

CUTOFF FOR THE EAST PROCESS

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ABSTRACT. The East process is a 1D kinetically constrained interacting particle system, introduced in the physics literature in the early 90's to model liquid-glass transitions. Spectral gap estimates of Aldous and Diaconis in 2002 imply that its mixing time on L sites has order L . We complement that result and show cutoff with an $O(\sqrt{L})$ -window.

The main ingredient is an analysis of the *front* of the process (its rightmost zero in the setup where zeros facilitate updates to their right). One expects the front to advance as a biased random walk, whose normal fluctuations would imply cutoff with an $O(\sqrt{L})$ -window. The law of the process behind the front plays a crucial role: Blondel showed that it converges to an invariant measure ν , on which very little is known. Here we obtain quantitative bounds on the speed of convergence to ν , finding that it is exponentially fast. We then derive that the increments of the front behave as a stationary mixing sequence of random variables, and a Stein-method based argument of Bolthausen ('82) implies a CLT for the location of the front, yielding the cutoff result.

Finally, we supplement these results by a study of analogous kinetically constrained models on trees, again establishing cutoff, yet this time with an $O(1)$ -window.

1. INTRODUCTION

The East process is a one-dimensional spin system that was introduced in the physics literature by Jäckle and Eisinger [2121, 21] in 1991 to model the behavior of cooled liquids near the glass transition point, specializing a class of models that goes back to [1919, 19]. Each site in \mathbb{Z} has a $\{0, 1\}$ -value (vacant/occupied), and, denoting this configuration by ω , the process attempts to update ω_x to 1 at rate $0 < p < 1$ (a parameter) and to 0 at rate $q = 1 - p$, only accepting the proposed update if $\omega_{x-1} = 0$ (a “kinetic constraint”).

It is the properties of the East process before and towards reaching equilibrium — it is reversible w.r.t. π , the product of Bernoulli(p) variables — which are of interest, with the standard gauges for the speed of convergence to stationarity being the inverse spectral-gap and the total-variation mixing time (gap^{-1} and T_{mix}) on a finite interval $\{0, \dots, L\}$, where we fix $\omega_0 = 0$ for ergodicity (postponing formal definitions to §2). That the spectral-gap is uniformly bounded away from 0 for any $p \in (0, 1)$ was first proved in a beautiful work of Aldous and Diaconis [33, 3] in 2002. This implies that T_{mix} is of order L for any fixed threshold $0 < \epsilon < 1$ for the total-variation distance from π .

For a configuration ω with $\sup\{x : \omega_x = 0\} < \infty$, call this rightmost 0 its *front* $X(\omega)$; key questions on the East process $\omega(t)$ revolve the law μ^t of the sites behind the front at time t , basic properties of which remain unknown. One can imagine that the front advances to the right as a biased walk, behind which $\mu^t \approx \pi$ (its trail is mixed). Indeed, if one (incorrectly!) ignores dependencies between sites as well as the randomness in the position of the front, it is tempting to conclude that μ^t converges to π , since upon updating a site x its marginal is forever set to Bernoulli(p). Whence, the positive vs. negative increments to $X(\omega)$ would have rates q (a 0-update at $X(\omega) + 1$) vs. pq (a 1-update at $X(\omega)$ with a 0 at its left), giving the front an asymptotic speed $v = q^2 > 0$.

Of course, ignoring the irregularity near the front is problematic, since it is precisely the distribution of those spins that governs the speed of the front (hence mixing). Still, just as a biased random walk, one expects the front to move at a positive speed with

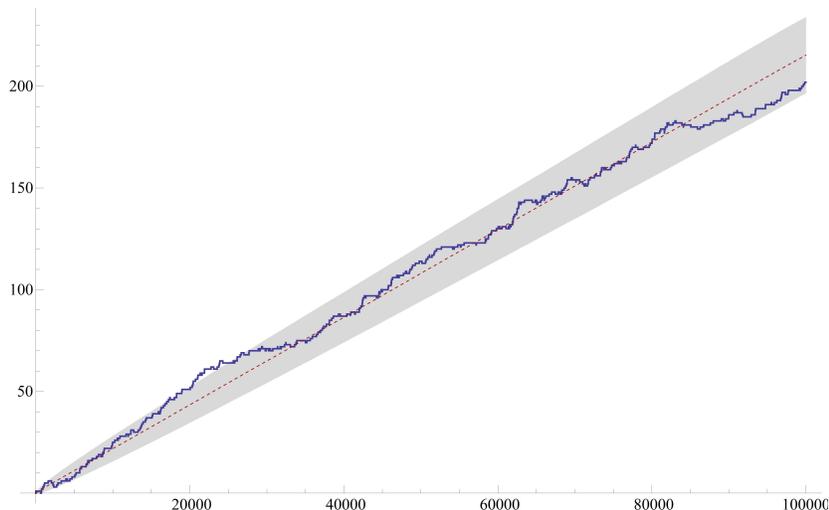


Figure 1: Trajectory of the front of an East process for $p = \frac{1}{4}$ along a time interval of 10^5 , vs. its mean and standard deviation window.

normal fluctuations, whence its concentrated passage time through an interval would imply total-variation *cutoff* — a sharp transition in mixing — within an $O(\sqrt{L})$ -window.

To discuss the behavior behind the front, let Ω_F denote the set of configurations ω^F on the negative half-line \mathbb{Z}_- with a fixed 0 at the origin, and let $\omega^F(t)$ evolve via the East process constantly re-centered (shifted by at most 1) to keep its front at the origin. Blondel [55, 5] showed (see Theorem 2.1) that the process $\omega^F(t)$ converges to an invariant measure ν , on which very little is known, and that $\frac{1}{t}X(\omega(t))$ converges in probability to a positive limiting value v as $t \rightarrow \infty$ (an asymptotic velocity) given by the formula

$$v = q - pq^* \quad \text{where} \quad q^* := \nu(\omega_{-1} = 0).$$

(We note that $q < q^* < q/p$ by the invariance of the measure ν and the fact that $v > 0$.)

The East process $\omega(t)$ of course entails the joint distribution of $\omega^F(t)$ and $X(\omega(t))$; thus, it is crucial to understand the dependencies between these as well as the rate at which $\omega^F(t)$ converges to ν as a prerequisite for results on the fluctuations of $X(\omega(t))$.

Our first result confirms the biased random walk intuition for the front of the East process $X(\omega(t))$, establishing a CLT for its fluctuations around vt (illustrated in Fig. 1).

Theorem 1. *There exists a non-negative constant $\sigma_* = \sigma_*(p)$ such that for all $\omega \in \Omega_F$,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} X(\omega(t)) = v \quad \mathbb{P}_\omega\text{-a.s.}, \quad (1.1)$$

$$\mathbb{E}_\omega [X(\omega(t))] = vt + O(1), \quad (1.2)$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}_\omega (X(\omega(t))) = \sigma_*^2. \quad (1.3)$$

Moreover, $X(\omega(t))$ obeys a central limit theorem:

$$\frac{X(\omega(t)) - vt}{\sqrt{t}} \xrightarrow{d} \mathcal{N}(0, \sigma_*^2) \quad \text{w.r.t. } \mathbb{P}_\omega \text{ as } t \rightarrow \infty. \quad (1.4)$$

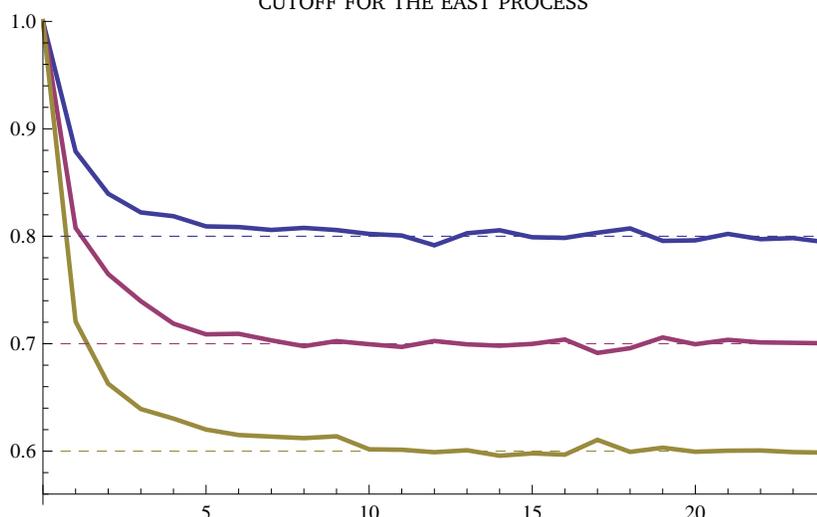


Figure 2: The invariant measure ν behind the front of the East process (showing $\nu(\omega_{-i} = 0)$) simulated via Monte-Carlo for $p \in \{0.2, 0.3, 0.4\}$.)

A key ingredient for the proof is a quantitative bound on the rate of convergence to ν , showing that it is exponentially fast (Theorem 3.1). We then show that the increments

$$\xi_n := X(\omega(n)) - X(\omega(n-1)) \quad (n \in \mathbb{N}) \quad (1.5)$$

behave (after an initial burn-in time) as a stationary sequence of weakly dependent random variables (Corollary 3.2), whence one can apply an ingenious Stein's-method based argument of Bolthausen [66, 6] from 1982 to derive the CLT.

Moving our attention to finite volume, recall that the *cutoff phenomenon* (coined by Aldous and Diaconis [22, 2]; see [11, 11616, 16] as well as [1414, 14] and the references therein) describes a sharp transition in the convergence of a finite Markov chain to stationarity: over a negligible period of time (the cutoff window) the distance from equilibrium drops from near 1 to near 0. Formally, a sequence of chains indexed by L has cutoff around t_L with window $w_L = o(t_L)$ if $T_{\text{mix}}(L, \epsilon) = t_L + O_\epsilon(w_L)$ for any fixed $0 < \epsilon < 1$.

It is well-known (see, e.g., [1515; 15, Example 4.46]) that a biased random walk with speed $v > 0$ on an interval of length L has cutoff at $v^{-1}L$ with an $O(\sqrt{L})$ -window due to normal fluctuations. Recalling the heuristics that depicts the front of the East process as a biased walk flushing a law $\mu^t \approx \pi$ in its trail, one expects precisely the same cutoff behavior. Indeed, the CLT in Theorem 1 supports a result exactly of this form.

Theorem 2. *The East process on $\Lambda = \{1, 2, \dots, L\}$ with parameter $0 < p < 1$ exhibits cutoff at $v^{-1}L$ with an $O(\sqrt{L})$ -window: for any fixed $0 < \epsilon < 1$ and large enough L ,*

$$T_{\text{mix}}(L, \epsilon) = v^{-1}L + O\left(\Phi^{-1}(1 - \epsilon)\sqrt{L}\right),$$

where Φ is the c.d.f. of $\mathcal{N}(0, 1)$ and the implicit constant in the $O(\cdot)$ depends only on p .

While these new results relied on a refined understanding of the convergence of the process behind the front to its invariant law ν (shown in Fig. 2), various basic questions on ν remain unanswered. For instance, are the single-site marginals of ν monotone in the distance from the front? What are the correlations between adjacent spins? Can one explicitly obtain $q^* = \nu(\omega_{-1} = 0)$, thus yielding an expression for the velocity v ? For the latter, we remark that the well-known upper bound on T_{mix} in terms

of the spectral-gap (Eq. (2.2)), together with Theorem 2, gives the lower bound (cf. also [1111, 11])

$$v \geq \limsup_{L \rightarrow \infty} \frac{\text{gap}(\mathcal{L}_{[0,L]})}{\log(1/(p \wedge q))} = \frac{\text{gap}(\mathcal{L})}{\log(1/(p \wedge q))}.$$

Finally, we accompany the concentration for $X(\omega(t))$ and cutoff for the East process by analogous results — including cutoff with an $O(1)$ -window — on the corresponding kinetically constrained models on trees, where a site is allowed to update (i.e., to be reset into a Bernoulli(p) variable) given a certain configuration of its children (e.g., all-zeros/at least one zero/etc.). These results are detailed in §5 (Theorems 5.1–5.2).

Remark. *The concentration and cutoff results for the kinetically constrained models on trees (Theorems 5.1–5.2) do not apply to every scale but rather to infinitely many scales, as is sometimes the case in the context of tightness for maxima of branching random walks or discrete Gaussian Free Fields; see, e.g., [77, 71717, 17] as well as the beautiful method in [88, 899, 9] to overcome this hurdle for certain branching random walks. Indeed, similarly to the latter, one of the models here gives rise to a distributional recursion involving the maximum of i.i.d. copies of the random variable of interest, plus a non-negative increment. Unfortunately, unlike branching random walks, here this increment is not independent of those two copies, and extending our analysis to every scale appears to be quite challenging.*

2. PRELIMINARIES AND TOOLS FOR THE EAST PROCESS

2.1. Setup and notation. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ and let $\Omega^* \subset \Omega$ consist of those configurations $\omega \in \Omega$ such that the variable $X(\omega) := \sup\{x : \omega_x = 0\}$ is finite. In the sequel, for any $\omega \in \Omega^*$ we will often refer to $X(\omega)$ as the *front* of ω . Given $\Lambda \subset \mathbb{Z}$ and $\omega \in \Omega$ we will write ω_Λ for the restriction of ω to Λ .

- (i) *The East process.* For any $\omega \in \Omega$ and $x \in \mathbb{Z}$ let $c_x(\omega)$ denote the indicator of the event $\{\omega_{x-1} = 0\}$. We will consider the Markov process $\{\omega(t)\}_{t \geq 0}$ on Ω with generator acting on local functions (i.e. depending on finitely many coordinates) $f : \Omega \mapsto \mathbb{R}$ given by

$$\mathcal{L}f(\omega) = \sum_{x \in \mathbb{Z}} c_x(\omega) [\pi_x(f)(\omega) - f(\omega)],$$

where $\pi_x(f)(\omega) := pf(\omega^{(x,1)}) + qf(\omega^{(x,0)})$ and $\omega^{(x,1)}, \omega^{(x,0)}$ are the configurations in Ω obtained from ω by fixing equal to 1 or to 0 respectively the coordinate at x . In the sequel the above process will be referred to as the *East process on \mathbb{Z}* and we will write $\mathbb{P}_\omega(\cdot)$ for its law when the starting configuration is ω . Average and variance w.r.t. to $\mathbb{P}_\omega(\cdot)$ will be denoted by $\mathbb{E}_\omega[\cdot]$ and $\text{Var}_\omega(\cdot)$ respectively. Similarly we will write $\mathbb{P}_\omega^t(\cdot)$ and $\mathbb{E}_\omega^t[\cdot]$ for the law and average at a fixed time $t > 0$. If the starting configuration is distributed according to an initial distribution η we will simply write $\mathbb{P}_\eta(\cdot)$ for $\int d\eta(\omega)\mathbb{P}_\omega(\cdot)$ and similarly for $\mathbb{E}_\eta[\cdot]$.

It is easily seen that the East process has the following graphical representation. To each $x \in \mathbb{Z}$ we associate a rate-1 Poisson process and, independently, a family of independent Bernoulli(p) random variables $\{s_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$. We assume independence as x varies in \mathbb{Z} . That fixes the probability space. Notice that almost surely all the occurrences $\{t_{x,k}\}_{k \in \mathbb{N}, x \in \mathbb{Z}}$ are different. On the above probability we construct a Markov process according to the following rules. At each time $t_{x,n}$ the site x queries the state of its own constraint c_x . If and only if the constraint is

satisfied ($c_x = 1$) then $t_{x,n}$ is called a *legal ring* and the configuration resets its value at site x to the value of the corresponding Bernoulli variable $s_{x,n}$. Using the graphical construction it is simple to see that if $\omega \in \Omega^*$ then

$$\mathbb{P}_\omega(\omega(t) \in \Omega^* \forall t \geq 0) = 1.$$

- (ii) *The half-line East process.* Consider now $a \in \mathbb{Z}$ and let Ω^a consist of those configurations $\omega \in \Omega$ with a *leftmost* zero at a . Clearly, for any $\omega \in \Omega^a$, $\mathbb{P}_\omega(\omega(t) \in \Omega^a \forall t > 0) = 1$ because $c_x(\omega) = 0$ for any $x \leq a$. We will refer to the corresponding process in Ω^a as the East process on the half-line (a, ∞) . Notice that in this case the variable at $a + 1$ will always be unconstrained because $c_a(\omega) = 1$ for all $\omega \in \Omega^a$. The corresponding generator will be denoted by $\mathcal{L}_{(a, \infty)}$.
- (iii) *The finite volume East process.* Finally, if $\Lambda \subset \mathbb{Z}$ is a discrete interval of the form $\Lambda = [a+1, \dots, a+L]$, the projection on $\Omega_\Lambda \equiv \{0, 1\}^\Lambda$ of the half-line East process on (a, ∞) is a continuous time Markov chain because each vertex $x \in \Lambda$ only queries the state of the spin to its left. In the sequel the above chain will be referred to as the *East process* in Λ . Let \mathcal{L}_Λ denote the corresponding generator.

The main properties of the above processes can be summarized as follows (cf. [1818, 18] for a survey). They are all ergodic and reversible w.r.t. to the product Bernoulli(p) measure π (on the corresponding state space). Their generators $\mathcal{L}, \mathcal{L}_{(a, \infty)}, \mathcal{L}_\Lambda$ are self-adjoint operators on $L^2(\pi)$ satisfying the following natural ordering:

$$\text{gap}(\mathcal{L}) \leq \text{gap}(\mathcal{L}_{(a, \infty)}) \leq \text{gap}(\mathcal{L}_\Lambda).$$

Remark. By translation invariance the value of $\text{gap}(\mathcal{L}_{(a, \infty)})$ does not depend on a and, similarly, $\text{gap}(\mathcal{L}_\Lambda)$ depends only on the cardinality of Λ .

