Markov Chain Analysis (math-ga 2932.001): homework assignment 1

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1. Let P be the transition kernel of a Markov chain on a finite state space Ω. We say that y is *accessible* from x — denoted $x \to y$ — if there exists some t such that $P^t(x,y) > 0$, and that x, y *communicate* if $x \to y$ and $y \to x$.

   (i) Show that if x, y communicate then they have the same period, and that the chain is irreducible iff every pair of states $x \neq y$ communicate.

   (ii) A state x is *essential* if $y \to x$ for all y with $x \to y$, and it is *inessential* otherwise. Prove that if x, y communicate then they are either both essential or both inessential.

   (iii) Show that there exists at least one essential state.

2. Recall that simple random walk (SRW) on $\{0, \ldots, n\}$ has $\mathbb{E}_k[\tau_0 \wedge \tau_n] = k(n-k)$, so SRW on $\mathbb{Z}_n$ has $\mathbb{E}_1 \tau_0 = n - 1$. Reobtain the latter using the stationary distribution of the SRW.

3. Let P be a transition kernel of an aperiodic irreducible Markov chain on a finite state space Ω with stationary distribution π. Recall that $d(t) = \max_x \|P^t(x, \cdot) - \pi\|_{\mathrm{tv}}$.

   (i) Show that $\|\mu P - \nu P\|_{\mathrm{tv}} \leq \|\mu - \nu\|_{\mathrm{tv}}$ for any μ, ν, and conclude that $d(t)$ is non-increasing in t, and moreover, that $\|P^t(x, \cdot) - \pi\|_{\mathrm{tv}}$ is non-increasing in t for any starting state x.

   (ii) Show that for $p \geq 1$ and $f : \Omega \to \mathbb{R}$ the function $p \mapsto (\int |f(x)|^p d\pi)^{1/p}$ is non-decreasing. Conclude that if

   $$d^{(p)}(t) := \max_x \left(\sum_y \left| \frac{P^t(x,y)}{\pi(y)} - 1 \right|^p \pi(y) \right)^{1/p}$$

   (notice $d(t) = \frac{1}{2} d^{(1)}(t)$ in this notation) then $d^{(p)}(t) \leq d^{(q)}(t)$ for any $1 \leq p \leq q$.

   (iii)* For $p \in \{1, 2, \infty\}$, prove that $d^{(p)}(t+s) \leq d^{(p)}(t) d^{(p)}(s)$ holds for any $s, t \geq 0$.

4*. Recall that the Metropolis chain for legal colorings of a graph with maximal degree Δ using $q \geq \Delta + 2$ colors (the chain which repeatedly selects a uniform vertex i and a uniform color c as its proposed new color, and accepts the move if c is not currently occupied by a neighbor of vertex i) converges to the uniform distribution over such colorings.

   Prove or disprove: if $q \geq \Delta + 2$ then the following process generates a uniform legal q-coloring of a graph with maximum degree Δ: take a uniform permutation π of the vertices, then proceed sequentially: at step i, assign vertex $\pi(i)$ a uniformly (and independently) chosen color out of those that had not already been assigned to any of its neighbors.