Transport in sheared stochastic magnetic fields

E. Vanden Eijnden[a)] and R. Balescu
Association EURATOM-Etat Belge, Physique Statistique, Plasma et Optique NL C.P. 231, Université Libre de Bruxelles, Campus Plaine, 1050 Bruxelles, Belgium

(Received 13 September 1996; accepted 7 November 1996)

The transport of test particles in a stochastic magnetic field with a sheared component is studied. Two stages in the particle dynamics are distinguished depending on whether the collisional effects perpendicular to the main field are negligible or not. Whenever the perpendicular collisions are unimportant, the particles show a subdiffusive behavior which is slower in the presence of shear. The particle dynamics is then inhomogeneous and non-Markovian and no diffusion coefficient may be properly defined. When the perpendicular collision frequency is small, this subdiffusive stage may be very long. In the truly asymptotic stage, however, the perpendicular collisions must be accounted for and the particle motion eventually becomes diffusive. Here again, however, the shear is shown to reduce the anomalous diffusion coefficient of the system. © *1997 American Institute of Physics.* [S1070-664X(97)03102-9]

I. INTRODUCTION

Because of its relevance for controlled fusion, the problem of plasma dynamics in the presence of stochastic magnetic fields has been intensively studied in recent years.[1–12] However, a full description of these phenomena has not yet been proposed. In this paper, we study the anomalous diffusion of test particles in stochastic fields. We are mainly concerned with the influence of a sheared component of the magnetic field on the particle motion. In a previous study,[11] we have shown that, in contrast to traditional wisdom, even a weak shear can strongly modify the statistical properties of the magnetic lines topology. Rather than simply reduce the value of the magnetic lines diffusion coefficient, as might be guessed from qualitative arguments, the shear may change the nature of the diffusion process itself and lead to a subdiffusive behavior of the magnetic lines. We analyze here the effect of such sheared configurations of the magnetic field on particle transport.

In strongly magnetized plasmas, the particles are almost tied to the magnetic lines and the collisional diffusion in the direction perpendicular to the magnetic field is a small effect which produces rare jumps of the particle from one magnetic line to another. As a result, the motion of the test particle is strongly influenced by the magnetic configuration. In particular, if the magnetic lines spreading is slowed down, as observed in a sheared configuration, the particle transport should be affected. We here confirm this picture. For weak perpendicular collision frequency, we distinguish two steps for the dynamics of the test particle. During the first one, the perpendicular collisions can be neglected. The statistics of the test particle trajectory is then the combination of the statistics of the particle motion on the magnetic line and the statistics of the line position itself. For unsheared configurations, the particle motion during this stage is known to be subdiffusive, with a mean square displacement growing like $t^{1/2}$. We show here that the shear slows down the process, yielding t^{α}, with $\alpha<1/2$, for the mean square displacement. Second, during the final stage of the particle dynamics, the perpendicular collisions, however small, eventually dominate the subdiffusive effects and must be taken into account. The particle motion is then equivalent to a random walk, the step length of which is shown to be reduced by the shear. As a result, the anomalous diffusion coefficient describing the transport in the system is smaller in the presence of the shear.

II. BASIC MODEL

In this paper, we consider a magnetic field having the following form:

$$\mathbf{B}(\mathbf{r})=B_0\{\hat{\mathbf{z}}+xL_s^{-1}\hat{\mathbf{y}}+\mathbf{b}(\mathbf{r})\},\tag{1}$$

where $\{\mathbf{r}\}=\{\mathbf{r}_\perp,z\}=\{x,y,z\}$. Here, $B_0\{\hat{\mathbf{z}}+xL_s^{-1}\hat{\mathbf{y}}\}$ is the sheared main component of the field and L_s is the shear length. This so-called *sheared slab* representation has to be understood as a local approximation, which mimics the field around a rational surface $x=0$ existing in real (i.e., periodical) systems. Equation (1) is thus only valid for $|x|\ll L_s$. In Eq. (1), $B_0\mathbf{b}$ is a fluctuating component perturbing the magnetic field. For simplicity, we assume that this fluctuation has only a x component depending only on $\{y,z\}$:

$$\mathbf{b}(\mathbf{r})=b(y,z)\hat{\mathbf{x}}.\tag{2}$$

This peculiar choice is often used in the literature (see for instance Refs. 1 and 5) and will be illustrative enough for our purpose. Notice also that Eq. (2) automatically fulfills the zero divergence constraint $\partial b/\partial x=0$. The fluctuation b is taken to be a Gaussian random process fully specified by its correlation