As mentioned before, the fact that $\text{gap}(\mathcal{L}) > 0$ (but only for $p \sim 1$) was first proved by Aldous and Diaconis [33, 3], where it was further shown that

$$e^{-(\frac{1}{\log 2} + o(1)) \log^2(1/q)} \leq \text{gap}(\mathcal{L}) \leq e^{-(\frac{1}{2 \log 2} + o(1)) \log^2(1/q)} \quad \text{as } q \downarrow 0, \quad (2.1)$$

the order of the exponent in the lower bound matching non-rigorous predictions in the physics literature. The positivity of $\text{gap}(\mathcal{L})$ was rederived and extended to all $p \in (0, 1)$ in [1010, 10] by different methods, and the correct asymptotics of the exponent as $q \downarrow 0$ — matching the *upper bound* in (2.1) — was very recently established in [1111, 11]. It is easy to check (e.g., from [1010, 10]) that $\lim_{p \rightarrow 0} \text{gap}(\mathcal{L}) = 1$, a fact that will be used later on.

For the East process in Λ it is natural to consider its mixing times $T_{\text{mix}}(L, \epsilon)$, $\epsilon \in (0, 1)$, defined by

$$T_{\text{mix}}(L, \epsilon) = \inf \left\{ t : \max_{\omega \in \Omega_\Lambda} \|P_\omega^t(\cdot) - \pi\| \leq \epsilon \right\},$$

where $\|\cdot\|$ denotes total-variation distance. It is a standard result for reversible Markov chains (see e.g. [44, 42222, 222424, 24]) that

$$T_{\text{mix}}(L, \epsilon) \leq \frac{1}{2} \text{gap}(\mathcal{L}_\Lambda)^{-1} \left(2 + \log \frac{1}{\pi_\Lambda^*} \right) \log \frac{1}{\epsilon}, \quad (2.2)$$

where $\pi_\Lambda^* := \min_{\omega \in \Omega_\Lambda} \pi(\omega)$. In particular $T_{\text{mix}}(L, \epsilon) \leq c(p)L \log 1/\epsilon$. A lower bound which also grows linearly in the length L of the interval Λ follows easily from the *finite speed of information propagation*: If we run the East model in Λ starting from the configuration of $\omega \equiv 1$ except for a zero at the origin, then, in order to create zeros near the right boundary of Λ a sequence of order L of successive rings of the Poisson clocks at consecutive sites must have occurred. That happens with probability $O(1)$ iff we allow a time which is linear in L (see §2.4 and in particular Lemma 2.6).

2.2. The process behind the front. Given two probability measures ν, μ on Ω and $\Lambda \subset \mathbb{Z}$ we will write $\|\mu - \nu\|_\Lambda$ to denote the total variation distance between the marginals of μ and ν on $\Omega_\Lambda = \{0, 1\}^\Lambda$.

When the process starts from a initial configuration $\omega \in \Omega^*$ with a front, it is convenient to define a new process $\{\omega^F(t)\}_{t \geq 0}$ on $\Omega_F := \{\omega \in \Omega^* : X(\omega) = 0\}$ as *the process as seen from the front* [55, 5]. Such a process is obtained from the original one by a random shift $-X(\omega(t))$ which forces the front to be always at the origin. More precisely we define on Ω_F the Markov process with generator $\mathcal{L}^F = \mathcal{L}^E + \mathcal{L}^S$ given by

$$\begin{aligned} \mathcal{L}^E f(\omega) &= \sum_{x < 0} c_x(\omega) [\pi_x(f)(\omega) - f(\omega)], \\ \mathcal{L}^S f(\omega) &= (1-p) [f(\vartheta^- \omega) - f(\omega)] + p c_0(\omega) [f(\vartheta^+ \omega) - f(\omega)], \end{aligned}$$

where

$$(\vartheta^\pm \omega)_x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ \omega_{x \mp 1} & \text{otherwise.} \end{cases}$$

That is, the generator \mathcal{L}^F incorporates the moves of the East process behind the front plus ± 1 shifts corresponding to whenever the front itself jumps forward/backward.

Remark. *The same graphical construction that was given for the East process $\omega(t)$ applies to the process $\omega^F(t)$: this is clear for the East part of the generator \mathcal{L}^E ; for the shift part \mathcal{L}^S , simply apply a positive shift ϑ^+ when there is a ring at the origin and the corresponding Bernoulli variable is one. If the Bernoulli variable is zero, operate a negative shift ϑ^- .*

With this notation, the main result of Blondel [55, 5] can be summarized as follows.

Theorem 2.1 ([55, 5]). *The front of the East process, $X(\omega(t))$, and the process as seen from the front, $\omega^F(t)$, satisfy the following:*

- (i) *There exists a unique invariant measure ν for the process $\{\omega^F(t)\}_{t \geq 0}$. Moreover, $\|\nu - \pi\|_{(-\infty, -x]}$ decreases exponentially fast in $x > 0$.*
- (ii) *Let $q^* := \nu(\omega_{-1} = 0)$ and let $v = q - pq^*$. Then $v > 0$ and for any $\omega \in \Omega_F$,*

$$\lim_{t \rightarrow \infty} \frac{X(\omega(t))}{t} \xrightarrow{\mathbb{P}_\omega} v.$$

Thus, if the East process has a front at time $t = 0$ then it will have a front at any later time. The latter progresses in time with an asymptotically constant speed v .

2.3. Local relaxation to equilibrium. In this section we review the main technical results on the local convergence to the stationary measure π for the (infinite volume) East process. The key message here is that *each* vacancy in the starting configuration, in a time lag t , induces the law π in an interval in front of its position of length proportional to t . That explains why the distance between the invariant measure ν and π deteriorates when we approach the front from behind.

Definition 2.2. *Given a configuration $\omega \in \Omega$ and an interval I we say that ω satisfies the **Strong Spacing Condition (SSC)** in I if the largest sub-interval of I where ω is identically equal to one has length at most $10 \log |I| / (|\log p| \wedge 1)$. Similarly, given $\delta, \epsilon \in (0, 1/4)$, we will say that ω satisfies the **(δ, ϵ) -Weak Spacing Condition (WSC)** in I if the largest sub-interval of I where ω is identically equal to one has length at most $\delta |I|^\epsilon$.*

For brevity, we will omit the (δ, ϵ) dependence in WSC case when these are made clear from the context.

Proposition 2.3. *There exist universal positive constants c^* , m independent of p such that the following holds. Let $\Lambda = [1, 2, \dots, \ell]$ and let $\omega \in \Omega$ be such that $\omega_0 = 0$. Further let $\Delta(\omega)$ be largest between the maximal spacing between two consecutive zeros of ω in Λ and the distance of the last zero of ω from the vertex ℓ . Then*

$$\|\mathbb{P}_\omega^t - \pi\|_\Lambda \leq \ell (c^*/q)^{\Delta(\omega)} e^{-t(\text{gap}(\mathcal{L}) \wedge m)}.$$

To prove this proposition, we need the following lemma.

Lemma 2.4. *There exist universal positive constants c^* , m independent of p such that the following holds. Fix $\omega \in \Omega$ with $\omega_0 = 0$, let $\ell \in \mathbb{N}$ and let $f : \Omega_{(-\infty, \ell]} \mapsto \mathbb{R}$ with $\|f\|_\infty \leq 1$. Let also $\pi_\ell(f)$ denote the new function obtained by averaging f w.r.t. the marginal of π over the spin at $x = \ell$. Then,*

$$|\mathbb{E}_\omega [f(\omega(t)) - \pi_\ell(f)(\omega(t))] | \leq (c^*/q)^\ell e^{-t(\text{gap}(\mathcal{L}) \wedge m)}. \quad (2.3)$$

Remark. *If we replaced the r.h.s. of (2.3) with $(2\sqrt{2}/(p \wedge q))^\ell e^{-t \text{gap}(\mathcal{L})}$, then the statement would coincide with that in [55; 5, Proposition 4.3]. Notice that as $p \downarrow 0$, the term c^*/q does not blow up—unlike $2\sqrt{2}/(p \wedge q)$ —and as remarked below (2.1), $\text{gap}(\mathcal{L})$ stays bounded away from 0. Hence, as $p \downarrow 0$, the time after which the r.h.s. in (2.3) becomes small is bounded from above by $C_0 \times \ell$ for some universal $C_0 > 0$ not depending on p . This fact will be crucially used in the proofs of some of the theorems to follow.*

Proof of Lemma 2.4. As mentioned in the remark using [55; 5, Proposition 4.3] it suffices to assume that $p < 1/3$. Fix ω as in the lemma and let $\Omega_{(-\infty, \ell]}^\omega$ be the set of all configurations $\omega' \in \Omega_{(-\infty, \ell]}$ which coincides with ω on the half line $(-\infty, 0]$. The special configuration in $\Omega_{(-\infty, \ell]}^\omega$ which is identically equal to one in the interval $[1, \ell]$ will be denoted by ω^* . Observe that, using reversibility together with the fact that the updates in $(-\infty, 0]$ do not check the spins to the right of the origin,

$$\begin{aligned} \sum_{\omega' \in \Omega_{(-\infty, \ell]}^\omega} \pi_{[1, \ell]}(\omega') \mathbb{E}_{\omega'} [f(\omega'(t))] &= \mathbb{E}_\omega [\pi_{[1, \ell]}(f)(\omega(t))] \\ \sum_{\omega' \in \Omega_{(-\infty, \ell]}^\omega} \pi_{[1, \ell]}(\omega') \mathbb{E}_{\omega'} [\pi_\ell(f)(\omega'(t))] &= \mathbb{E}_\omega [\pi_{[1, \ell]}(f)(\omega(t))]. \end{aligned} \quad (2.4)$$

Using the graphical construction as a grand coupling for the processes with initial condition in $\Omega_{(-\infty, \ell]}^\omega$, it is easy to verify that, at the hitting time τ_ℓ of the set $\{\omega' \in \Omega_{(-\infty, \ell]} : \omega'_\ell = 0\}$ for the process started from ω^* , the processes starting from *all* possible initial conditions in $\Omega_{(-\infty, \ell]}^\omega$ have coupled. Let $\omega' \in \Omega_{(-\infty, \ell]}^\omega$ be distributed according to $\pi_{[1, \ell]}$. Then using the grand coupling,

$$\begin{aligned} |\mathbb{E}_\omega [f(\omega(t)) - \pi_\ell(f)(\omega(t))] | &= |\mathbb{E}_{\omega, \pi_{[1, \ell]}} [f(\omega(t)) - f(\omega'(t)) + \pi_\ell(f)(\omega'(t)) - \pi_\ell(f)(\omega(t))] | \\ &\leq 4 \sup_{\omega' \in \Omega_{(-\infty, \ell]}^\omega} \mathbb{P}(\exists x \in [1, \ell] : \omega_x(t) \neq \omega'_x(t)) \\ &\leq 4\mathbb{P}_{\omega^*}(\tau_\ell > t) \\ &\leq 4\mathbb{P}_{\omega^*}(X(\omega^*(t)) < \ell). \end{aligned}$$

The first equality follows by adding and subtracting $\mathbb{E}_\omega [\pi_{[1, \ell]}(f)(\omega(t))]$ from the l.h.s. and then using (2.4). The rest of the inequalities are immediate from the above discussion. In order to bound the above probability, we observe that the front $X(\omega^*(t))$, initially at $x = 0$, can be coupled to an asymmetric random walk $\xi(t)$, with q (resp. p) as jump rate to the right (resp. left), in such a way that $X(\omega^*(t)) \geq \xi(t)$ for all $t \geq 0$. Since

we have assumed that $p < 1/3$, by standard hitting time estimates for biased random walk there exist universal constants c, m such that, for $t \geq cl$, the above probability is smaller than e^{-mt} . \square

Proof of Proposition 2.3. Let $\omega \in \Omega$ be such that $\omega_0 = 0$. Then

$$\begin{aligned} & \max_{\substack{f: \Omega_\Lambda \rightarrow \mathbb{R} \\ \|f\|_\infty \leq 1}} |\mathbb{E}_\omega [f(\omega(t)) - \pi(f)]| \\ & \leq \max_{\substack{f: \Omega_\Lambda \rightarrow \mathbb{R} \\ \|f\|_\infty \leq 1}} |\mathbb{E}_\omega [f(\omega(t)) - \pi_\ell(f)(\omega(t))]| + \max_{\substack{f: \Omega_\Lambda \rightarrow \mathbb{R} \\ \|f\|_\infty \leq 1}} |\mathbb{E}_\omega [\pi_\ell(f)(\omega(t)) - \pi(f)]| \\ & \leq (c^*/q)^{\Delta(\omega)} e^{-t(\text{gap}(\mathcal{L}) \wedge m)} + \max_{\substack{f: \Omega_\Lambda \rightarrow \mathbb{R} \\ \|f\|_\infty \leq 1}} |\mathbb{E}_\omega [\pi_\ell(f)(\omega(t)) - \pi(f)]|, \end{aligned}$$

where we applied the above lemma to the shifted configuration in which the origin coincides with the rightmost zero in Λ of ω .

We now observe that the new function $\pi_\ell(f)$ depends only on the first $\ell - 1$ coordinates of ω and that $\|\pi_\ell(f)\|_\infty \leq 1$. Thus we can iterate the above bound $(\ell - 1)$ times to get that

$$\|\mathbb{P}_\omega^t - \pi\|_\Lambda \leq 2 \max_{\substack{f: \Omega_\Lambda \rightarrow \mathbb{R} \\ \|f\|_\infty \leq 1}} |\mathbb{E}_\omega [f(\omega(t)) - \pi(f)]| \leq \ell (c^*/q)^{\Delta(\omega)} e^{-t(\text{gap}(\mathcal{L}) \wedge m)}. \quad \square$$

Corollary 2.5. Fix $\omega \in \Omega^*$, $\ell \in \mathbb{N}$ and let $I_\omega^\ell = [X(\omega), X(\omega) + \ell - 1]$. Then

$$\sup_{\omega \in \Omega^*} \|\mathbb{P}_\omega^t - \pi\|_{I_\omega^\ell} \leq (c^*/q)^\ell e^{-t(\text{gap}(\mathcal{L}) \wedge m)}. \quad (2.5)$$

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega \left(\omega(t) \text{ does not satisfy SSC in } I_\omega^\ell \right) \leq \ell (c^*/q)^\ell e^{-t(\text{gap}(\mathcal{L}) \wedge m)} + \ell^{-9}. \quad (2.6)$$

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega \left(\omega(t) \text{ does not satisfy WSC in } I_\omega^\ell \right) \leq (c^*/q)^\ell e^{-t(\text{gap}(\mathcal{L}) \wedge m)} + \ell p^{\delta \ell^\varepsilon / 2}. \quad (2.7)$$

Proof. By construction, $\Delta_{I_\omega^\ell}(\omega) = \ell$ for any $\omega \in \Omega^*$. Thus the first statement follows at once from Proposition 2.3. The other two statements follow from the fact that

$$\pi(\{\omega : \omega \text{ does not satisfy SSC in } [1, \dots, \ell]\}) \leq \ell^{-9}$$

and

$$\pi(\{\omega : \omega \text{ does not satisfy the WSC in } [1, \dots, \ell]\}) \leq \ell p^{\delta \ell^\varepsilon / 2}. \quad \square$$

2.4. Finite speed of information propagation. As the East process is an interacting particle system whose rates are bounded by one, it is well known that in this case information can only travel through the system at finite speed. A quantitative statement of the above general fact goes as follows.

Lemma 2.6. For $x < y \in \mathbb{Z}$ and $0 \leq s < t$, define the “linking event” $F(x, y; s, t)$ as the event that there exists a ordered sequence $s \leq t_x < t_{x+1} < \dots < t_y < t$ or $s \leq t_y < t_{y-1} < \dots < t_x < t$ of rings of the Poisson clocks associated to the corresponding sites in $[x, y] \cap \mathbb{Z}$. Then there exists a constant v_{\max} such that, for all $|y - x| \geq v_{\max}(t - s)$,

$$\mathbb{P}(F(x, y; s, t)) \leq e^{-|x-y|}.$$

Proof. The probability of $F(x, y; s, t)$ is equal to the probability that a Poisson process of intensity 1 has at least $|x - y|$ instances within time $t - s$. \square

Remark 2.7. An important consequence of the above lemma is the following fact. Let $0 < s < t$ and let \mathcal{F}_s be the σ -algebra generated by all the rings of the Poisson clocks and all the coin tosses up to time s in the graphical construction of the East process. Fix $x < y < z$ and let A, B be two events depending on $\{\omega_a\}_{a \leq x}$ and $\{\omega_a\}_{a \geq z}$ respectively. Then

$$\begin{aligned} & \mathbb{P}_\omega (\{\omega(t) \in A \cap B\} \cap F(y, z; s, t)^c \mid \mathcal{F}_s) \\ &= \mathbb{P}_\omega (\{\omega(t) \in A\} \mid \mathcal{F}_s) \mathbb{P}_\omega (\{\omega(t) \in B\} \cap F(y, z; s, t)^c \mid \mathcal{F}_s). \end{aligned}$$

This is because: (i) on the event $F(y, z; s, t)^c$ the occurrence of the event B does not depend anymore on the Poisson rings and coin tosses to the left of y ; (ii) the occurrence of the event A depends only on the Poisson rings and coin tosses to the left of x because of the oriented character of the East process.

