same argument as in Claim B.1 and boundedness
of $J_{h_0}$, we find that $\sum_{\Gamma \in A} J_{h_0} X(\Gamma)^{\mathcal{P}h}(\Gamma)$ converges uniformly for $y \in \mathbb{Z}^2 \setminus \{0\}$, and thus we can differentiate $F(y)$ under the sum to get

$$0 = \nabla F(y) = \sum_{\Gamma \in A} J_{h_0} X(\Gamma)^{\mathcal{P}h}(\Gamma) = J_{h_0} \sum_{\Gamma \in A} X(\Gamma)^{\mathcal{P}h}(\Gamma) = J_{h_0}^\mathcal{P}h \{X(\Gamma)\},$$

i.e., $E^h \{X(\Gamma)\} \in \text{Ker}(J_{h_0})$. Differentiating the relation $h_0 \cdot y = \tau_\beta(y)$, we see that $y \in \text{Ker}(J_{h_0})$, and so if $J_{h_0}$ is non-degenerate we get that $E^h \{X(\Gamma)\}$ is collinear to $y$, i.e., $E^h \{X(\Gamma)\} = \alpha y$ for some scalar $\alpha = \alpha(y)$. Finally, the non-degeneracy condition can be removed: since $\tau_\beta$ is analytic (outside of the origin), the points at which $J_{h_0}$ is fully degenerate form a discrete set, and since collinearity must hold outside of such set and $E^h \{X(\Gamma)\}$ is continuous, we get the result for any $y \in \mathbb{R}^d \setminus \{0\}$.

It remains to show the exponential tail decay (2.26), which actually holds for the weights $W^h$ for any $h \in W$. Below, $\nu_1 > 0$ and $\nu_2 > 0$ denote constants independent of $h$ and $\beta$, while $C_1 > 0$ and $C_2 > 0$ may depend on $\beta$ but not $h$. We express the left-hand side of (2.26) as follows:

$$\sum_{\Gamma \in A \cup A_R} W^h(\Gamma) \leq \sum_{|y|_1 \leq k/2} \sum_{|y|_1 \leq k/2} e^{h \cdot y} G(y \mid |\text{Cpts}(\Gamma)| = 0) \leq e^{-(c\beta - \epsilon)|y|_1} \leq C_1 e^{-\nu_1 \beta k}.$$  

The second term in (B.5) is bounded by

$$\sum_{|y|_1 < k/2} e^{h \cdot y} G(y \mid |\gamma| \geq k) \leq \sum_{|y|_1 < k/2} e^{|\gamma|} \left( e^{-\nu_0 \beta k/|y|_1} \right) G(y) \leq C_2 e^{-\nu_0 \beta k} \sum_{|y|_1 < k/2} e^{|y|_1} \leq C_2 e^{-\nu_2 \beta k},$$

where the first inequality uses (2.21) and $h \cdot y \leq \tau_\beta(y)$, and the second inequality uses (B.1).

**Proof of Eq. (5.17).** Define the partition function over contours $\gamma \subset \mathbb{Z}^2$ such that $\gamma : 0 \to y$:

$$G^\text{full}_D(y) := \sum_{\gamma : 0 \to y} q_D(\gamma).$$

Eq. (2.13) gives us $q_D(\gamma) \leq \exp(6e^{-\chi_\beta} |\gamma|) q(\gamma)$, from which we find

$$G^\text{full}_D(y) \leq \sum_{\gamma : 0 \to y} e^{6e^{-\chi_\beta} |\gamma|} q(\gamma) \leq \sum_{\gamma : 0 \to y} e^{6e^{-\chi_\beta} |\gamma|} G(y) + \sum_{\gamma : 0 \to y} e^{6e^{-\chi_\beta} |\gamma|} q(\gamma) \leq e^{7e^{-\chi_\beta} |y|_1} G(y),$$

where in the final inequality we used Eq. (2.21). We then obtain the analogue of Eq. (B.1) by applying Eq. (B.1) to the above: for any $\epsilon \in (0, 1)$, there exists $\beta_0 := \beta_0(\epsilon) > 0$ such that for all $\beta \geq \beta_0$ and for all $y \in \mathbb{Z}^2$,

$$G^\text{full}_D(y) \leq Ce^{-\tau_\beta(y) + \epsilon |y|_1}$$  

for some $C := C(\epsilon, \beta) > 0$. Similarly, we obtain the length and contour cone-points bound Lemma 2.11 and the animal cone-points bound Proposition 2.12 with $G^\text{full}_D(y)$ replacing $G(y)$ there. With these bounds, we may re-run the proof of Eq. (2.26), replacing $W^h(\Gamma)$ with $e^{h \cdot X(\Gamma)} q_D(\Gamma)$.  

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P. Caddeo  
Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.  
*Email address*: patrizio.caddeo@courant.nyu.edu

Y. H. Kim  
Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.  
*Email address*: yujin.kim@courant.nyu.edu

E. Lubetzky  
Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA.  
*Email address*: eyal@courant.nyu.edu