$$\mathscr{B}(y,z)=\langle b(y+y',z+z')b(y',z')\rangle_b.\tag{3}$$

Here, $\langle\ \ldots\ \rangle_b$ denotes the ensemble average over the fluctuating magnetic field and we assume that the function $\mathscr{B}$ has intensity β^2 and correlation lengths $\lambda_\parallel$ and $\lambda_\perp$ in the directions, respectively, parallel and perpendicular to $B_0\hat{\mathbf{z}}$. For modeling the plasma, we adopt the model discussed in Ref. 8 and single out a test particle. Collisions with the other particles are described as random variations of the test particle

[a)]Electronic mail: evdende@ulb.ac.be

 1070-664X/97/4(2)/270/7/$10.00

velocity which may occur in directions either parallel or perpendicular to the magnetic field (1); we refer to them as ''parallel'' and ''perpendicular'' collisions and denote the corresponding random velocity component by $\eta_\parallel$ and $\boldsymbol{\eta}_\perp=\{\eta_x,\eta_y\}$. Within the guiding center approximation, the trajectory of the test particle is specified by

$$\frac{dx(t)}{dt}=b[y(t),z(t)]\eta_\parallel(t)+\eta_x(t), \tag{4a}$$

$$\frac{dy(t)}{dt}=x(t)L_s^{-1}\eta_\parallel(t)+\eta_y(t), \tag{4b}$$

$$\frac{dz(t)}{dt}=\eta_\parallel(t), \tag{4c}$$

with initial conditions $\mathbf{r}(0)=\mathbf{r}_0$. Due to the random velocities, Eqs. (4) are of Langevin type. They must be completed with the statistics of $\boldsymbol{\eta}=\{\eta_\parallel,\boldsymbol{\eta}_\perp\}$, which is supposed to be the centered Gaussian process defined by[13]

$$\begin{aligned}
&\langle \eta_\parallel(t)\eta_\parallel(t')\rangle_\eta=\chi_\parallel\delta(t-t'),\\
&\langle \eta_x(t)\eta_x(t')\rangle_\eta=\langle \eta_y(t)\eta_y(t')\rangle_\eta=\chi_\perp\delta(t-t'),\\
&\langle \eta_\parallel(t)\eta_x(t')\rangle_\eta=\langle \eta_\parallel(t)\eta_y(t')\rangle_\eta=\langle \eta_x(t)\eta_y(t')\rangle_\eta=0,
\end{aligned} \tag{5}$$

in which $\langle\ \dots\ \rangle_\eta$ denotes an ensemble average over collisions and $\chi_\parallel=V_T^2/2\nu$ and $\chi_\perp=V_T^2\nu/2\Omega^2$ are the classical collisional diffusion coefficients ($V_T=\sqrt{2T/m}$ is the thermal velocity and $\Omega=eB/mc$ is the Larmor frequency). For strongly magnetized plasmas one has $\chi_\parallel\gg\chi_\perp$. It is also assumed that there is no correlation between the magnetic field and the collisions:

$$\langle b\,\boldsymbol{\eta}\rangle_{b,\eta}=0. \tag{6}$$

For obtaining the statistical properties of the test particle dynamics, we study the stochastic Liouville equation (or hybrid kinetic equation) associated with Eqs. (4):

$$\partial_t f+\partial_z\eta_\parallel f+\partial_x(\eta_\parallel b+\eta_x)f+\partial_y(\eta_\parallel xL_s^{-1}+\eta_y)f=0, \tag{7}$$

with initial condition $f(\mathbf{r};0)=\delta(\mathbf{r}-\mathbf{r}_0)$ ($\partial_t f\equiv\partial f/\partial t$ and so on). With this peculiar initial condition, the function $f(\mathbf{r};t)$ represents the probability of finding the test particle at position $\mathbf{r}$ and time t in a given realization of $\boldsymbol{\eta}$ and b, knowing that it was located at $\mathbf{r}_0$ at time $t=0$. The solution of Eq. (7) is thus

$$f(\mathbf{r};t)=\delta[\mathbf{r}-\mathbf{r}(t)], \tag{8}$$

where $\mathbf{r}(t)=\{x(t),y(t),z(t)\}$ is the solution of the Langevin equations (4). This follows from the uniqueness of the test particle trajectory in each realization of b and $\boldsymbol{\eta}$. The ''true'' probability distribution function of the system is $F=\langle f\rangle_{b,\eta}$, the equation of which is obtained by double-averaging of the kinetic equation (7):