The finite speed of information propagation, together with the results of [33, 3], implies the following rough bound on the position of the front $X(\omega(t))$ for the East process started from $\omega \in \Omega^*$ (also see, e.g., [55; 5, Lemma 3.2]).

Lemma 2.8. *There exists constants $v_{\min} > 0$ and $\gamma > 0$ such that*

$$\sup_{\omega \in \Omega^*} \mathbb{P}_\omega (X(\omega(t)) \in [X(\omega) + v_{\min}t, X(\omega) + v_{\max}t]) \geq 1 - e^{-\gamma t}.$$

Remark 2.9. *When $p \downarrow 0$ one can obtain the above statement with $v_{\min} \rightarrow 1$ and γ uniformly bounded away from 0 by using our Proposition 2.3 instead of [55; 5, Proposition 4.3] in the proof of [55; 5, Lemma 3.2].*

The second consequence of the finite speed of information propagation is a kind of mixing result behind the front $X(\omega(t))$ for the process started from $\omega \in \Omega^*$. We first need few additional notation.

Definition 2.10. *For any $a \in \mathbb{Z}$, we define the shifted configuration $\vartheta_a \omega$ by*

$$\vartheta_a \omega_x = \omega_{x+a}, \quad \forall x \in \mathbb{Z}.$$

Proposition 2.11. *Let $\Lambda \subset (-\infty, -\ell] \cap \mathbb{Z}$ and let $B \subset \{0, 1\}^\Lambda$. Assume $\ell \geq 2v_{\max}(t - s)$. Then for any $\omega \in \Omega^*$ and any $a \in \mathbb{Z}$ the following holds:*

$$\begin{aligned} & \left| \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_s \right] - \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mid \mathcal{F}_s \right] \mathbb{E}_\omega \left[\mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_s \right] \right| \\ &= O(e^{-\ell}). \end{aligned}$$

To see what the proposition roughly tells we first assume that the front at time s is at 0. Then the above result says that at a later time t any event supported on $(-\infty, -\ell]$ is almost independent of the location of the front.

Proof. Recall the definition of the event $F(x, y; s, t)$ from Lemma 2.6 and let

$$\begin{aligned} B_1 &:= F(X(\omega(s)) - \ell, X(\omega(s)) - \ell/2 - 1; s, t) \\ B_2 &:= F(X(\omega(s)) - \ell/2, X(\omega(s)); s, t). \end{aligned}$$

We now write

$$\begin{aligned} \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mathbb{1}_{\{X(\omega(t))=a\}} &= \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mathbb{1}_{\{B_1^c\}} \mathbb{1}_{\{B_2^c\}} \\ &\quad + \mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)] \in B\}} \mathbb{1}_{\{X(\omega(t))=a\}} \left[1 - \mathbb{1}_{\{B_1^c\}} \mathbb{1}_{\{B_2^c\}} \right]. \end{aligned}$$

We first note that given \mathcal{F}_s for any $a < X(\omega(s)) - \ell/2$,

$$\mathbb{1}_{\{X(\omega(t))=a\}} \mathbb{1}_{\{B_2^c\}} = 0,$$

and hence

$$\mathbb{E}_\omega \left[\mathbb{1}_{\{X(\omega(t))=a\}} \mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s \right] = 0.$$

Thus, we may assume that $a \geq X(\omega(s)) - \ell/2$. Now

$$\begin{aligned} & \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mathbb{1}_{\{B_1^c\}} \mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s \right] \\ &= \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_{X(\omega(s))}[\omega(t)]_\Lambda \in B\}} \mathbb{1}_{\{B_1^c\}} \mid \mathcal{F}_s \right] \mathbb{E}_\omega \left[\mathbb{1}_{\{X(\omega(t))=a\}} \mathbb{1}_{\{B_2^c\}} \mid \mathcal{F}_s \right] \end{aligned}$$

because under the assumption that $a \geq X(\omega(s)) - \ell/2$, the two events are functions of an independent set of variables in the graphical construction (cf. Remark 2.7). By Lemma 2.6 we know that $\mathbb{P}(B_i^c \mid \mathcal{F}_s) \leq e^{-\ell}$, $i = 1, 2$ and the proof is complete. \square

3. THE LAW BEHIND THE FRONT OF THE EAST PROCESS

Our main result in this section is a quantitative estimate on the rate of convergence as $t \rightarrow \infty$ of the law μ_ω^t of the process seen from the front to its invariant measure ν . Consider the process $\{\omega^F(t)\}_{t \geq 0}$ seen from the front (recalling §2.2) and let μ_ω^t be its law at time t when the starting configuration is ω .

Theorem 3.1. *For any $p \in (0, 1)$ there exist $\alpha \in (0, 1)$ and $v^* > 0$ such that*

$$\sup_{\omega \in \Omega_F} \|\mu_\omega^t - \nu\|_{[-v^*t, 0]} = O(e^{-t^\alpha}).$$

Moreover, α and v^* can be chosen uniformly as $p \rightarrow 0$.

A corollary of this result — which will be key in the proof of Theorem 1 — is to show that, for any $\omega \in \Omega_F$, the increments in the position of the front (the variables ξ_n below) behave asymptotically as a stationary sequence of weakly dependent random variables with exponential moments.

Fix $\Delta^1 > 0$ and let $t_n = n\Delta$ for $n \in \mathbb{N}$. Define

$$\xi_n := X(\omega(t_n)) - X(\omega(t_{n-1})),$$

so that

$$X(\omega(t)) = \sum_{n=1}^{N_t} \xi_n + [X(\omega(t)) - X(\omega(t_N))], \quad N = \lfloor t/\Delta \rfloor. \quad (3.1)$$

Recall also that α, v^* are the constants appearing in Theorem 3.1.

Corollary 3.2. *Let $f : \mathbb{R} \mapsto [0, \infty)$ be such that $e^{-|x|} f^2(x) \in L^1(\mathbb{R})$. Then*

$$C_f \equiv \sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_1)^2] < \infty. \quad (3.2)$$

Moreover, there exists a constant $\gamma > 0$ such that

$$\sup_{\omega \in \Omega_F} |\mathbb{E}_\omega [f(\xi_n)] - \mathbb{E}_\nu [f(\xi_1)]| = O(e^{-\gamma n^\alpha}) \quad \forall n \geq 1, \quad (3.3)$$

$$\sup_{\omega \in \Omega_F} |\text{Cov}_\omega(\xi_j, \xi_n) - \text{Cov}_\nu(\xi_1, \xi_{n-j})| = O(e^{-\gamma j^\alpha}) \wedge O(e^{-\gamma(n-j)^\alpha}) \quad \forall j < n \quad (3.4)$$

and

$$\sup_{\omega \in \Omega_F} |\text{Cov}_\omega(f(\xi_j), f(\xi_n))| = O(e^{-\gamma(n-j)^\alpha}), \quad \forall j \leq n, \quad (3.5)$$

¹In the sequel we will always use the letter Δ to denote a time lag. Its value will depend on the context and will be specified in advance.

where the constants in the r.h.s. of (3.3) and (3.5) depend on f only through the constant C_f . Finally, for any $k, n \in \mathbb{N}$ such that $v^*k > nv_{\max}$ and for any bounded $F : \mathbb{R}^n \mapsto \mathbb{R}$,

$$\sup_{\omega \in \Omega_{\mathbb{F}}} \left| \mathbb{E}_{\omega} \left[F(\xi_k, \xi_{k+1}, \dots, \xi_{k+n-1}) \right] - \mathbb{E}_{\nu} \left[F(\xi_1, \xi_2, \dots, \xi_n) \right] \right| = O(e^{-\gamma t_k^{\alpha}}). \quad (3.6)$$

To prove Theorem 3.1 we will require a technical result, Theorem 3.3 below, which can informally be summarized as follows:

- Starting from $\omega \in \Omega^*$, at any fixed large time t , with high probability the configuration satisfies WSC apart from an interval behind the front $X(\omega(t))$ of length proportional to t^{ϵ} .
- If the above property is true at time t , then at a later time $t' = t + \text{const} \times t^{\epsilon}$ the law of the process will be very close to π apart from a small interval behind the front where the strong spacing property will occur with high probability.

Formally, fix a constant κ to be chosen later on and $t > 0$. Let $\ell \equiv t^{\epsilon}$, where ϵ appears in the WSC and let $t_{\ell} = t - \kappa\ell/v_{\min}$. Let \mathcal{S}_{ℓ} denotes the set of configurations which fail to satisfy SSC in the interval $[-3(v_{\max}/v_{\min})\kappa\ell, -\kappa\log\ell] \cap \mathbb{Z}$ and let $\mathcal{W}_{\ell,t}$ be those configurations which fail to satisfy WSC in the interval $[-v_{\min}t, -\ell] \cap \mathbb{Z}$.

Theorem 3.3. *It is possible to choose δ small enough and κ large enough depending only on p in such a way that for all t large enough the following holds:*

$$\sup_{\omega \in \Omega_{\mathbb{F}}} \mu_{\omega}^t(\mathcal{W}_{\ell,t}) = O(e^{-t^{\epsilon/2}}), \quad (3.7)$$

$$\sup_{\omega \in \Omega_{\mathbb{F}}} \mu_{\omega}^t(\mathcal{S}_{\ell} \mid \mathcal{F}_{t_{\ell}}) = O(t^{-7\epsilon}) + \mathbb{1}_{\mathcal{W}_{\ell,t}}(\omega(t_{\ell})), \quad (3.8)$$

$$\sup_{\omega \in \Omega_{\mathbb{F}}} \|\mu_{\omega}^t(\cdot \mid \mathcal{F}_{t_{\ell}}) - \pi\|_{[-v_{\min}t, -3(v_{\max}/v_{\min})\kappa\ell]} = O(e^{-t^{\epsilon/2}}) + \mathbb{1}_{\mathcal{W}_{\ell,t}}(\omega(t_{\ell})). \quad (3.9)$$

Moreover, κ stays bounded as $p \downarrow 0$.

3.1. Non-equilibrium properties of the law behind the front: Proof of Theorem 3.3.

We begin by proving (3.7). Bounding $\sup_{\omega \in \Omega_{\mathbb{F}}} \mu_{\omega}^t(\mathcal{W}_{\ell,t})$ from above is equivalent to bounding $\sup_{\omega \in \Omega_{\mathbb{F}}} \mathbb{P}_{\omega}(\omega(t) \in \mathcal{W}_{\ell,t}^*)$ from above, where $\mathcal{W}_{\ell,t}^*$ denotes the set of configurations $\omega \in \Omega^*$ which do not satisfy the spacing condition in $[X(\omega) - v_{\min}t, X(\omega) - \ell]$.

Using Lemma 2.8, with probability greater than $1 - e^{-\gamma t}$ we can assume that $X(\omega(t)) \in [v_{\min}t, v_{\max}t]$. Next we observe that, for any $a \in [v_{\min}t, v_{\max}t]$, the events $\{X(\omega(t)) = a\}$ and $\{\omega(t) \in \mathcal{W}_{\ell,t}^*\}$ imply that there exists $x \in \mathbb{Z}$ with the following properties:

- $0 \leq x \leq a - \ell$;
- The hitting time $\tau_x := \inf\{s > 0 : X(\omega(s)) = x\}$ is smaller than t ;
- $\omega(t)$ is identically equal to one in the interval $I_x := [x, x + \delta(v_{\min}t)^{\epsilon}/2]$;
- The linking event $F(x, a; \tau_x, t)$ defined in Lemma 2.6 occurred.

In conclusion, using twice a union bound (once for the choice of $a \in [v_{\min}t, v_{\max}t]$ and once for the choice of $x \in [0, a - \ell]$) together with the strong Markov property at time

τ_x , we get

$$\begin{aligned}
& \mathbb{P}_\omega(\omega(t) \in \mathcal{W}_{\ell,t}^*) \\
& \leq e^{-\gamma t} + \sum_{a=v_{\min}t}^{v_{\max}t} \sum_{x=0}^{a-\ell} \mathbb{P}(F(x, a; \tau_x, t)) \mathbf{1}_{\{|x-a| \geq v_{\max}(t-\tau_x)\}} \\
& \quad + \sum_{a=v_{\min}t}^{v_{\max}t} \sum_{x=0}^{a-\ell} \left[\|\mathbb{P}_{\omega(\tau_x)}^t - \pi\|_{I_x} + p^{\frac{\delta}{2}(v_{\min}t)^\epsilon} \right] \mathbf{1}_{\{|x-a| \leq v_{\max}(t-\tau_x)\}} \\
& \leq (v_{\max}t)^2 \left[e^{-\gamma t} + 2e^{-\ell} + \sqrt{2}\delta(v_{\min}t)^\epsilon \left(\frac{c^*}{q} \right)^{\frac{\delta}{2}(v_{\min}t)^\epsilon} e^{-\frac{\ell}{v_{\max}}(\text{gap}(\mathcal{L}) \wedge m)} + p^{\frac{\delta}{2}(v_{\min}t)^\epsilon} \right].
\end{aligned}$$

Above we used Lemma 2.6 in the case $|x-a| \geq v_{\max}(t-\tau_x)$ and (2.5) of Corollary 2.5 otherwise. The statement (3.7) now follows by taking δ small enough.

We now prove (3.8). As before we give the result in the East process setting (*i.e.* for the law $\mathbb{P}_\omega^t(\cdot | \mathcal{F}_s)$ and \mathcal{S}_ℓ replaced by its random shifted version \mathcal{S}_ℓ^*). We decompose the interval $[X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\ell, X(\omega(t)) - \kappa\log\ell] \cap \mathbb{Z}$ where we want SSC to hold into $[X(\omega(t_\ell)), X(\omega(t)) - \kappa\log\ell]$ and $[X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\ell, X(\omega(t_\ell))]$. Note that by Lemma 2.8 we can ignore the events $\{X(\omega(t_\ell)) > X(\omega(t)) - \kappa\log\ell\}$ and $\{X(\omega(t)) - 3(v_{\max}/v_{\min})\kappa\ell > X(\omega(t_\ell))\}$.

We now proceed in two steps: (1) we show that SSC occurs with high probability in the first interval. Here we do not use the condition that $\omega(t_\ell) \notin \mathcal{W}_{\ell,t}^*$. (2) we prove the same statement for the second interval. Here instead the fact that $\omega(t_\ell) \notin \mathcal{W}_{\ell,t}^*$ will be crucial.

- *Step (1).* Let $\Delta \equiv 5 \log \ell / (|\log p| \wedge 1)$. For any intermediate time $s \in [t_\ell, t - (\kappa/v_{\max}) \log \ell]$, Corollary 2.5 together with the Markov property at time s show that

$$\begin{aligned}
& \mathbb{P}_\omega(\omega(t)_x = 1 \forall x \in [X(\omega(s)), X(\omega(s)) + \Delta] | \mathcal{F}_s) \\
& \leq \|\mathbb{P}_\omega(\cdot | \mathcal{F}_s) - \pi\|_{[X(\omega(s)), X(\omega(s)) + \Delta]} + \pi(\omega_x = 1 \forall x \in [X(\omega(s)), X(\omega(s)) + \Delta]) \\
& \leq \Delta \left(\frac{c^*}{q} \right)^\Delta e^{-(t-s)(\text{gap}(\mathcal{L}) \wedge m)} + p^{|\Delta|} = O(t^{-10\epsilon}). \tag{3.10}
\end{aligned}$$

Above we used the fact that $t-s \geq \kappa/v_{\max} \log \ell$. Hence, κ can be chosen depending only on p such that (3.10) holds and κ stays bounded as $p \downarrow 0$.

We now take the union of the random intervals $[X(\omega(s)), X(\omega(s)) + \Delta]$ over discrete times s of the form $s_j = t_\ell + j/\ell^2$, $j = 0, 1, \dots, n$ and n such that $s_n = t - (\kappa/v_{\max}) \log \ell$. Thus $n = O(\ell^3) = O(t^{3\epsilon})$. The aim here is to show that, with high probability, the above union is actually an interval containing the target one $[X(\omega(t_\ell)), X(\omega(t)) - \kappa\log\ell]$, with the additional property that it does not contain a sub-interval of length Δ where $\omega(t)$ is constantly equal to one (which will then imply (3.8), with room to spare).

We now upper bound the probability that the set $\cup_{j=0}^n [X(\omega(s_j)), X(\omega(s_j)) + \Delta]$ is not an interval. It is an easy observation that if $X(\omega(s_n)) > X(\omega(s_0))$ then the aforementioned event occurs if $X(\omega(s_{j+1})) - X(\omega(s_j)) \leq \Delta$ for $j = 0, 1, \dots, n$. Now by the lower bound in Lemma 2.8

$$\mathbb{P}(X(\omega(s_n)) > X(\omega(s_0))) \geq 1 - e^{-ct^\epsilon}$$

for some constant c . Also

$$\begin{aligned} & \sum_j \mathbb{P}_\omega(X(\omega(s_{j+1})) - X(\omega(s_j)) \geq \Delta) \\ & \leq \sum_j \mathbb{E} [\mathbb{P}_\omega(F(X(\omega(s_j)), X(\omega(s_j)) + \Delta; s_j, s_{j+1}) \mid \mathcal{F}_{s_j})] \\ & \leq ne^{-\Delta} = O(t^{-8\epsilon}). \end{aligned}$$

Above $F(X(\omega(s_j)), X(\omega(s_j)) + \Delta; s_j, s_{j+1})$ is the linking event and we used Lemma 2.6 because $\Delta \gg (s_{j+1} - s_j)$.

