Markov Chain Analysis (math-ga 2932.001): homework assignment 2

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1. Let P be the transition kernel of a Markov chain on a metric space (Ω, d), and suppose that there exists $\theta < 1$ such that $d_{\mathcal{K}}(P(x,\cdot), P(y,\cdot)) \leq \theta d(x,y)$ for all x, y, where $d_{\mathcal{K}}$ is the Kanotorovich metric. Show that $d_{\mathcal{K}}(\mu P, \nu P) \leq \theta d_{\mathcal{K}}(\mu, \nu)$ for any two distributions μ, ν on Ω.

2. Let G be a connected d-regular graph on the vertex set $V = \{1, \ldots, n\}$, fix $0 < p < 1$ and let (X_t) be the Markov chain on $\{\pm 1\}^V$ where each at step a uniformly chosen vertex updates its vote as follows: with probability p it copies the vote of a uniformly chosen neighbor, and with probability $1-p$ it selects new a uniform $\{\pm 1\}$-value. Show that $t_{\mathrm{mix}}(\frac{1}{4}) = O(n \log n)$.

3. Recall that τ is a strong stationary time for (X_t) if $\mathbb{P}(X_\tau = y,\, \tau = t) = \pi(y)\mathbb{P}(\tau = t)$ for any y and t. Prove that this is equivalent to having $\mathbb{P}(X_t = y,\, \tau \leq t) = \pi(y)\mathbb{P}(\tau \leq t)$ for any y, t.

4. Prove the following statement: If G is a connected non-bipartite graph (so random walk on G is ergodic), $(X_t), (Y_t)$ are two independent simple random walks on G and $Y_0 \sim \pi$ (with π the stationary distribution) then $\tau = \min\{t : X_t = Y_t\}$ is *not* a stationary time for (X_t).

5*. A Markov chain P on a state space Ω is *transitive* if for any pair of states $x, y \in \Omega$ there is a bijection $\psi_{x,y} : \Omega \to \Omega$ with $\psi_{x,y}(x) = y$ and $P(\psi_{x,y}(u), \psi_{x,y}(v)) = P(u,v)$ for all u, v.

   (i) Show that if P is transitive, its time-reversal $\hat{P}$ satisfies $\|P^t(x,\cdot) - \pi\|_{\mathrm{tv}} = \|\hat{P}^t(x,\cdot) - \pi\|_{\mathrm{tv}}$ for any t (in class we proved this identity for the special case of random walks on groups).

   (ii) Show that if P is random walk on a transitive connected graph G (in particular P is transitive) then $\mathbb{E}_a \tau_b = \mathbb{E}_b \tau_a$ for any $a, b \in \Omega$, and give an example of a graph G and two vertices a, b on which this identity fails for the lazy random walk.

   (iii) If P is transitive and, in addition, for any x, y there exists $\Psi_{x,y}$ as above which further has $\Psi_{x,y}(y) = x$, clearly $\mathbb{E}_a \tau_b = \mathbb{E}_b \tau_a$ for any a, b. Find a small transitive graph (say, on at most 25 vertices) where for some x, y there is no such $\Psi_{x,y}$ for simple random walk.

6*. Let (X_t) be simple random walk on $\mathbb{Z}_n$, and let τ be 0 with probability $\frac{1}{n}$ and the *cover-time* $(\min\{t : \{X_0, \ldots, X_t\} = \mathbb{Z}_n\})$ with probability $\frac{n-1}{n}$. Prove or disprove: τ is a stationary time.