$$\partial_t F=-\nabla\cdot\boldsymbol{\Gamma}, \tag{9}$$

where the particle flux $\boldsymbol{\Gamma}=\{\Gamma_x,\Gamma_y,\Gamma_z\}$ is

$$\begin{aligned}
&\Gamma_x=\langle(\eta_\parallel b+\eta_x)f\rangle_{b,\eta},\\
&\Gamma_y=\langle(\eta_\parallel xL_s^{-1}+\eta_y)f\rangle_{b,\eta},\\
&\Gamma_z=\langle\eta_\parallel f\rangle_{b,\eta}.
\end{aligned} \tag{10}$$

Of course, expressions (10) used in Eq. (9) do not provide a closed equation for the distribution function F, but start a hierarchy. The closure of Eqs. (9) and (10) is proposed in Sec. III. It is also worth noting at this point that, though fully specifying the transport in the system, the fluxes (10) need not be of the form $\boldsymbol{\Gamma}=-\mathsf{D}\cdot\nabla F$ and, in particular, it may be impossible to define the diffusion tensor D.

III. THE DENSITY PROFILE

For strongly magnetized plasmas, the parallel and perpendicular collisional processes have very different characteristic times. In particular, the action of the perpendicular collisions significantly affects the motion of the test particle only for very long times. This allows one to consider the perpendicular collisions as a perturbation, negligible for small and intermediate times, and studying first Eqs. (4) for $\boldsymbol{\eta}_\perp=0$. The effect of $\boldsymbol{\eta}_\perp$ will be considered in Sec. V. Neglecting perpendicular collisions, the random velocity $\eta_\parallel$ modeling the parallel collisions can be eliminated from Eqs. (4) by dividing the first two by the third. A similar procedure has been used in Ref. 12; this transformation of Eqs. (4) leads to

$$\frac{dx(z)}{dz}=b[y(z),z], \tag{11a}$$

$$\frac{dy(z)}{dz}=x(z)L_s^{-1}, \tag{11b}$$

with initial conditions $x(z_0)=x_0$ and $y(z_0)=y_0$. Here the z coordinate plays the role of an effective time. We can thus define a distribution function $g(x,y;z)$ which yields the probability that a line passes in $\{x,y\}$ at ''time'' z in a given realization of b. This distribution function obeys the Liouville equation

$$\partial_z g+\partial_x bg+\partial_y xL_s^{-1}g=0, \tag{12}$$

with initial condition $g(x,y;z_0)=\delta(x-x_0)\delta(y-y_0)$. Equation (12) has the solution

$$g(x,y;z)=\delta[x-x(z)]\delta[y-y(z)]. \tag{13}$$

By comparing Eqs. (8) and (13), we thus conclude

$$f(x,y,z;t)=g(x,y;z)\delta[z-z(t)]. \tag{14}$$

We now average this expression on both b and $\eta_\parallel$. Due to assumption (6), the two averages can be performed independently in the right-hand side, yielding

$$F(x,y,z;t)\equiv G(x,y;z)\mathscr{P}(z;t), \tag{15}$$

where

$$G=\langle g\rangle_b,\quad \mathscr{P}=\langle\delta[z-z(t)]\rangle_\eta. \tag{16}$$

The $z(t)$ trajectory appearing in $\mathscr{P}$ is obtained by direct solution of Eq. (4c) and averaging yields

$$\mathscr{P}(z;t)=\frac{1}{\sqrt{4\pi\chi_{\|}t}}e^{-(z-z_0)^2/4\chi_{\|}t}. \tag{17}$$

Next, we use Eq. (15) for deriving the density profile defined by:

$$n(x;t)=\int\int dydz\ F(x,y,z;t)$$
$$=\int_{-\infty}^{\infty}dz\ H(x;z)\mathscr{P}(z;t), \tag{18}$$

where

$$H(x;z)=\int_{-\infty}^{\infty}dyG(x,y;z). \tag{19}$$

The density profile n specifies the statistical properties of the particle dynamics in the x direction only. Correspondingly, the function (19) yields the statistics of the magnetic lines in the x direction and was studied in Ref. 11 for sheared magnetic field configurations. Restricting ourselves to the quasilinear regime, $\beta\lambda_{\|}\ll\lambda_{\perp}$, we obtain for H the Smoluchowsky equation