Moreover, Lemma 2.8 implies that κ can be chosen (bounded as $p \downarrow 0$), such that with probability greater than

$$1 - e^{-\gamma(t-s_0)} - e^{-\gamma(t-s_n)} = 1 - O(t^{-10\epsilon}),$$

the front $X(\omega(t))$ satisfies

$$X(\omega(t)) \leq X(\omega(s_n)) + v_{\max}(t - s_n) \leq X(\omega(s_n)) + \kappa \log \ell.$$

Thus

$$[X(\omega(t_\ell)), X(\omega(t)) - \kappa \log \ell] \subset \cup_{j=0}^n [X(\omega(s_j)), X(\omega(s_j)) + \Delta].$$

with probability $1 - O(t^{-10\epsilon})$.

Finally, using (3.10) and union bound, the probability that there exists $j \leq n$ such that $\omega(t)$ is identically equal to one in $[X(\omega(s_j)), X(\omega(s_j)) + \Delta]$ is $O(t^{-7\epsilon})$ uniformly in the configuration at time t_ℓ .

In conclusion we proved that SSC holds with probability $1 - O(t^{-8\epsilon})$ in an interval containing $[X(\omega(t_\ell)), X(\omega(t)) - \kappa \log \ell]$. The first step is complete.

- *Step (2)*. Let $x^* = \max\{x \leq X(\omega(t_\ell)) - 3\kappa(v_{\max}/v_{\min})\ell : \omega(t_\ell)_x = 0\}$. Since $\omega(t_\ell) \notin \mathcal{W}_{\ell,t}^*$ such a zero exists. Moreover, $\omega(t_\ell) \notin \mathcal{W}_{\ell,t}^*$ implies that $\omega(t_\ell)$ has a zero in every sub-interval of $[x^*, X(\omega(t_\ell)) - \ell]$ of length $\delta t^\epsilon = \delta \ell$. Hence we can apply Proposition 2.3 to the interval $[x^*, X(\omega(t_\ell))]$ to get that

$$\|\mathbb{P}_\omega^t(\cdot \mid \mathcal{F}_{t_\ell}) - \pi\|_{[x^*, X(\omega(t_\ell))]} = O(e^{-t^\epsilon/2}),$$

by choosing κ large enough. Since by Remark 2.9 $v_{\min} \rightarrow 1$ as $p \downarrow 0$, we can choose κ to be bounded as $p \downarrow 0$. Also

$$\pi(\omega : \omega \text{ violates SSC in } [x^*, X(\omega(t_\ell))]) = O(t^{-7\epsilon}).$$

Thus we have proved that SSC holds in $[x^*, X(\omega(t_\ell))]$ with probability $1 - O(t^{-7\epsilon})$.

Finite speed of propagation in the form of Lemma 2.8 guarantees that, with probability $1 - O(e^{-\gamma(t-t_\ell)})$, $x^* < X(\omega(t)) - 2\kappa(v_{\max}/v_{\min})\ell$. The proof of (3.8) is complete.

It remains to prove (3.9). Let $\Lambda := [-v_{\min}t, -3(v_{\max}/v_{\min})\kappa\ell] \cap \mathbb{Z}$ and let $A \subset \{0, 1\}^\Lambda$. Recall Definition 2.10 of the shifted configuration $\vartheta_a \omega$ and that $t_\ell = t - \kappa\ell/v_{\min}$. Then (3.9) follows once we show that

$$|\mathbb{P}_\omega(\vartheta_{X(\omega(t))} \omega(t)_\Lambda \in A \mid \mathcal{F}_{t_\ell}) - \pi(A)| \leq e^{-t^\epsilon/2}$$

whenever $\omega(t_\ell)$ satisfies WSC in the interval $I = [X(\omega(t_\ell)) - v_{\min}t, X(\omega(t_\ell)) - \ell]$. This property is assumed henceforth. Let us decompose $\mathbb{P}_\omega(\vartheta_{X(\omega(t))} \omega(t)_\Lambda \in A \mid \mathcal{F}_{t_\ell})$ according to the value of the front:

$$\mathbb{P}_\omega(\vartheta_{X(\omega(t))} \omega(t)_\Lambda \in A \mid \mathcal{F}_{t_\ell}) = \sum_{a \in \mathbb{Z}} \mathbb{E}_\omega [\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell}].$$

Using Lemma 2.8, $0 < X(\omega(t)) - X(\omega(t_\ell)) \leq v_{\max}(t - t_\ell)$ occurs with probability greater than $1 - e^{-\gamma(t-t_\ell)}$. Thus

$$\begin{aligned} & \sum_{a \in \mathbb{Z}} \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell} \right] \\ = & \sum_{\substack{a \in \mathbb{Z} \\ 0 < a - X(\omega(t_\ell)) \leq v_{\max}(t - t_\ell)}} \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell} \right] + e^{-\gamma(t-t_\ell)}. \end{aligned}$$

By definition, the event $\{\vartheta_a \omega(t)_\Lambda \in A\}$ is the same as the event $\{\omega(t)_{\Lambda+a} \in A\}$. Using the restriction that $|a - X(\omega(t_\ell))| \leq v_{\max}(t - t_\ell)$, the choice of Λ and the fact that $(v_{\max}/v_{\min})\kappa\ell \geq v_{\max}(t - t_\ell)$, we get $\Lambda + a \subset (-\infty, X(\omega(t_\ell)) - 2(v_{\max}/v_{\min})\kappa\ell]$. Thus, the event $\{\omega(t)_{\Lambda+a} \in A\}$ satisfies the hypothesis of Proposition 2.11, which can then be applied to each term in the above sum to get

$$\begin{aligned} & \sum_{\substack{a \in \mathbb{Z} \\ 0 < a - X(\omega(t_\ell)) \leq v_{\max}(t - t_\ell)}} \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell} \right] \\ = & \sum_{\substack{a \in \mathbb{Z} \\ 0 < a - X(\omega(t_\ell)) \leq v_{\max}(t - t_\ell)}} \mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mid \mathcal{F}_{t_\ell} \right] \mathbb{E}_\omega \left[\mathbb{1}_{\{X(\omega(t))=a\}} \mid \mathcal{F}_{t_\ell} \right] + O(\ell e^{-\ell}). \end{aligned}$$

Finally we claim that, for any a such that $0 < a - X(\omega(t_\ell)) \leq v_{\max}(t - t_\ell)$, if δ is chosen small enough and κ large enough depending on p (bounded as $p \downarrow 0$),

$$\mathbb{E}_\omega \left[\mathbb{1}_{\{\vartheta_a \omega(t)_\Lambda \in A\}} \mid \mathcal{F}_{t_\ell} \right] = \pi(A) + O(e^{-t^{\epsilon/2}}). \quad (3.11)$$

To prove it we apply Proposition 2.3 to the interval $I = [X(\omega(t_\ell)) - v_{\min}t, X(\omega(t_\ell)) - \ell]$ to get that

$$\|\mathbb{P}_\omega^t(\cdot \mid \mathcal{F}_{t_\ell}) - \pi\|_I \leq |I| \left(\frac{c^*}{q} \right)^{\delta|I|^\epsilon} e^{-(t-t_\ell)(\text{gap}(\mathcal{L}) \wedge m)}, \quad (3.12)$$

where $|I| = O(t)$ is the length of I , since by assumption $\omega(t_\ell)$ satisfies WSC in I . Because of our choice of the parameters (ℓ, t_ℓ) the r.h.s. of (3.12) is $O(e^{-t^{\epsilon/2}})$ if δ, κ are chosen small enough and large enough respectively depending on p . Since by Remark 2.9 $v_{\min} \rightarrow 1$ as $p \downarrow 0$, κ can be chosen to be bounded as $p \downarrow 0$.

The claim now follows because $\{\omega : \vartheta_a \omega \in A\} \subset \{0, 1\}^{\Lambda+a}$, with

$$\begin{aligned} \Lambda + a &= [-v_{\min}t + a, -3(v_{\max}/v_{\min})\kappa\ell + a] \\ &\subset [X(\omega(t_\ell)) - v_{\min}t_\ell, X(\omega(t_\ell)) - (v_{\max}/v_{\min})\kappa\ell] \subset I, \end{aligned}$$

together with the translation invariance of π expressed by $\pi(\{\omega : \vartheta_a \omega \in A\}) = \pi(A)$. This establishes (3.9) and concludes the proof of Theorem 3.3. Notice that at all points in the proof, κ was chosen to be bounded as $p \downarrow 0$. \square

3.2. On the rate of convergence to the invariant measure ν : Proof of Theorem 3.1.

The proof is based on a coupling argument. There exists $v^* > 0$ such that, for any t large enough and for any pair of starting configurations $(\omega, \omega') \in \Omega_F \times \Omega_F$,

$$\|\mu_\omega^t - \mu_{\omega'}^t\|_{[-v^*t, 0]} \leq c' e^{-t^\alpha}, \quad (3.13)$$

with (c', α) independent of (ω, ω') . Also v^*, α can be chosen uniformly as $p \downarrow 0$. Once this step is established and using the invariance of the measure ν under the action of

the semigroup $e^{t\mathcal{L}^F}$,

$$\begin{aligned} \|\mu_\omega^t - \nu\|_{[-v^*t, 0]} &= \|\mu_\omega^t - \int d\nu(\omega') \mu_{\omega'}^t\|_{[-v^*t, 0]} \\ &\leq \int d\nu(\omega') \|\mu_\omega^t - \mu_{\omega'}^t\|_{[-v^*t, 0]} \leq c' e^{-t^\alpha}. \end{aligned}$$

We now prove (3.13). We first fix a bit of notation.

Given $\epsilon \in (0, 1)$ and a large $t > 0$, let $\Delta_1 = (\kappa/v_{\min})t^\epsilon$ where κ is the constant appearing in Theorem 3.3, let $\Delta_2 = \kappa\epsilon \log t$ and define $\Delta = \Delta_1 + \Delta_2$. We then set

$$t_0 = (1 - \epsilon)t, \quad t_n = t_{n-1} + \Delta, \quad n = 1, \dots, N, \quad N = \lfloor \epsilon t / \Delta \rfloor.$$

It will be convenient to refer to the time lag $[t_{n-1}, t_n)$ as the n^{th} -round. In turn we split each round into two parts: from t_{n-1} to $s_n := t_{n-1} + \Delta_1$ and from s_n to t_n . We will refer to the first part of the round as the *burn-in part* and to the second part as the *mixing part*. We also set $I_n = [-v_{\min}t_n + 2v_{\max}\Delta_n, 0]$. Observe that $I_n \neq \emptyset$ for any $n \leq N + 1$ if ϵ is chosen smaller than v_{\min}/v_{\max} and t is large enough depending on ϵ .

Next, for any pair (μ, μ') of probability measures on a finite set, we denote by $\text{MC}(\mu, \mu')$ their *maximal coupling*, namely the one that achieves the variation distance between μ, μ' in the variational formula (see, e.g., [2222, 22])

$$\|\mu - \mu'\| = \inf\{M(\omega \neq \omega') : M \text{ a coupling of } \mu, \mu'\}.$$

If (μ, μ') are probability measures on Ω and Λ is a finite subset of \mathbb{Z} , we define the Λ -*maximal coupling* $\text{MC}_\Lambda(\mu, \mu')$ as follows:

- a) first sample $(\omega_\Lambda, \omega'_\Lambda)$ according to the maximal coupling of the marginals of μ, μ' on Ω_Λ ;
- b) then sample *independently* $(\omega_{\mathbb{Z}\setminus\Lambda}, \omega'_{\mathbb{Z}\setminus\Lambda})$ according to their respective conditional distribution $\mu(\cdot | \omega_\Lambda), \mu'(\cdot | \omega'_\Lambda)$.

Finally the *basic coupling* for the East process will be the one in which two configurations evolve according to the graphical construction using the same Poisson clocks and the same coin tosses.

We are now ready to recursively construct the coupling $M_{\omega, \omega'}^t$ of $\mu_\omega^t, \mu_{\omega'}^t$ satisfying (3.13). For lightness of notation, in the sequel the starting configurations (ω, ω') will be sometimes omitted.

Definition 3.4 (The coupling $M_{\omega, \omega'}^t$). *We first define a family $\{M^{(n)}\}$ of couplings for $\{(\mu_\omega^{t_n}, \mu_{\omega'}^{t_n})\}_{n=0}^N$ as follows. $M^{(0)}$ is the trivial product coupling. Given $M^{(n)}$, the coupling $M^{(n+1)}$ at time t_{n+1} is constructed according to the following algorithm:*

- (a) *Sample $(\omega(t_n), \omega'(t_n))$ from $M^{(n)}$. If they coincide in the interval I_n then let them evolve according to the basic coupling for a time lag Δ ;*
- (b) *otherwise, sample $(\omega(s_n), \omega'(s_n))$ at the end of the burn-in part of round $(n + 1)$ via the Λ_n -maximal coupling MC_{Λ_n} for the laws $\mu_\omega^{s_n}(\cdot | \mathcal{F}_{t_n})$ and $\mu_{\omega'}^{s_n}(\cdot | \mathcal{F}_{t_n})$ at the configurations $(\omega(t_n), \omega'(t_n))$ from step (a). Here $\Lambda_n = [-v_{\min}s_n, -3(v_{\max}/v_{\min})\kappa t^\epsilon]$.*
 - (i) *If $(\omega(s_n), \omega'(s_n))$ are not equal in the interval Λ_n , then let them evolve for the mixing part of the round (i.e., from time s_n to time t_{n+1}) via the basic coupling.*

(ii) If instead they agree on Λ_n , then search for the rightmost common zero of $(\omega(s_n), \omega'(s_n))$ in Λ_n and call x_* its position. If there is no such a zero, define x_* to be the right boundary of Λ_n . Next sample a Bernoulli random variable ξ with $\text{Prob}(\xi = 1) = e^{-2\Delta_2}$. The value $\xi = 1$ has to be interpreted as corresponding to the event that the two Poisson clocks associated to x_* and to the origin in the graphical construction did not ring during the mixing part of the round.

(1) If $\xi = 1$, set $\omega(t_{n+1})_{x_*} = \omega(s_n)_{x_*}$ and similarly for ω' . The remaining part of the configurations at time t_{n+1} is sampled using the basic coupling to the left of x_* and the maximal coupling for the East process in the interval $[x_* + 1, -1]$ with boundary condition at x_* equal to $\omega(s_n)_{x_*}$.

(2) If $\xi = 0$ we let evolve $(\omega(s_n), \omega'(s_n))$ with the basic coupling conditioned to have at least one ring either at x_* or at the origin or both.

The final coupling $M_{\omega, \omega'}^t$ will be obtained by first sampling $(\omega(t_N), \omega'(t_N))$ from $M^{(N)}$ and then by applying the basic coupling for the time lag $(t - t_N)$.

It is easy to check that $\{M^{(n)}\}$ is indeed a family of couplings for $\{(\mu_\omega^{t_n}, \mu_{\omega'}^{t_n})\}_{n=0}^N$. Define now

$$p_n := M^{(n)}(\omega \neq \omega' \text{ in the interval } I_n)$$

and recall that ϵ is the exponent entering in the definition of the round length Δ .

Claim 3.5. *There exist $\epsilon_0 > 0$ such that, for all $\epsilon < \epsilon_0$ and all t large enough depending on ϵ ,*

$$p_N = O(e^{-t^\alpha}),$$

for some positive $\alpha = \alpha(\epsilon)$.

Proof. The claim follows from the recursive inequality:

$$p_{n+1} \leq C e^{-t^{\epsilon/2}} + p_n(1 - e^{-2\Delta_2}/2), \quad (3.14)$$

for some constant C . In fact, if we assume (3.14) and recall that $e^{-2\Delta_2} = t^{-2\kappa\epsilon}$, we get

$$\begin{aligned} p_N &\leq C e^{-t^{\epsilon/2}} [1 + (1 - e^{-2\Delta_2}/2) + (1 - e^{-2\Delta_2}/2)^2 + \dots] + (1 - e^{-2\Delta_2}/2)^N \\ &\leq 2C e^{-t^{\epsilon/2}} t^{2\kappa\epsilon} + (1 - e^{-2\Delta_2}/2)^N = O(e^{-t^{\epsilon/3}}), \end{aligned}$$

provided that $1 - \epsilon(1 + 2\kappa) > \epsilon/3$, i.e. $\epsilon < 3/(4 + 6\kappa)$, since $N > ct^{1-\epsilon}$ for some constant c . Notice crucially that since κ was bounded as $p \downarrow 0$ in the statement of Theorem 3.3, ϵ_0 and $\alpha(\epsilon)$ can be chosen uniformly as $p \downarrow 0$.

To prove (3.14) we use Lemma 2.6 together with Theorem 3.3. We begin by examining the possible occurrence of two very unlikely events each of which will contribute to the constant term in (3.14).

- The first possibility is that $\omega(t_n) = \omega'(t_n)$ in the interval I_n and $F(a_n, a_{n+1}; t_n, t_{n+1})$ occurred. Here $a_n = -v_{\min}t_n + 2v_{\max}\Delta n$ is the left boundary of I_n and similarly for a_{n+1} . The linking event could in fact move possible discrepancies between $\omega(t_n), \omega'(t_n)$ sitting outside I_n to the inside of I_{n+1} . Since $|a_n - a_{n+1}| \geq v_{\max}(t_{n+1} - t_n)$, Lemma 2.6 shows that this case gives a contribution to p_{n+1} which is $O(e^{-|a_n - a_{n+1}|}) = O(e^{-v_{\max}t^\epsilon})$.
- The second possibility is that either $\omega(t_n)$ or $\omega'(t_n)$ do not satisfy the (δ, ϵ) -weak spacing condition in $[-v_{\min}t_n, -t_n^\epsilon]$. The bound (3.7) of Theorem 3.3 shows that the contribution of such a case is $O(e^{-t^{\epsilon/2}})$.

Having discarded the occurrence of the above “extremal” situations, we now assume that $(\omega(t_n), \omega'(t_n))$ are such that: (i) they are different in the interval I_n ; (ii) they satisfy the (δ, ϵ) -weak spacing condition in $[-v_{\min}t_n, -t_n^\epsilon]$. It will be useful to denote by \mathcal{G}_n the set of pairs $(\omega, \tilde{\omega})$ fulfilling (i) and (ii) above.

We will show that, *uniformly* in $(\omega, \tilde{\omega}) \in \mathcal{G}_n$, the probability that at the end of the round $(\omega(\Delta), \tilde{\omega}(\Delta))$ are not coupled inside the interval I_{n+1} is smaller than $(1 - \frac{1}{2}e^{-2\Delta_2})$. That clearly proves the second term in (3.14).

To prove that, recall the definition of the Λ_n -maximal coupling MC_{Λ_n} , fix $(\omega, \tilde{\omega}) \in \mathcal{G}_n$ and consider the event \mathcal{B} that:

- (i) at the end of the *burn-in* part of the round $\omega(\Delta_1) = \tilde{\omega}(\Delta_1)$ in Λ_n ,
- (ii) the vertex x_* appearing in (ii) of step (b) of Definition 3.1 is within $\epsilon \log t$ from the right boundary of Λ_n and $\omega(\Delta_1)_{x_*} = \tilde{\omega}(\Delta_1)_{x_*} = 0$,
- (iii) $\omega(\Delta_1)$ and $\tilde{\omega}(\Delta_1)$ satisfy SSC in the interval $[-3(v_{\max}/v_{\min})\kappa t^\epsilon, -\kappa \epsilon \log t]$.

Theorem 3.3 proves that, uniformly in $\omega, \tilde{\omega} \in \mathcal{G}_n$,

$$MC_{\Lambda_n}(\mathcal{B}) \geq 1 - O(e^{-t^{\epsilon/2}}) - O(t^{-7\epsilon}) - p^{\epsilon \log t} = 1 - O(p^{\epsilon \log t}).$$

The first error term takes into account the variation distance from π of the marginals in Λ_n of $\mathbb{P}_{\tilde{\omega}}^{\Delta_1}$ and $\mathbb{P}_{\omega}^{\Delta_1}$, the second error term bounds the probability that either $\omega(\Delta_1)$ or $\tilde{\omega}(\Delta_1)$ do not satisfy the SSC condition in the interval $[-3(v_{\max}/v_{\min})\kappa t^\epsilon, -\kappa \epsilon \log t]$ and the third term bounds the π -probability that the event in item (ii) does not occur.

Next we claim that, for any κ large enough and any $z \in \Lambda_n$ at distance at most $\epsilon \log t$ from the right boundary of Λ_n ,

$$\begin{aligned} & \sup_{\omega, \tilde{\omega} \in \mathcal{G}_n} \mathbb{P}(\omega(\Delta) \neq \tilde{\omega}(\Delta) \text{ in } I_{n+1} \mid \mathcal{B}, \{x_* = z\}, \{\xi = 1\}) \\ & \leq e^{-|a_n - a_{n+1}|} + 3\kappa t^\epsilon \left(\frac{c^*}{q}\right)^{\epsilon \log t} e^{-\Delta_2(\text{gap}(\mathcal{L}) \wedge m)} = O(t^{-2\epsilon}). \end{aligned} \quad (3.15)$$

The first term in the r.h.s. is the probability that the linking event $F(a_n, a_{n+1}; \Delta_1, \Delta)$ occurred. The second term comes from Proposition 2.3 and it bounds from above the probability that, under the maximal coupling for the East process in the interval $[x_* + 1, -1]$ and in a time lag Δ_2 , we see a discrepancy.

In conclusion, the probability that $\omega(\Delta) = \tilde{\omega}(\Delta)$ in I_{n+1} is larger than

$$MC_{\Lambda_n}(\mathcal{B})(1 - o(1))\mathbb{P}(\xi = 1) \geq \frac{1}{2}e^{-2\Delta_2},$$

thus proving the claim. \square

We are now in a position to finish the proof of Theorem 3.1. Let $v^* \equiv v_{\min} - 3\epsilon v_{\max}$ and let $a_N = -v_{\min}t_N + \epsilon v_{\max}t$ be the left boundary of the interval $I_N = [a_N, 0]$. Since by Remark 2.9, v_{\min} converges to 1 as $p \downarrow 0$, v^* can be chosen uniformly as $p \downarrow 0$.

Pick two configurations $\omega(t_N), \omega'(t_N)$ at time t_N and make them evolve under the basic coupling until time t . Clearly the events $\{\omega(t_N)_x = \omega'(t_N)_x \forall x \in I_N\}$ and $\{\exists x \in [-v^*t, 0] : \omega_x(t) \neq \omega'_x(t)\}$ imply the linking event $F(a_N, -v^*t; t_N, t)$ from Lemma 2.6. By construction $|v^*t - a_N| = \epsilon v_{\max}t \geq v_{\max}(t - t_N)$ for large enough t . Therefore,

$$\begin{aligned} M_{\omega, \omega'}^t(\exists x \in [-v^*t, 0] : \omega_x \neq \omega'_x) & \leq p_N + \mathbb{P}(F(a_N, -v^*t; t_N, t)) \\ & \leq O(e^{-t^\alpha}) + e^{-\epsilon v_{\max}t}, \end{aligned}$$

as required. Moreover, by the proof of Claim 3.5, α can be chosen uniformly as $p \downarrow 0$. Thus we are done. \square

3.3. Mixing properties of the front increments: Proof of Corollary 3.2. To prove (3.2) we observe that, for any $n \geq v_{\max}\Delta$, the event $|\xi_1| \geq n$ implies the occurrence of the linking event $F(0, n; 0, \Delta)$. Lemma 2.6 now gives that

$$\mathbb{E}_\omega [f(\xi_1)^2] \leq \max_{|x| \leq v_{\max}\Delta} f(x)^2 + \sum_{n \geq v_{\max}\Delta} f(n+1)^2 e^{-n} < \infty. \quad (3.16)$$

In order to prove (3.3) we apply the Markov property at time t_{n-1} and write

$$\mathbb{E}_\omega [f(\xi_n)] = \int d\mu_\omega^{t_{n-1}}(\omega') \mathbb{E}_{\omega'} [f(\xi_1)].$$

At this stage we would like to appeal to Theorem 3.1 to get the sought statement. However Theorem 3.1 only says that, for any t large enough, μ_ω^t is very close to the invariant measure ν in the interval $[-v^*t, 0]$. In order to overcome this problem, for any $\omega \in \Omega_F$ and any $t > 0$ we define $\Phi_t(\omega) \in \Omega_F$ as that configuration which is equal to ω in $[-v^*t, 0]$ and identically equal to 1 elsewhere. Then, under the basic coupling, the front at time t starting from $\Phi_t(\omega)$ is different from the front starting from ω iff the linking event $F(-v^*t, 0; 0, \Delta)$ occurred.

In conclusion, if $v^*t_{n-1} \geq v_{\max}\Delta$,

$$\begin{aligned} & \sup_{\omega \in \Omega_F} \left| \int d\mu_\omega^{t_{n-1}}(\omega') \mathbb{E}_{\omega'} [f(\xi_1)] - \int d\mu_\omega^{t_{n-1}}(\omega') \mathbb{E}_{\Phi_{t_{n-1}}(\omega')} [f(\xi_1)] \right| \\ & \leq \mathbb{P}(F(-v^*t_{n-1}, 0; 0, \Delta))^{1/2} \sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_1)^2]^{1/2} \\ & \leq e^{-v^*t_{n-1}/2} \sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_1)^2]^{1/2}. \end{aligned}$$

We can now apply Theorem 3.1 to get that

$$\begin{aligned} & \left| \int d\mu_\omega^{t_{n-1}}(\omega') \mathbb{E}_{\Phi_{t_{n-1}}(\omega')} [f(\xi_1)] - \mathbb{E}_\nu [f(\xi_1)] \right| \\ & \leq \left[\sup_{\omega \in \Omega_F} \|\mu_\omega^{t_{n-1}} - \nu\|_{[-v^*t_{n-1}, 0]}^{1/2} + e^{-v^*t_{n-1}/2} \right] \sup_{\omega \in \Omega_F} \mathbb{E}_\omega [f(\xi_1)^2]^{1/2} = O(e^{-t_{n-1}^\alpha}). \end{aligned}$$

To prove (3.4) suppose first that $v^*(j-1) \geq v_{\max}(n-j)$ where v^* is the constant appearing in Theorem 3.1. Then we can use the Markov property at time t_{j-1} and repeat the previous steps to get the result. If instead $v^*(j-1) \leq v_{\max}(n-j)$ it suffices to write

$$\text{Cov}_\omega(\xi_j, \xi_n) = \text{Cov}_\omega(\xi_j, \mathbb{E}_\omega[\xi_n | \mathcal{F}_{t_j}])$$

and apply (3.3) to $\mathbb{E}_\omega[\xi_n | \mathcal{F}_{t_j}]$ to get that in this case

$$\sup_{\omega \in \Omega_F} |\text{Cov}_\omega(\xi_j, \xi_n)| = O(e^{-\gamma(n-j)^\alpha}) \quad (3.17)$$

for some constant γ depending on v^*, v_{\max} . Following the exact steps as above after replacing ξ_j, ξ_n by $f(\xi_j), f(\xi_n)$ yields (3.5). Finally, (3.6) follows from exactly the same steps leading to the proof of (3.3). \square

4. PROOFS OF MAIN RESULTS

4.1. Proof of Theorem 1. We begin with the proofs of (1.1) and (1.2).

As far as (1.2) is concerned, this follows directly from observing that

$$\frac{d}{dt} \mathbb{E}_\omega [X(\omega(t))] = q - p\mu_\omega^t(\omega(-1) = 0) = v + O(e^{-t^\alpha}).$$

Appealing to (3.1) and Corollary 3.2 we get immediately that for any $\omega \in \Omega_F$

$$\mathbb{E}_\omega \left[((X(\omega(t)) - vt)/t)^4 \right] = O(t^{-2})$$

and (1.1) follows at once.

We next prove (1.3). Using Corollary 3.2 with $f(x) = x^2$, we get that, for any n large enough,

$$\text{Var}_\omega(\xi_n) = \text{Var}_\nu(\xi_1) + O(e^{-n^\alpha}).$$

Hence

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left[\sum_{n=1}^{N_t} \text{Var}_\omega(\xi_n) + \text{Var}_\omega(X(\omega(t)) - X(\omega(t_N))) \right] = \Delta^{-1} \text{Var}_\nu(\xi_1).$$

Moreover, (3.4) implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{2}{t} \left[\sum_{j < n}^{N_t} \text{Cov}_\omega(\xi_j, \xi_n) + \sum_{n=1}^{N_t} \text{Cov}_\omega(\xi_n, X(\omega(t)) - X(\omega(t_N))) \right] \\ = \frac{2}{\Delta} \sum_{n \geq 2} \text{Cov}_\nu(\xi_1, \xi_n), \end{aligned}$$

the series being absolutely convergent because of (3.17). In conclusion, for any $\omega \in \Omega_F$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}_\omega(X(\omega(t))) = \Delta^{-1} \left[\text{Var}_\nu(\xi_1) + 2 \sum_{n \geq 2} \text{Cov}_\nu(\xi_1, \xi_n) \right]. \quad (4.1)$$

Next we show that for p small enough the r.h.s. of (4.1) is positive. We first observe that there exists $c = c(p)$ such that $\limsup_{p \rightarrow 0^+} c(p) < \infty$ and

$$\sup_{\Delta} \sum_{n \geq 2} |\text{Cov}_\nu(\xi_1, \xi_n)| \leq c(p). \quad (4.2)$$

To prove (4.2) assume without loss of generality that $\Delta \in \mathbb{N}$ and write $\xi_1 = \sum_{i=1}^{\Delta} \xi'_i$ and $\xi_n = \sum_{i=(n-1)\Delta+1}^{n\Delta} \xi'_i$, where the increments ξ'_i 's refer to a unit time lag. Thus

$$\sum_{n \geq 2} |\text{Cov}_\nu(\xi_1, \xi_n)| \leq \sum_{n \geq 2} \sum_{i=1}^{\Delta} \sum_{j=(n-1)\Delta+1}^{n\Delta} |\text{Cov}_\nu(\xi'_i, \xi'_j)|$$

The claim now follows from (3.4) together with the fact that the constants α, v^* are uniformly bounded away from zero as $p \rightarrow 0$.

Thus, in order to show that the r.h.s. of (4.1) is positive, it is enough to show that it is possible to choose Δ and p such that $\text{Var}_\nu(\xi_1) > \limsup c(p)$.

Recall that $q^* = \nu(\omega_{-1} = 0)$. Then a little computation shows that

$$\begin{aligned} \frac{d}{dt} \text{Var}_\nu(X(\omega(t))) &= q + pq^* - 2p \text{Cov}_\nu(X(\omega(t)), \mathbb{1}_{\{\omega(t) \in \Omega^{**}\}}) \\ &\geq q + pq^* - 2p \text{Var}_\nu(X(\omega(t)))^{1/2} (q^*(1 - q^*))^{1/2} \end{aligned} \quad (4.3)$$

$$\geq q + pq^* - p \text{Var}_\nu(X(\omega(t)))^{1/2}, \quad (4.4)$$

where $\Omega^{**} = \{\omega \in \Omega^* : \omega_{X(\omega)-1} = 0\}$.

If $[\text{Var}_\nu(\xi_1)]^{1/2} \leq \frac{q+pq^*}{2p}$ for all $\Delta > 0$, then (4.4) implies that

$$\lim_{\Delta \rightarrow \infty} \text{Var}_\nu(\xi_1) = \infty.$$

Otherwise there exists $\Delta > 0$ such that $[\text{Var}_\nu(\xi_1)]^{1/2} \geq \frac{q+pq^*}{2p}$; hence, the desired inequality (1.3) follows by taking p small enough.

It remains to prove (1.4). If $\sigma^* = 0$, then necessarily

$$\sup_{\Delta} \text{Var}_\nu(\xi_1) < \infty.$$

In this case the Chebyshev inequality suffices to prove that, for any $\omega \in \Omega_{\mathbb{F}}$,

$$(X(\omega(t)) - vt)/\sqrt{t} \xrightarrow{\mathbb{P}_\omega} 0, \quad \text{as } t \rightarrow \infty.$$

If instead $\sigma^* > 0$, we appeal to an old result on the central limit theorem for mixing stationary random fields [66, 6]. Unfortunately our mixing result, as expressed e.g. in Corollary 3.2 (cf. (3.6)), is not exactly what is needed there and we have to go through some of the steps of [66, 6] to prove the sought statement.

Consider the sequence $\{\xi_j\}$ defined above (with e.g. $\Delta = 1$) and let $\bar{\xi}_j := \xi_j - v\Delta$. Further let $S_n = \sum_{j=1}^n \bar{\xi}_j$. It suffices to prove that, for all $\omega \in \Omega_{\mathbb{F}}$, the law of $S_n/\sigma_*\sqrt{n}$ converges to the normal law $\mathcal{N}(0, 1)$. As in [66, 6] let $f_N(x) = \max[\min(x, N), -N]$ and let $\tilde{f}_N(x) := x - f_N(x)$. Clearly $\text{Var}(\tilde{f}_N(\bar{\xi}_j)) \rightarrow 0$ as $N \rightarrow \infty$ uniformly in j .

Then Corollary 3.2 (3.5) implies that

$$\mathbb{E}_\omega \left[\frac{\sum_{j=1}^n \tilde{f}_N(\bar{\xi}_j) - \mathbb{E}_\omega[\tilde{f}_N(\bar{\xi}_j)]}{n^{1/2}} \right]^2 = \frac{1}{n} \sum_{j,k=1}^n \text{Cov}_\omega(\tilde{f}_N(\bar{\xi}_j), \tilde{f}_N(\bar{\xi}_k))$$

converges to 0 as $N \rightarrow \infty$ uniformly in n . Hence it is enough to prove the result for the truncated variables $f_N(\bar{\xi}_j)$. For lightness of notation we assume henceforth that the $\bar{\xi}_j$'s are bounded.