$$\partial_z H(x;z)=\partial_x[D(x)\partial_x H(x;z)], \tag{20}$$

with initial and boundary conditions $H(x;z_0)=\delta(x-x_0)$ and $H(\pm\infty;z)=0$. In Eq. (20), it is worth stressing that the diffusion coefficient is x dependent:

$$D(x)=\int_0^{\infty}dz\mathscr{B}(xL_s^{-1}z,z). \tag{21}$$

The derivation of Eq. (20) is standard and may be found in Ref. 11 (see also Refs. 14 and 15). Because $\mathscr{B}(y,z)$ depends on y and z only through the ratios $y/\lambda_{\perp}$ and $z/\lambda_{\|}$ and has intensity β^2, Eq. (21) implies that

$$D(x)=D_m\theta(x\lambda_s^{-1}), \tag{22}$$

where $D_m=\int_0^{\infty}dz\mathscr{B}(0,z)\propto\beta^2\lambda_{\|}$ is the quasilinear magnetic line diffusion coefficient. The function θ has the properties $\theta(0)=1$ and $\theta(u)\approx 0$ for $u>1$. In Eq. (22),

$$\lambda_s=L_s\frac{\lambda_{\perp}}{\lambda_{\|}}, \tag{23}$$

represents the actual characteristic length introduced by the shear in the model. For realistic problems ($\lambda_{\perp}\ll\lambda_{\|}$), λ_s is much smaller than L_s, which implies that the shear influences the small scales even for rather large shear length. In particular, for small ratios $\lambda_{\perp}/\lambda_{\|}$, the particle may experience radial excursions such that *both* the basic weak shear condition $|x|\ll L_s$ [see Eq. (1)] and the condition $|x|>\lambda_s$ are simultaneously fulfilled. For such situations, the x dependence of $D(x)$ cannot be neglected.

Obviously, the explicit form of $D(x)$ depends on the explicit form of $\mathscr{B}$. Within a non-self-consistent treatment, the latter must be specified *a priori*. Without losing much generality, we here assume an asymptotic power law dependence for D:

$$D(x)\sim D_m(x\lambda_s^{-1})^{-\gamma},\quad |x|>\lambda_s \tag{24}$$

with $\gamma>0$. In Ref. 11, we considered a peculiar form of $\mathscr{B}$ yielding the value $\gamma=3$ but which cannot be considered as generic.

For $|z-z_0|>\lambda_s^2/D_m$, the diffusion equation (20) implies that H has spread over an x range much larger than λ_s. The solution of Eq. (20) may then be approximated by the exact solution of this equation with D given by relation (24) for all x:

$$H=c(D_m\lambda_s^{\gamma}|z-z_0|)^{-1/(2+\gamma)}$$
$$\times\exp(-|x|^{2+\gamma}/(2+\gamma)^2D_m\lambda_s^{\gamma}|z-z_0|), \tag{25}$$

where the pure number c is a normalization constant. We use Eqs. (17), (18), and (25) for specifying the density profile. Due to the condition $|z-z_0|>\lambda_s^2/D_m$, the error introduced by using expression (25) for all z in Eq. (18) becomes negligible only when the standard deviation of the Gaussian packet (17) is much larger than the z range where Eq. (25) is not valid. This introduces the following condition:

$$t>\frac{\lambda_s^4}{D_m^2\chi_{\|}}. \tag{26}$$

We stress that for all finite values of λ_s, arbitrarily large, the range (26) always exists. However, in the limit $L_s\to\infty$, $\gamma>0$ fixed, this range disappears, which implies that the shearless limit is singular.[16] Contrary to the solution (25), Eq. (20) remains valid in the limit $L_s\to\infty$; this situation was considered in Refs. 9 and 11. We also note parenthetically that the shearless limit may be obtained by taking $\gamma=0$ in Eqs. (24) and (25): this property remains true in all the results which follow. Moreover, for times much smaller than (26), a local approximation of the diffusion coefficient is relevant. Hence, for such short times, the $\gamma=0$ prescription also gives the correct results, provided both the substitutions $D_m\to D(x_0)$ and $x\to x-x_0$, where x_0 is the initial position of the particle, are performed.