Let now $\ell_n = n^{1/3}$ and let

$$S_{j,n} = \sum_{k=1}^n \mathbf{1}_{|k-j| \leq \ell_n} \bar{\xi}_k, \quad \alpha_n = \sum_{j=1}^n \mathbb{E}_\omega[\bar{\xi}_j S_{j,n}], \quad j \in \{1, \dots, n\}.$$

The decay of covariances (3.4) implies that $\alpha_n = \text{Var}_\omega(S_n) + o(1)$. Hence it is enough to show that $S_n/\sqrt{\alpha_n}$ is asymptotically normal. The main observation of [66, 6], in turn inspired by the Stein method [2525, 25], is that the latter property of $S_n/\sqrt{\alpha_n}$ follows if

$$\lim_{n \rightarrow \infty} \mathbb{E}_\omega \left[(i\lambda - S_n) e^{i\lambda \frac{S_n}{\sqrt{\alpha_n}}} \right] = 0, \quad \forall \lambda \in \mathbb{R}. \quad (4.5)$$

In turn (4.5) follows if (see [66; 6, Eqs. (4)–(5)])

$$\lim_{n \rightarrow \infty} \mathbb{E}_\omega \left[\left(1 - \frac{1}{\sqrt{\alpha_n}} \sum_{j=1}^n \bar{\xi}_j S_{j,n} \right)^2 \right] = 0 \quad (4.6)$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\alpha_n}} \mathbb{E}_\omega \left[\left| \sum_{j=1}^n \bar{\xi}_j \left(1 - e^{-i\lambda \frac{S_n}{\sqrt{\alpha_n}}} - i\lambda S_{j,n} \right) \right| \right] = 0 \quad (4.7)$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=1}^n \mathbb{E}_\omega \left[\bar{\xi}_j e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} \right] = 0. \quad (4.8)$$

As in [66, 6], the mixing properties (3.4) and (3.6) easily prove that (4.6) and (4.7) hold. As far as (4.8) is concerned the formulation of Theorem 3.1 forces us to argue a

bit differently than [66, 6]. We first observe that, using the boundedness of the variables $\bar{\xi}_j$'s, (4.8) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n \mathbb{E}_\omega \left[\bar{\xi}_j e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} \right] = 0, \quad \forall \lambda \in \mathbb{R}. \quad (4.9)$$

Fix two numbers M and L with $L \leq M/10$ (eventually they will be chosen logarithmically increasing in n) and write

$$\begin{aligned} e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} &= \sum_{m=0}^M \frac{(i\lambda)^m}{m!} \left(\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \\ &+ \sum_{m=M+1}^{\infty} \frac{(i\lambda)^m}{m!} \left(\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \mathbb{1}_{\{|(S_n - S_{j,n})/\sqrt{\alpha_n}| \leq L\}} \\ &+ \left[e^{i\lambda \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}}} - \sum_{m=0}^M \frac{(i\lambda)^m}{m!} \left(\frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right)^m \right] \mathbb{1}_{\{|(S_n - S_{j,n})/\sqrt{\alpha_n}| > L\}} \\ &=: Y_1^{(j)} + Y_2^{(j)} + Y_3^{(j)}. \end{aligned}$$

Let us first examine the contribution of $Y_2^{(j)}$ and $Y_3^{(j)}$ to the covariance term (4.9). Using the boundedness of the variables $\{\bar{\xi}_j\}_{j=1}^n$ there exists a positive constant c such that:

$$\begin{aligned} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n |\mathbb{E}_\omega [\bar{\xi}_j Y_2^{(j)}]| &\leq c \sqrt{n} \frac{L^{M+1}}{M!}, \\ \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n |\mathbb{E}_\omega [\bar{\xi}_j Y_3^{(j)}]| &\leq c \sqrt{n} \max_j \mathbb{E}_\omega \left[e^{2|\lambda| \frac{|S_n - S_{j,n}|}{\sqrt{\alpha_n}}} \right]^{1/2} \mathbb{P}_\omega \left(\left| \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right| > L \right). \end{aligned}$$

Lemma 4.1. *There exists $c > 0$ such that, for all n large enough and any $\beta = O(\log n)$,*

$$\mathbb{E}_\omega \left[e^{\beta \frac{|S_n - S_{j,n}|}{\sqrt{\alpha_n}}} \right] \leq 2e^{c\beta^2}. \quad (4.10)$$

Moreover, there exists $c' > 0$ such that, for all n large enough and all $L \leq \log n$,

$$\mathbb{P}_\omega \left(\left| \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right| > L \right) \leq e^{-c'L^2}. \quad (4.11)$$

Assume for the moment the lemma and choose $L = M/10$ and $M = \log n$. We can conclude that

$$\frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n |\mathbb{E}_\omega [\bar{\xi}_j (Y_2^{(j)} + Y_3^{(j)})]| \leq C \sqrt{n} \left[e^{-c'L^2} + \frac{L^{M+1}}{M!} \right],$$

so that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n |\mathbb{E}_\omega [\bar{\xi}_j (Y_2^{(j)} + Y_3^{(j)})]| = 0.$$

We now examine the contribution of $Y_1^{(j)}$ to (4.9). Recall

$$S_n - S_{j,n} = \sum_{\substack{1 \leq i \leq n \\ |i-j| > \ell_n}} \bar{\xi}_i.$$

Thus clearly,

$$\frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n \mathbb{E}_\omega \left[\bar{\xi}_j (Y_1^{(j)}) \right] = \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n \sum_{m=1}^M \left(\frac{i\lambda}{\sqrt{n}} \right)^m \sum_{\substack{i_1, \dots, i_m \\ \min_k |i_k - j| \geq \ell_n}} \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^m \bar{\xi}_{i_k} \right],$$

where the labels i_1, \dots, i_m run in $\{1, 2, \dots, n\}$.

Lemma 4.2. *Let $M = \log n$. Then, for any $m \leq M$, any $j \in \{\ell_n, \dots, n\}$ and any $\{i_1, \dots, i_m\}$ satisfying $\min_k |i_k - j| \geq \ell_n$, it holds that*

$$\left| \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^m \bar{\xi}_{i_k} \right] \right| = O(e^{-n^{\alpha/6}}).$$

Here α is the mixing exponent appearing in Theorem 3.1.

Assuming the lemma we get immediately that also

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\alpha_n}} \sum_{j=\ell_n}^n \mathbb{E}_\omega \left[\bar{\xi}_j (Y_1^{(j)}) \right] = 0$$

and (4.9) is established. In conclusion, (1.4) would follow from Lemmas 4.1–4.2.

Proof of Lemma 4.1. Let us begin with (4.10). For simplicity we prove that, for any constant $\beta = O(\log n)$, $\mathbb{E}_\omega [\exp(\beta S_n / \sqrt{n})] \leq e^{c\beta^2}$ for some constant $c > 0$. Similarly one could proceed for $\mathbb{E}_\omega [\exp(-\beta S_n / \sqrt{n})]$ and get that

$$\mathbb{E}_\omega [\exp(\beta |S_n| / \sqrt{n})] \leq \mathbb{E}_\omega [\exp(\beta S_n / \sqrt{n})] + \mathbb{E}_\omega [\exp(-\beta S_n / \sqrt{n})] \leq 2e^{c\beta^2}.$$

We partition the discrete interval $\{1, 2, \dots, n\}$ into disjoint blocks of cardinality $n^{1/3}$. Given a integer κ , by applying the Cauchy-Schwarz inequality a finite number of times depending on κ , it is sufficient to prove the result for S_n replaced by the sum $S_B^{(\kappa)}$ of the $\bar{\xi}_j$'s restricted to an arbitrary collection \mathcal{B} of blocks with the property that any two blocks in \mathcal{B} are separated by at least κ blocks.

Fix one such collection \mathcal{B} and let B be the rightmost block in \mathcal{B} . Let n_B be the largest label in \mathcal{B} which is not in the block B and let $t_B = n_B \Delta$ be the corresponding time. Further let $Z_B = \sum_{j \in B} \bar{\xi}_j$. If $c\kappa > v_{\max}$ where c is the constant appearing in Theorem 3.1, we can appeal to (3.6) to obtain

$$\mathbb{E}_\nu [\exp(\beta Z_B / \sqrt{n}) | \mathcal{F}_{t_B}] = \mathbb{E}_\nu [\exp(\beta Z_B / \sqrt{n})] + O(e^{-n^{\alpha/3}} e^{\beta n^{-1/6}}).$$

Using the trivial bound $Z_B / \sqrt{n} = O(n^{-2/3})$ we have

$$\mathbb{E}_\nu [\exp(c Z_B / \sqrt{n})] = 1 + \frac{\beta^2}{2n} \text{Var}_\nu(Z_B) + O(\beta^3 n^{-7/6}) \text{Var}_\nu(Z_B),$$

where $\text{Var}_\nu(Z_B) = O(n^{1/3})$ thanks to (3.4). Above we used the trivial bound

$$\mathbb{E}_\nu [|Z_B|^3] \leq c n^{1/3} \text{Var}_\nu(Z_B).$$

In conclusion, using the apriori bound $\beta \leq \log n$, we get that

$$\mathbb{E}_\omega [\exp(\beta Z_B / \sqrt{n}) | \mathcal{F}_{t_B}] \leq 1 + c \frac{\beta^2}{n^{2/3}}.$$

The Markov property and a simple iteration imply that,

$$\mathbb{E}_\nu [\exp(\beta S_B^{(\kappa)} / \sqrt{n})] \leq \left[1 + c \frac{\beta^2}{n^{2/3}} \right]^{|\mathcal{B}|} \leq \exp(c' \beta^2),$$

uniformly in the cardinality $|\mathcal{B}|$ of the collection. The bound (4.10) is proved.

The bound (4.11) follows at once from (4.10) and the exponential Chebyshev inequality

$$\mathbb{P}_\omega \left(\left| \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right| > L \right) \leq e^{-\beta L} \mathbb{E}_\omega \left[\exp \left(\left| \frac{(S_n - S_{j,n})}{\sqrt{\alpha_n}} \right| \right) \right],$$

with $\beta = \varepsilon L$, ε being a sufficiently small constant. \square

Proof of Lemma 4.2. Fix $j \in [1, \dots, n]$ and $m \leq \log n$, together with a choice of labels $1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n$ such that $\min_k |i_k - j| \geq \ell_n$. Let $t_{i_k} = i_k \Delta$.

• If $i_m \leq j - \ell_n$ then we can apply the Markov property at time t_{i_m} together with Corollary 3.2 to get

$$\left| \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^m \bar{\xi}_{i_k} \right] \right| \leq e^{-n^{\alpha/3}} \mathbb{E}_\omega \left[\prod_{i=k}^m |\bar{\xi}_{i_k}| \right] \leq c^m e^{-n^{\alpha/3}}.$$

• If instead there exists $b \leq m - 1$ such that $i_b < j < i_{b+1}$ we need to distinguish between two sub-cases.

(a) For all $k \geq b + 2$ it holds that $i_k - i_{k-1} \leq n^{1/6}$ and in particular $t_m - t_{b+1} \leq n^{1/6} \Delta$. In this case the fact that $t_{b+1} - j \Delta \geq \ell_n$ and $v_{\max}(t_m - t_{b+1}) \ll \ell_n$ together with (3.6), imply that

$$\mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^m \bar{\xi}_{i_k} \right] = \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^b \bar{\xi}_{i_k} \right] \left[\mathbb{E}_{\nu'} \left[\prod_{i=b+1}^m \bar{\xi}_{i_k} \right] + O \left(e^{-n^{\alpha/3}} \text{poly}(n) \right) \right].$$

The conclusion of the lemma then follows from the previous case $i_m \leq j - \ell_n$.

(b) We now assume that $k^* := \max\{k \geq b + 1 : i_{k+1} \geq i_k + n^{1/6}\} < n$. By repeating the previous step with the Markov property applied at time $t_{i_{k^*}}$ we get

$$\mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^m \bar{\xi}_{i_k} \right] = \mathbb{E}_\omega \left[\bar{\xi}_j \prod_{i=k}^{k^*} \bar{\xi}_{i_k} \right] \left(\mathbb{E}_{\nu'} \left[\prod_{i=k^*+1}^m \bar{\xi}_{i_k} \right] + O \left(e^{-n^{\alpha/3}} \text{poly}(n) \right) \right).$$

By iterating the above procedure we can reduce ourselves to case (a) and get the sought result. \square

As Lemmas 4.1–4.2 imply (1.4), this concludes the proof of Theorem 1. \square

Remark 4.3. The above proof also established that the limiting variance σ_*^2 is strictly positive for all p small enough.

4.2. Proof of Theorem 2. Given the interval $\Lambda = [1, \dots, L]$ and $\omega \in \Omega_\Lambda$, let $\mathbb{P}_\omega^{\Lambda, t}$ be the law of the process started from ω . Recall that

$$\|\mathbb{P}_\omega^{\Lambda, t} - \mathbb{P}_{\omega'}^{\Lambda, t}\| = \inf \{ M(\omega(t)) \neq \omega'(t) : M \text{ a coupling of } \mathbb{P}_\omega^{\Lambda, t} \text{ and } \mathbb{P}_{\omega'}^{\Lambda, t} \},$$

and introduce the hitting time

$$\tau(L) = \inf \{ t : X(\omega(t)) = L \},$$

where the initial configuration is identically equal to one (in the sequel 1). It is easy to check (see, e.g., [1818, 18]) that at time $\tau(L)$ the basic coupling (cf. §2.1) has coupled all initial configurations. Thus

$$d_{\text{TV}}(t) \leq \sup_{\omega, \omega'} \|\mathbb{P}_\omega^{\Lambda, t} - \mathbb{P}_{\omega'}^{\Lambda, t}\| \leq \mathbb{P}^\Lambda(\tau(L) > t).$$

Using the graphical construction, up to time $\tau(L)$ the East process in Λ started from the configuration **1** coincides with the infinite East process started from the configuration $\omega^* \in \Omega_F$ with a single zero at the origin. Therefore

$$\mathbb{P}_1^\Lambda(\tau(L) > t) \leq \mathbb{P}_{\omega^*}(X(\omega(t)) < L),$$

thus establishing a bridge with Theorem 1. Recall now the definition of σ_* from Theorem 1 and distinguish between the two cases $\sigma_* > 0$ and $\sigma_* = 0$.

- The case $\sigma_* > 0$. Here we will show that

$$T_{\text{mix}}(L, \epsilon) = v^{-1}L + (1 + o(1)) \frac{\sigma_*}{v^{3/2}} \Phi^{-1}(1 - \epsilon) \sqrt{L}. \quad (4.12)$$

For $s \in \mathbb{R}$, let $t_\star = L/v + s\sqrt{L}$. Then (1.3) implies that

$$\mathbb{P}_{\omega^*}(X(\omega(t_\star)) < L) = \mathbb{P}_{\omega^*}\left(\frac{X(\omega(t_\star)) - vt_\star}{\sqrt{L/v}} < -v^{3/2}s\right) \rightarrow \Phi\left(-\frac{v^{3/2}s}{\sigma_*}\right)$$

as $L \rightarrow \infty$. Hence,

$$\limsup_{L \rightarrow \infty} d_{\text{TV}}(L/v + s\sqrt{L}) \leq \Phi\left(-\frac{v^{3/2}s}{\sigma_*}\right). \quad (4.13)$$

To prove a lower bound on the total variation norm, set $a_L = \log L$ (any diverging sequence which is $o(\sqrt{L})$ would do here) and define the event

$$A_t = (\omega_x(t) = 1 \text{ for all } x \in (L - a_L, L]).$$

Then

$$\mathbb{P}_1^\Lambda(A_t) \geq \mathbb{P}_{\omega^*}(X(\omega(t)) \leq L - a_L) \quad \text{and} \quad \pi(A_t) = p^{a_L} = o(1),$$

and so any lower bound on $\mathbb{P}_{\omega^*}(X(\omega(t_\star)) \leq L - a_L)$ would translate to a lower bound on $d_{\text{TV}}(t_\star)$ up to an additive $o(1)$ -term. Again by (1.3),

$$\mathbb{P}_{\omega^*}(X(\omega(t_\star)) \leq L - a_L) = \mathbb{P}_{\omega^*}\left(\frac{X(\omega(t_\star)) - vt_\star}{\sqrt{L/v}} \leq -v^{3/2}s - a_L \sqrt{v/L}\right) \rightarrow \Phi\left(-\frac{v^{3/2}s}{\sigma_*}\right)$$

as $L \rightarrow \infty$. Thus we conclude that

$$\liminf_{L \rightarrow \infty} d_{\text{TV}}(L/v + s\sqrt{L}) \geq \Phi\left(-\frac{v^{3/2}s}{\sigma_*}\right). \quad (4.14)$$

Eq. (4.12) now follows from (4.13) and (4.14) by choosing $s = \sigma_* v^{-3/2} \Phi^{-1}(1 - \epsilon)$.

- The case $\sigma_* = 0$. Here a similar argument shows that

$$T_{\text{mix}}(L, \epsilon) = v^{-1}L + O_\epsilon(1),$$

using the fact (following the results in §3) that $\sup_\omega \sup_t \text{Var}_\omega(X(\omega(t))) < \infty$ if $\sigma_* = 0$.

This concludes the proof of Theorem 2. \square

5. CUTOFF AND CONCENTRATION FOR CONSTRAINED MODELS ON TREES

In this section we consider constrained oriented models on regular trees and prove strong concentration results for hitting times which are the direct analog of the hitting time $\tau(L)$ define in §4.2 for the East process. As a consequence we derive a strong cutoff result for the “maximally constrained model” (see below).

5.1. Kinetically constrained models on trees. Let \mathbb{T} be the k -ary rooted tree, $k \geq 2$, in which each vertex x has k children. We will denote by r the root and by \mathbb{T}_L the subtree of \mathbb{T} consisting of the first L -levels starting from the root.

In analogy to the East process, for a given integer $1 \leq j \leq k$ consider the constrained oriented process OFA-jf on $\Omega = \{0, 1\}^{\mathbb{T}}$ (cf. [2323, 23]) in which each vertex waits an independent mean one exponential time and then, provided that j among its children are in state 0, updates its spin variable ω_x to 1 with probability p and to 0 with probability $q = 1 - p$. It is known that this process exhibits an ergodicity breakdown above a certain critical probability $p = p_c(k, j)$ (defined more precisely later). In this paper we will only examine the two extreme cases $j = 1$ and $j = k$ which will be referred to in the sequel as the *minimally* and *maximally* constrained models.