The density profile (18) is explicitly evaluated in Sec. IV. It is interesting to note that it obeys the evolution equation (derived in Appendix A)

$$\partial_t n(x;t)=\partial_x\int_0^t dt_1K(x,t_1)\partial_x n(x;t-t_1), \tag{27}$$

where the function K is:

$$K(x;t)=-\sqrt{\frac{\chi_{\|}}{4\pi t^3}}\,D(x). \tag{28}$$

Due to the dependence $t^{-3/2}$ of the function K, no time-Markovianization applied to Eq. (27) would yield a relevant approximation for the solution of this equation.[9] The dynamics of the test particle is thus non-Markovian as long as the effect of the perpendicular collisions is negligible and no diffusion coefficient exists.

IV. THE SUBDIFFUSIVE INTERMEDIATE STAGE

We now consider Eq. (18) and derive the moments of this profile:

$$\langle x^p(t)\rangle_{b,\eta}=\int_{-\infty}^{\infty}dx\; x^p n(x;t), \quad p\geqslant 0. \tag{29}$$

The latter can be calculated exactly by interchanging the order of the integrations in Eqs. (18) and (29). We obtain $\langle x^{2p+1}(t)\rangle_{b,\eta}=0$ and

$$\langle x^{2p}(t)\rangle_{b,\eta}\propto(D_m^2\lambda_s^{2\gamma}\chi_\parallel t)^{p/(2+\gamma)}. \tag{30}$$

Equation (30) should be compared with the shearless result $\langle x^{2p}(t)\rangle_{b,\eta}\propto(D_m^2\chi_\parallel t)^{p/2}$ (see Refs. 7, 9 and 12). The expression (30) thus implies that the already subdiffusive behavior of the particles is further slowed down by the shear. Notice also that the substitution $|z-z_0|\propto\sqrt{t}$ in the moments of the magnetic line profile (25), though giving the correct power dependence on t of $\langle x^{2p}(t)\rangle_{b,\eta}$, would miss the differences between the statistical properties of the particle motion on the magnetic line and the statistical properties of the magnetic line itself. This is obvious from the definition (18) of the profile: $n(x;t)\neq H(x;|z-z_0|=c\sqrt{t})$ and was previously noted in Refs. 7, 9 and 12 for the shearless system.

Next, for an analytical approximation of the profile (18) itself, we change the integration variable into

$$u=\left(\frac{2\chi_\parallel t|x|^{2+\gamma}}{(2+\gamma)^2D_m\lambda_s^\gamma}\right)^{-1/3}(z-z_0), \tag{31}$$

and we introduce the following similarity variable:

$$q=\frac{x}{(2[2+\gamma]^4D_m^2\lambda_s^{2\gamma}\chi_\parallel t)^{1/2(2+\gamma)}}\propto\frac{x}{t^{1/2(2+\gamma)}}. \tag{32}$$

Equation (18) then becomes

$$n(q;t)=c\,\frac{|q|^{(1+\gamma)/3}}{(D_m^2\lambda_s^{2\gamma}\chi_\parallel t)^{1/2(2+\gamma)}}\times\int_0^\infty du\,u^{-1/(2+\gamma)}e^{-|q|^{2(2+\gamma)/3}(u^2/2+1/u)}. \tag{33}$$

For $|q|\gg 1$, i.e., $|x|\gg t^{1/2(2+\gamma)}$, Eq. (33) can be evaluated by the steepest descent method:

$$n(q;t)\propto(D_m^2\lambda_s^{2\gamma}\chi_\parallel t)^{-1/2(2+\gamma)}\times|q|^{-1/3}e^{-(3/2)|q|^{2(2+\gamma)/3}}. \tag{34}$$

Equation (34) gives an approximation of the profile (33) which is irrelevant around the origin $x=0$. By removing the $q^{-1/3}$ algebraic prefactor and constraining the profile to be normalized, we can, however, obtain a uniform approximation of n:

$$n_{\mathrm{AS}}(q;t)=c(D_m^2\lambda_s^{2\gamma}\chi_\parallel t)^{-1/2(2+\gamma)}e^{-(3/2)|q|^{2(2+\gamma)/3}}. \tag{35}$$

Equation (35) is a typical stretched exponential, with stretching factor $|q|^{2(2+\gamma)/3}\propto|x|^{2(2+\gamma)/3}t^{-1/3}$.