The finite volume version of the OFA-jf process is a continuous time Markov chain on $\Omega_{\mathbb{T}_L} = \{0, 1\}^{\mathbb{T}_L}$. In this case, in order to guarantee irreducibility, the variables at leaves of \mathbb{T}_L are assumed to be unconstrained. As in the case of the East process, the product Bernoulli(p) measure π is the unique reversible measure and the same graphical construction described in §2.1 holds in this new context.

5.2. New Results. We are now in a position to state our results for the minimally and maximally constrained finite volume OFA-jf models. Recall that

$$T_{\text{mix}}(L, \varepsilon) := \inf\{t : \max_{\omega \in \Omega_{\mathbb{T}_L}} \|\mu_{\omega}^t - \pi\| \leq \varepsilon\}, \quad \varepsilon \in (0, 1)$$

and define $T_{\text{hit}}(L) := \mathbb{E}[\tau(L)]$, where $\tau(L)$ is the first legal ring for the root for the OFA-jf process on $\Omega_{\mathbb{T}_L}$ started from the configuration identically equal to one. Our first result addresses the concentration of $\tau(L)$. Recall that $O_{\delta}(\cdot)$ denotes that the implicit constant may depend on δ .

Theorem 5.1. *The following hold for the centered variable $\tau(L) - T_{\text{hit}}(L)$, denoted $\bar{\tau}(L)$.*

- (i) *Consider either the minimally or the maximally constrained model and fix $p < p_c$. For any fixed $\delta > 0$, if $n \in \mathbb{N}$ is large enough there exists $L_n \in [n, (1 + \delta)n]$ such that*

$$\mathbb{E}|\bar{\tau}(L_n)| = O_{\delta}(1).$$

- (ii) *Consider the maximally constrained model and choose $p = p_c$. For any fixed $\delta > 0$, if $n \in \mathbb{N}$ is large enough then there exists $L_n \in [n, (1 + \delta)n]$ such that*

$$\mathbb{E}|\bar{\tau}(L_n)| = O_{\delta}(L_n^{-1} T_{\text{hit}}(L_n)).$$

The second result concerns the cutoff phenomenon.

Theorem 5.2. *Consider the maximally constrained model.*

- (i) *If $p < p_c$ then for any $\delta > 0$ and any large enough n there exists $L_n \in [n, (1 + \delta)n]$ such that*

$$|T_{\text{mix}}(L_n, \varepsilon) - T_{\text{hit}}(L_n)| = O_{\varepsilon, \delta}(1) \quad \forall \varepsilon \in (0, 1).$$

- (ii) *If $p = p_c$ then for any $\delta > 0$ and any large enough n there exists $L_n \in [n, (1 + \delta)n]$ such that*

$$|T_{\text{mix}}(L_n, \varepsilon) - T_{\text{hit}}(L_n)| = O_{\varepsilon, \delta}(L_n^{-1} T_{\text{hit}}(L_n)) \quad \forall \varepsilon \in (0, 1).$$

5.3. Previous work. Before proving our results we recall the main findings of [2323, 23] and [1212, 12]. We now formally define the critical density for the OFA-jf model:

$$p_c = \sup\{p \in [0, 1] : 0 \text{ is simple eigenvalue of } \mathcal{L}\},$$

where \mathcal{L} is the generator of the process. The regime $p < p_c$ is called the *ergodic regime* and we say that an *ergodicity breaking transition* occurs at the critical density p_c .

Let

$$g_p(\lambda) := p \sum_{i=k-j+1}^k \binom{k}{i} \lambda^i (1-\lambda)^{k-i}$$

be the natural bootstrap percolation recursion map (cf. [2323, 23]) associated to the OFA-jf process and let

$$\tilde{p} := \sup\{p \in [0, 1] : \lambda = 0 \text{ is the unique fixed point of } g_p(\lambda)\}.$$

In [2323, 23] it was proved that $p_c = \tilde{p}$ and that $p_c \in (0, 1)$ for $j \geq 2$ and $p_c = 1$ for $j = 1$. Notice that, for $j = k$, the value \tilde{p} coincides with the site percolation threshold on \mathbb{T} so that $p_c = \tilde{p} = 1/k$.

Consider now the finite volume OFA-jf process on $\Omega_{\mathbb{T}_L}$ and let μ_ω^t be the law of the process at time t when the initial configuration is ω . Further let h_ω^t be the relative density of μ_ω^t w.r.t the reversible stationary measure π . Define the family of mixing times $\{T_a(L)\}_{a \geq 1}$ by

$$T_a(L) := \inf \left\{ t \geq 0 : \max_{\omega} \pi (|h_\omega^t - 1|^a) \leq 1/4 \right\}.$$

Notice that $T_1(L)$ coincides with the usual mixing time $T_{\text{mix}}(L)$ of the chain (see, e.g., [2222, 22]) and that, for any $a \geq 1$, one has $T_1(L) \leq T_a(L)$. Further let $T_{\text{rel}}(L)$ be the relaxation time of the chain, ie the inverse of the spectral gap of the generator $\mathcal{L}_{\mathbb{T}_L}$.

Theorem 5.3 ([2323, 23]).

(i) Assume $p < p_c$ and consider the finite volume OFA-jf model on $\Omega_{\mathbb{T}_L}$. Then

$$\sup_L T_{\text{rel}}(L) < \infty.$$

If instead $p > p_c$ then $T_{\text{rel}}(L)$ is exponentially large in L .

(ii) For all $p \in (0, 1)$ there exists a constant $c > 0$ such that

$$T_2(L+1) - T_2(L) \leq c T_{\text{rel}}(L).$$

In particular

$$T_{\text{mix}}(L) \leq T_2(L) \leq c T_{\text{rel}}(L) L$$

The second result concerns the critical behavior $p = p_c$.

Theorem 5.4 ([1212, 12]). Consider the maximally constrained model $j = k$ and choose $p = p_c$. Then there exists $\beta \geq 2$ and $c > 0$ such that

$$c^{-1} L^2 \leq T_{\text{rel}}(L) \leq c L^\beta.$$

Moreover,

$$c^{-1} L T_{\text{rel}}(L) \leq T_{\text{mix}}(L) \leq T_2(L) \leq c L T_{\text{rel}}(L).$$

5.4. Proof of Theorem 5.1. We first need a preliminary result saying that, for infinitely many values of L , the increments of $T_{\text{hit}}(L)$ can be controlled by the corresponding relaxation time.

Lemma 5.5. There exists a constant c_1 such that, for all $\delta > 0$ and all n large enough, the following holds.

(a) In the maximally constrained model at $p \leq p_c$

$$\max \left(T_{\text{hit}}(L_n) - T_{\text{hit}}(L_n - 1), T_{\text{hit}}(L_n + 1) - T_{\text{hit}}(L_n) \right) \leq \frac{c_1}{\delta} T_{\text{rel}}((1 + \delta)n),$$

for some $L_n \in [n, (1 + \delta)n]$.

(b) *In the minimally constrained model*

$$T_{\text{hit}}(L_n + 1) - T_{\text{hit}}(L_n) \geq -\frac{c_1}{\delta} T_{\text{rel}}((1 + \delta)n)$$

for some $L_n \in [n, (1 + \delta)n]$.

Proof. Fix δ and $n \geq 1/\delta$ and consider the maximally constrained model. Using part (ii) of Theorem 5.3,

$$T_{\text{mix}}(n) \leq T_2(n) \leq c \sum_{i=1}^n T_{\text{rel}}(i) \leq cn T_{\text{rel}}(n), \quad (5.1)$$

where we used the fact that $T_{\text{rel}}(i) \leq T_{\text{rel}}(n)$ for all $i \leq n$. Fix now $c_1 > 0$ and suppose that, for all $i \in [n, (1 + \delta)n - 1]$,

$$\max\left(T_{\text{hit}}(i + 1) - T_{\text{hit}}(i), T_{\text{hit}}(i + 1) - T_{\text{hit}}(i)\right) \geq \frac{c_1}{\delta} T_{\text{rel}}((1 + \delta)n).$$

In particular

$$T_{\text{hit}}((1 + \delta)n) \geq c_1 n T_{\text{rel}}((1 + \delta)n) / 2.$$

On the other hand, using the results in [11, 1], there exists a constant $\lambda = \lambda(p)$ such that

$$T_{\text{hit}}((1 + \delta)n) \leq \lambda T_{\text{mix}}((1 + \delta)n). \quad (5.2)$$

In conclusion, using Theorem 5.3,

$$\begin{aligned} T_{\text{rel}}((1 + \delta)n) &\leq \frac{2}{c_1 n} T_{\text{hit}}((1 + \delta)n) \leq \frac{2\lambda}{c_1 n} T_{\text{mix}}((1 + \delta)n) \\ &\leq \frac{2\lambda c(1 + \delta)}{c_1} T_{\text{rel}}((1 + \delta)n), \end{aligned}$$

and we reach a contradiction by choosing $c_1 > 2\lambda c(1 + \delta)$.

Similarly, in the minimally constrained case, assume

$$T_{\text{hit}}(i + 1) - T_{\text{hit}}(i) \leq -\frac{c_1}{\delta} T_{\text{rel}}((1 + \delta)n), \quad \forall i \in [n, (1 + \delta)n - 1],$$

so that

$$0 \leq T_{\text{hit}}((1 + \delta)n) \leq T_{\text{hit}}(L) - c_1 n T_{\text{rel}}((1 + \delta)n).$$

Using again Theorem 5.3 together with (5.2) we get

$$\begin{aligned} T_{\text{rel}}((1 + \delta)n) &\leq \frac{1}{c_1 n} T_{\text{hit}}(L) \\ &\leq \frac{\lambda}{c_1 n} T_{\text{mix}}(L) \leq \frac{c\lambda}{c_1 n} L T_{\text{rel}}(L) \leq \frac{\lambda c(1 + \delta)}{c_1} T_{\text{rel}}((1 + \delta)n). \end{aligned}$$

and again we reach a contradiction by choosing $c_1 > \lambda c(1 + \delta)$. \square

5.4.1. *Proof of theorem 5.1 for the maximally constrained model.* The key observation here is that, for any $L \in \mathbb{N}$, the hitting time $\tau(L + 1)$ is stochastically larger than the maximum between k independent copies $\{\tau^{(i)}(L)\}_{i=1}^k$ of the hitting time $\tau(L)$. That follows immediately by noting that:

- starting from the configuration identically equal to 1, a vertex x can be updated only after the first time at which all its k -children have been updated;
- the projection of the OFA-jf process on the sub-trees rooted at each one of the children of the root of \mathbb{T}_{L+1} are independent OFA-jf processes on \mathbb{T}_L .

Henceforth, the proof follows from a beautiful argument of Dekking and Host that was used in [1717, 17] to derive tightness for the minima of certain branching random walks.

$$\begin{aligned}
T_{\text{hit}}(L+1) &\geq \mathbb{E}\left[\max_{i=1,\dots,k} \tau^{(i)}(L)\right] \\
&\geq \frac{1}{2}\mathbb{E}\left[\tau^{(1)}(L) + \tau^{(2)}(L) + |\tau^{(1)}(L) - \tau^{(2)}(L)|\right] \\
&= T_{\text{hit}}(L) + \frac{1}{2}\mathbb{E}\left[|\tau^{(1)}(L) - \tau^{(2)}(L)|\right] \\
&\geq T_{\text{hit}}(L) + \frac{1}{2}\mathbb{E}\left[|\bar{\tau}^{(1)}(L)|\right],
\end{aligned}$$

since whenever X', X'' are i.i.d. copies of a variable one has $\mathbb{E}|X' - X''| \geq \mathbb{E}|X' - \mathbb{E}X'|$ by conditioning on X'' and then applying Cauchy-Schwarz. Altogether,

$$\mathbb{E}\left[|\bar{\tau}^{(1)}(L)|\right] \leq 2(T_{\text{hit}}(L+1) - T_{\text{hit}}(L)). \quad (5.3)$$

The conclusion of the theorem now follows from Lemma 5.5 and Theorem 5.3. \square

5.4.2. *Proof of theorem 5.1 for the minimally constrained model.* In this case we define

$$\tau_{\min}(L) := \min_{i=1,\dots,k} \tau^{(i)}(L),$$

where $\tau^{(i)}(L)$ is the first time that the i^{th} -child of the root of \mathbb{T}_{L+1} is updated and we write

$$T_{\text{hit}}(L+1) \leq \mathbb{E}\left[\tau_{\min}(L)\right] + \sup_L \sup_{\omega \in \mathcal{G}_L} \mathbb{E}_\omega\left[\tau(L)\right],$$

with \mathcal{G}_L the set of configurations in $\Omega_{\mathbb{T}_L}$ with $\omega_r = 1$ and at least one zero among the children of the children of the root r .

Lemma 5.6. $\sup_L \sup_{\omega \in \mathcal{G}_L} \mathbb{E}_\omega \tau(L) < \infty$.

Assuming the lemma we write

$$\begin{aligned}
T_{\text{hit}}(L+1) &\leq \mathbb{E}\left[\tau_{\min}(L)\right] + c \\
&\leq \frac{1}{2}\mathbb{E}\left[\tau^{(1)}(L) + \tau^{(2)}(L) - |\tau^{(1)}(L) - \tau^{(2)}(L)|\right] + c \\
&= T_{\text{hit}}(L) - \frac{1}{2}\mathbb{E}\left[|\tau^{(1)}(L) - \tau^{(2)}(L)|\right] + c.
\end{aligned}$$

Thus

$$\mathbb{E}\left[|\bar{\tau}(L)|\right] \leq \mathbb{E}\left[|\tau^{(1)}(L) - \tau^{(2)}(L)|\right] \leq 2(T_{\text{hit}}(L) - T_{\text{hit}}(L+1)) + 2c.$$

Hence, if $L_n \in [n, (1+\delta)n]$ satisfies property (b) of Lemma 5.5, we get

$$\mathbb{E}\left[|\bar{\tau}^{(1)}(L)|\right] \leq 2\frac{c_1}{\delta}T_{\text{rel}}((1+\delta)n) + 2c.$$

The conclusion of the theorem now follows from Theorem 5.3. \square

Proof of Lemma 5.6. Fix L and $\omega \in \mathcal{G}_L$ and observe that

$$\begin{aligned}
&\mathbb{P}_\omega(\omega_r(t) = 1) \\
&= \mathbb{P}_\omega(\omega_r(t) = 1 \mid \tau(L) \geq t)\mathbb{P}_\omega(\tau(L) \geq t) + \mathbb{P}_\omega(\omega_r(t) = 1 \mid \tau(L) < t)\mathbb{P}_\omega(\tau(L) < t) \\
&= (1-p)\mathbb{P}_\omega(\tau(L) \geq t) + p.
\end{aligned}$$

That is because $\omega_r = 1$ at time $t = 0$ while it is a Bernoulli(p) random variable given that the root has been updated at least once. Thus

$$\mathbb{E}_\omega[\tau(L)] \leq \frac{1}{1-p} \int_0^\infty dt |\mathbb{P}_\omega(\omega_r(t) = 1) - p|.$$

In order to bound from above the above integral we closely follow the strategy of [1313; 13, §4]. In what follows, for any finite subtree \mathcal{T} of \mathbb{T} , we will refer to the *children* of \mathcal{T} as the vertices of $\mathbb{T} \setminus \mathcal{T}$ with their parent in \mathcal{T} . Using the graphical construction, for all times $t \geq 0$ we define a (random) *distinguished* tree \mathcal{T}_t according to the following algorithm:

- (i) \mathcal{T}_0 coincides with the root together with those among its children which have at least one zero among their children (*i.e.* they are unconstrained).
- (ii) $\mathcal{T}_t = \mathcal{T}_0$ until the first “legal” ring at time t_1 at one of the children of \mathcal{T}_0 , call it x_0 .
- (iii) $\mathcal{T}_{t_1} = \mathcal{T}_0 \cup \{x_0\}$.
- (iv) Iterate.

Exactly as in [1313; 13, §4.1], one can easily verify the following key properties of the above construction:

- (a) for all $t \geq 0$ each leaf of \mathcal{T}_t is unconstrained *i.e.* there is a zero among its children;
- (b) if at time $t = 0$ the variables $\{\omega_x\}_{x \in \mathcal{T}_0}$ are not fixed by instead are i.i.d with law π , then, conditionally on $\{\mathcal{T}_s\}_{s \leq t}$, the same is true for the variables $\{\omega_x(t)\}_{x \in \mathcal{T}_t}$.
- (c) For all $i \geq 1$, given \mathcal{T}_{t_i} and t_i , the law of the random time $t_{i+1} - t_i$ does not depend on the variables (clock rings and coin tosses) of the graphical construction in \mathcal{T}_{t_i} .

As in [1313; 13, Eqs. (4.8) and (4.10)], the above properties imply that

$$\text{Var}_\pi(\mathbb{E}_\omega[\omega_r(t) \mid \{\mathcal{T}_s\}_{s \leq t}]) \leq e^{-2t/T_{\text{rel}}(L)}.$$

Therefore,

$$\begin{aligned} \sup_{\omega \in \mathcal{G}_L} |\mathbb{E}_\omega[\omega_r(t) - p]| &\leq \sup_{\omega \in \mathcal{G}_L} \mathbb{E}_\omega \left| \mathbb{E}_\omega[\omega_r(t) - p \mid \{\mathcal{T}_s\}_{s \leq t}] \right| \\ &\leq \left(\frac{1}{p \wedge q} \right)^{|\mathcal{T}_0|} \sup_{\omega \in \mathcal{G}_L} \mathbb{E}_\omega \left[\sum_{\omega \in \Omega_{\mathcal{T}_0}} \pi(\omega) \left| \mathbb{E}_\omega(\omega_r(t) - p \mid \{\mathcal{T}_s\}_{s \leq t}) \right| \right] \\ &\leq \left(\frac{1}{p \wedge q} \right)^{|\mathcal{T}_0|} \sup_{\omega \in \mathcal{G}_L} \mathbb{E}_\omega \left[\text{Var}_\pi(\mathbb{E}_\omega(\omega_r(t) \mid \{\xi_s\}_{s \leq t}))^{1/2} \right] \\ &\leq \left(\frac{1}{p \wedge q} \right)^{|\mathcal{T}_0|} e^{-t/T_{\text{rel}}(L)}. \end{aligned}$$

By Theorem 5.3 we have that $\sup_L T_{\text{rel}}(L) < \infty$, and the proof is complete. \square

Consider the maximally constrained process on $\Omega_{\mathbb{T}_{L+1}}$ and let $\tau^{\max}(L)$ be the first time at which all the children of the root have been updated at least once starting from the configuration identically equal to one. For a given $\omega \in \Omega_{\mathbb{T}_{L+1}}$ and $x \in \mathbb{T}_{L+1}$, further let $\mathcal{C}_\omega(x)$ be the maximal subtree rooted at x where ω is equal to one. Finally, recall that $\mathbb{P}(\cdot)$ denotes the basic coupling given by the graphical construction and that $\omega(t)$ denotes the process at time t started from the initial configuration ω .

Lemma 5.7. *There exists some $c > 0$ such that*

$$\max_{\omega \in \Omega_{\mathbb{T}_{L+1}}} \mathbb{P}(|\mathcal{C}_{\omega(\tau^{\max}(L))}(r)| \geq n) \leq c \pi \left(|\mathcal{C}_\omega(r)| \geq \frac{n-2}{k-1} \right),$$

and in particular,

$$\max_{\omega \in \Omega_{\mathbb{T}_{L+1}}} \mathbb{E} |\mathcal{C}_{\omega(\tau^{\max}(L))}(r)| \leq c \sum_{\omega} \pi(\omega) |\mathcal{C}_{\omega}(r)|.$$

Proof. Recall that under the basic coupling all the starting configurations have coupled by time $\tau^{\max}(L)$. Hence,

$$\begin{aligned} \mathbb{P}(\exists \omega \in \Omega_{\mathbb{T}_{L+1}} : |\mathcal{C}_{\omega(\tau^{\max}(L))}(r)| \geq n) &= \sum_{\omega} \pi(\omega) \mathbb{P}(|\mathcal{C}_{\omega(\tau^{\max}(L))}(r)| \geq n) \\ &\leq k \sum_{\omega} \pi(\omega) \mathbb{P}\left(|\mathcal{C}_{\omega(\tau^{(1)}(L))}(r)| \geq n, \tau^{\max}(L) = \tau^{(1)}(L)\right) \\ &\leq k \sum_{\omega} \pi(\omega) \mathbb{P}\left(|\mathcal{C}_{\omega(\tau^{(1)}(L))}(r)| \geq n\right), \end{aligned}$$

where $\tau^{(1)}(L)$ is the first time that the first (in some chosen order) child of the root has been updated starting from all ones. By construction, at time $\tau^{(1)}(L)$ the first child has all its children equal to zero. Therefore the event $\{|\mathcal{C}_{\omega(\tau^{(1)}(L))}(r)| \geq n\}$ implies that there exists some other child x of the root such that $\mathcal{C}_{\omega(\tau^{(1)}(L))}(x)$ has cardinality at least $(n-2)/(k-1)$. Using reversibility and the independence between $\tau^{(1)}(L)$ and the process in the subtree of depth L rooted at x together with a union bound over the choice of x , we conclude that

$$\sum_{\omega} \pi(\omega) \mathbb{P}\left(|\mathcal{C}_{\omega(\tau^{(1)}(L))}(r)| \geq n\right) \leq (k-1)\pi\left(|\mathcal{C}_{\omega}(r)| \geq \frac{n-2}{k-1}\right).$$

The statement of the lemma follows at once by summing over n . \square

Using the lemma we can now prove the analogue of Lemma 5.6

Lemma 5.8. *Fix any positive integer ℓ . For all $p \leq p_c$ there exists $c = c(\ell, p)$ such that*

$$\begin{aligned} (i) \quad T_{\text{hit}}(L + \ell) &\leq \mathbb{E}[\tau^{\max}(L)] + cT_{\text{rel}}(L) && \text{if } p < p_c, \\ (ii) \quad T_{\text{hit}}(L + \ell) &\leq \mathbb{E}[\tau^{\max}(L)] + cLT_{\text{rel}}(L) && \text{if } p = p_c. \end{aligned}$$

Moreover, for any $d > 0$,

$$\mathbb{P}(\tau(L + \ell) - \tau^{\max}(L) \geq dT_{\text{rel}}) = \begin{cases} O(d^{-1}) & \text{if } p < p_c, \\ O(d^{-1/3}) & \text{if } p = p_c. \end{cases} \quad (5.4)$$

Proof. For simplicity we give a proof for the case $\ell = 1$. The general proof is similar and we omit the details. We first claim that, starting from $\omega \in \Omega_{\mathbb{T}_{L+1}}$, one has

$$\mathbb{E}_{\omega}[\tau(L + 1)] \leq c|\mathcal{C}_{\omega}|T_{\text{rel}}(L) \quad (5.5)$$

for some constant c , where $|\mathcal{C}_{\omega}|$ denotes the cardinality of \mathcal{C}_{ω} . If we assume the claim, the strong Markov property implies that

$$T_{\text{hit}}(L + 1) \leq \mathbb{E}[\tau^{\max}(L)] + c\mathbb{E}[|\mathcal{C}_{\omega(\tau^{\max}(L))}|]T_{\text{rel}}(L)$$

where all expectations are computed starting from all ones. Using Lemma 5.7,

$$\mathbb{E}[|\mathcal{C}_{\omega(\tau^{\max}(L))}|] \leq c' \sum_{\omega} \pi(\omega) |\mathcal{C}_{\omega}(r)|$$

for some constant c' and parts (i) and (ii) of the lemma follow by standard results on percolation on regular trees (see, e.g., [2020, 20]).

To prove (5.5) we proceed exactly as in Lemma 5.6. We first write

$$\mathbb{E}_\omega [\tau(L+1)] \leq \frac{1}{1-p} \int_0^\infty dt |\mathbb{P}_\omega(\omega_r(t) = 1) - p|$$

and then we apply the results of [1313; 13, §4] to get that

$$|\mathbb{P}_\omega(\omega_r(t) = 1) - p| \leq \min \left[1, \left(\frac{1}{p \wedge q} \right)^{|\mathcal{C}_\omega|} e^{-t/T_{\text{rel}}(L)} \right].$$

Thus,

$$\frac{1}{1-p} \int_0^\infty dt |\mathbb{P}_\omega(\omega_r(t) = 1) - p| \leq c |\mathcal{C}_\omega| T_{\text{rel}}(L)$$

for some constant c and (5.5) follows.

Lastly we prove (5.4). The subcritical case $p < p_c$ follows easily from (i) and Markov's inequality, while the critical case follows from (5.5). To see this, write

$$\begin{aligned} & \mathbb{P}(\tau(L+1) - \tau^{\max}(L) \geq d T_{\text{rel}}(L)) \\ &= \mathbb{P}(\tau(L+1) - \tau^{\max}(L) \geq d T_{\text{rel}}(L), |\mathcal{C}_{\omega(\tau^{\max}(L))}| \leq d^{2/3}) \\ &+ \mathbb{P}(\tau(L+1) - \tau^{\max}(L) \geq d T_{\text{rel}}(L), |\mathcal{C}_{\omega(\tau^{\max}(L))}| > d^{2/3}). \end{aligned}$$

Using Markov's inequality and (5.5),

$$\begin{aligned} & \mathbb{P}(\tau(L+1) - \tau^{\max}(L) \geq d T_{\text{rel}}(L), |\mathcal{C}_{\omega(\tau^{\max}(L))}| \leq d^{2/3}) \\ & \leq \frac{1}{d T_{\text{rel}}(L)} \mathbb{E} \left[\mathbf{1}_{\{|\mathcal{C}_{\omega(\tau^{\max}(L))}| \leq d^{2/3}\}} \mathbb{E}_{\omega(\tau^{\max}(L))} [\tau(L+1)] \right] \\ & \leq \frac{c}{d} \mathbb{E} \left[\mathbf{1}_{\{|\mathcal{C}_{\omega(\tau^{\max}(L))}| \leq d^{2/3}\}} |\mathcal{C}_{\omega(\tau^{\max}(L))}| \right] \leq c d^{-1/3}. \end{aligned}$$

The second term is also $O(d^{-1/3})$ using Lemma 5.7 and the fact that, for $p = p_c$,

$$\pi(|\mathcal{C}_\omega| \geq n) = O(1/\sqrt{n}). \quad \square$$

5.5. Proof of Theorem 5.2. Fix $\varepsilon \in (0, 1/2)$. Let $\{L_n\}$ be a sequence such that, for all n large enough,

$$\max\left(T_{\text{hit}}(L_n) - T_{\text{hit}}(L_n - 1), T_{\text{hit}}(L_n + 1) - T_{\text{hit}}(L_n)\right) \leq c T_{\text{rel}}(L_n), \quad (5.6)$$

for some constant c independent of n . The existence of such a sequence is guaranteed by Lemma 5.5. We begin by proving that

$$T_{\text{mix}}(L_n, \varepsilon) \leq T_{\text{hit}}(L_n) + O_\varepsilon(T_{\text{rel}}(L_n)). \quad (5.7)$$

Exactly as for the East process, one readily infers from the graphical construction that at time $\tau(L_n)$ all initial configurations $\omega \in \Omega_{\mathbb{T}_{L_n}}$ have coupled. Therefore (cf. §4.2),

$$\max_{\omega, \omega'} \left\| \mathbb{P}_\omega^{\mathbb{T}_{L_n}, t} - \mathbb{P}_{\omega'}^{\mathbb{T}_{L_n}, t} \right\| \leq \mathbb{P}(\tau(L_n) > t).$$

If $t = T_{\text{hit}}(L_n) + \Delta$, Markov's inequality together with (5.3) imply that

$$\begin{aligned} \mathbb{P}(\tau(L_n) > T_{\text{hit}}(L_n) + \Delta) & \leq \frac{1}{\Delta} \mathbb{E}(|\bar{\tau}(L_n)|) \leq \frac{2}{\Delta} [T_{\text{hit}}(L_n + 1) - T_{\text{hit}}(L_n)] \\ & \leq \frac{2}{\Delta} c T_{\text{rel}}(L_n). \end{aligned}$$

Inequality (5.7) now follows by choosing $\Delta = 2c T_{\text{rel}}(L_n)/\varepsilon$.

Next we prove the lower bound

$$T_{\text{mix}}(L_n, 1 - \varepsilon) \geq T_{\text{hit}}(L_n) - O_\varepsilon(T_{\text{rel}}(L_n)). \quad (5.8)$$

Start the process from the configuration ω identically equal to one and let $\tau^{\max}(L_n - \ell)$ be the time when all the vertices at distance ℓ from the root have been updated at least once. Conditionally on $\tau^{\max}(L_n - \ell) > t$, the root is connected by a path of 1's to some vertex at distance ℓ at time t . On the other hand, standard percolation results for $p \leq p_c$ imply that the π -probability of the above event is smaller than $\varepsilon/2$ provided that ℓ is chosen large enough. Therefore, for such value of ℓ ,

$$\|\mu_\omega^t - \pi\| \geq \mathbb{P}(\tau^{\max}(L_n - \ell) > t) - \varepsilon/2.$$

It remains to show that

$$\mathbb{P}(\tau^{\max}(L_n - \ell) > t) \geq 1 - \frac{\varepsilon}{2},$$

for $t = T_{\text{hit}}(L_n) - O_\varepsilon(T_{\text{rel}}(L))$.

We prove this by contradiction. Let $t = T_{\text{hit}}(L_n) - DT_{\text{rel}}$, where D is a constant to be specified later, and suppose that $\mathbb{P}(\tau^{\max}(L_n - \ell) > t) < 1 - \frac{\varepsilon}{2}$. Using Lemma 5.8 we can choose a large constant Δ independent of L_n such that

$$\mathbb{P}(\tau(L_n) - \tau^{\max}(L_n - \ell) \geq \Delta T_{\text{rel}}) \leq \varepsilon/4,$$

and hence, by a union bound,

$$\mathbb{P}(\tau(L_n) < t + \Delta T_{\text{rel}}) > \varepsilon/4.$$

However, for large enough D , this contradicts Theorem 5.1. Theorem 5.2 now follows from (5.7), (5.8), Theorems 5.3 and 5.4, and Lemma 5.5. \square

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REFERENCES

- [1] [cite.Aldous1D. Aldous, *Random walks on finite groups and rapidly mixing Markov chains*, Seminar on probability, XVII, 1983, pp. 243–297.
- [2] [cite.AD862D. Aldous and P. Diaconis, *Shuffling cards and stopping times*, Amer. Math. Monthly **93** (1986), 333–348.
- [3] [cite.AD023D. Aldous and P. Diaconis, *The asymmetric one-dimensional constrained Ising model: rigorous results*, J. Stat. Phys. **107** (2002), no. 5-6, 945–975.
- [4] [cite.AF4D. Aldous and J. Fill, *Reversible Markov Chains and Random Walks on Graphs*. In preparation, <http://www.stat.berkeley.edu/~aldous/RWG/book.html>.
- [5] [cite.Blondel5O. Blondel, *Front progression for the East model*, Stochastic Process. Appl. **123** (2013), 3430–3465.
- [6] [cite.Bolthausen6E. Bolthausen, *On the central limit theorem for stationary mixing random fields*, Ann. Probab. **10** (1982), no. 4, 1047.
- [7] [cite.BDZ7E. Bolthausen, J. D. Deuschel, and O. Zeitouni, *Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field*, Electron. Commun. Probab. **16** (2011), 114–119.
- [8] [cite.BZ18M. Bramson and O. Zeitouni, *Tightness for the minimal displacement of branching random walk*, J. Stat. Mech. Theory Exp. (2007), no. 7, P07010, 12.
- [9] [cite.BZ29M. Bramson and O. Zeitouni, *Tightness for a family of recursion equations*, Ann. Probab. **37** (2009), no. 2, 615–653.
- [10] [cite.CMRT10N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained spin models*, Probab. Theory Related Fields **140** (2008), no. 3-4, 459–504.
- [11] [cite.CFM11P. Chleboun, A. Faggionato, and F. Martinelli, *Time scale separation and dynamic heterogeneity in the low temperature East model*, Comm. Math. Phys., to appear.

- [12]]cite.CMRTtree12N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Mixing time of a kinetically constrained spin model on trees: power law scaling at criticality*, Probab. Theory Related Fields, to appear.
- [13]]cite.CMST13N Cancrini, F. Martinelli, R. Schonmann, and C. Toninelli, *Facilitated Oriented Spin Models: Some Non Equilibrium Results*, J. Stat. Phys. **138** (January 2010), no. 6, 1109–1123.
- [14]]cite.Diaconis14P. Diaconis, *The cutoff phenomenon in finite Markov chains*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), no. 4, 1659–1664.
- [15]]cite.DiFi15P. Diaconis and J. A. Fill, *Strong stationary times via a new form of duality*, Ann. Probab. **18** (1990), no. 4, 1483–1522.
- [16]]cite.DiSh16P. Diaconis and M. Shahshahani, *Generating a random permutation with random transpositions*, Z. Wahrsch. Verw. Gebiete **57** (1981), no. 2, 159–179.
- [17]]cite.DH9117F. M. Dekking and B. Host, *Limit distributions for minimal displacement of branching random walks*, Probab. Theory Related Fields **90** (1991), no. 3, 403–426.
- [18]]cite.East-survey18A. Faggionato, F. Martinelli, C. Roberto, and C. Toninelli, *The East model: recent results and new progresses*, Markov Processes and Related Fields, in press.
- [19]]cite.FH19G. H. Fredrickson and H. C. Andersen, *Kinetic Ising Model of the Glass Transition*, Phys. Rev. Lett. **53** (1984), no. 13, 1244–1247.
- [20]]cite.Grimmett20G. Grimmett, *Percolation*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 321, Springer-Verlag, Berlin, 1999.
- [21]]cite.JE9121J. Jäckle and S. Eisinger, *A hierarchically constrained kinetic Ising model*, Zeitschrift für Physik B Condensed Matter **84** (1991), no. 1, 115–124.
- [22]]cite.LPW22D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.
- [23]]cite.MT23F. Martinelli and C. Toninelli, *Kinetically constrained spin models on trees*, Ann. Appl. Probab. **23** (2013), no. 5, 1721–2160.
- [24]]cite.Saloff24L. Saloff-Coste, *Lectures on finite Markov chains*, Lectures on probability theory and statistics (Saint-Flour, 1996), Lecture Notes in Math., vol. 1665, Springer, Berlin, 1997, pp. 301–413.
- [25]]cite.Stein25C. Stein, *A bound for the error in the normal approximation to the distribution of a sum of dependent random variables*, Proc. of the Sixth Berkeley Symp. on Math. Statist. and Prob., Univ. California Press, 1972, pp. 583–602.

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