V. THE EFFECT OF PERPENDICULAR COLLISIONS

In this section, we analyze the effect of the perpendicular collisions on the test particle motion. For treating this problem, two possibilities may be considered:

(1) The first was initiated in Ref. 5 and finalized in Ref. 12. It amounts to study the stochastic Liouville equation (7) within the framework of the renormalization procedure provided by the direct interaction approximation. This systematic approach yields the various diffusion regimes of the particles in shearless configurations,[12] including the Rechester–Rosenbluth diffusion coefficient,[3] whose derivation is notably difficult. The techniques developed in Ref. 12 could be employed for the sheared geometry. It would, however, lead to a highly technical treatment which would obscure the description of the basic physical effects.

(2) A second, semiquantitative approach[3,4] amounts to describing the asymptotic stage of the test particle motion as a random walk, the length and time steps of which have to be identified. It is worth stressing that the results provided by this method agree with the ones of the previous more systematic treatment, which validates the semiquantitative approach. For clarity, it is thus best to analyze the effects of the perpendicular collisions within this framework, taking as a guideline the very clear exposition given by Isichenko[6] (see also Ref. 10).

So far, we assumed that the test particle sticks to a peculiar magnetic field line. The perpendicular collisional effects modify this picture by ejecting the particle from its initial field line. We stress that, however small, the perpendicular collisions generate a diffusive process which is thus faster than the subdiffusive effects considered in the previous sections. As a result, the particle dynamics eventually becomes diffusive in the x direction. Such a property implies that the description of the final stage of the dynamics is simpler than the intermediate subdiffusive stage (Sec. IV), the very nature of which is not known *a priori*. Actually, the diffusive final stage of the dynamics justifies a random walk picture which is specified as follows (further details may be found for instance in Refs. 8 and 14). One first defines a decorrelation time t_d, as the time needed for the test particle to move one perpendicular correlation length $\lambda_\perp$ away from its original line; after a time t_d the particle and the line are decorrelated (on the average). Successive steps of such a process form thus effectively a random walk, the time step of which is t_d and the length step of which is $\langle x^2(t_d)\rangle_{b,\eta}^{1/2}$ i.e., Eq. (30) evaluated at time $t=t_d$. The anomalous diffusion coefficient $\chi_\perp^{\mathrm{AN}}$ of the random walk is given by

$$\chi_\perp^{\mathrm{AN}}=\frac{\langle x^2(t_d)\rangle_{b,\eta}}{2t_d}. \tag{36}$$

Of course for $\langle x^2(t_d)\rangle_{b,\eta}^{1/2}$ to be given by Eq. (30), t_d must be greater than the characteristic time (26) associated with the shear-induced subdiffusive stage of the particle motion. For t_d smaller than Eq. (26), $\langle x^2(t_d)\rangle_{b,\eta}^{1/2}$ takes its shearless value.

We now evaluate the decorrelation time t_d. Two cases must be considered depending on whether the magnetic fluctuations or the classical collisions are primarily responsible for the decorrelation between the particle and the magnetic line:

(1) For situations [to be specified below, Eq. (41)] where the stochasticity of the background magnetic field can be neglected during the separation process, the decorrelation results from purely classical collisional effects. Hence t_d is given by

$$t_d \equiv t_\perp = \frac{\lambda_\perp^2}{2\chi_\perp}. \quad (37)$$

In order for this time to be greater than the time defined in Eq. (26), the following condition must be satisfied

$$\frac{D_m^2 \lambda_\perp^2}{2\lambda_s^4} \frac{\chi_\|}{\chi_\perp} > 1. \quad (38)$$

Inserting the value $t_d \equiv t_\perp$ into Eq. (36) and using Eq. (30), we obtain

$$\chi_\perp^{\mathrm{AN}} \propto (D_m^2 \lambda_s^{2\gamma} \lambda_\perp^{-(2+2\gamma)} \chi_\| \chi_\perp^{1+\gamma})^{1/(2+\gamma)}. \quad (39)$$

The shearless result is recovered by setting $\gamma=0$, which yields the Kadomtsev–Pogutse[4] diffusion coefficient $\chi_\perp^{\mathrm{KP}} = D_m \lambda_\perp^{-1} (\chi_\| \chi_\perp)^{1/2}$. We notice that Eq. (38) can be reorganized into $\lambda_s^{2\gamma} < (D_m \lambda_\perp \sqrt{\chi_\| / \chi_\perp})^\gamma$. Using this inequality in Eq. (39) shows that $\chi_\perp^{\mathrm{AN}} < \chi_\perp^{\mathrm{KP}}$ i.e., the effect of the shear is a reduction of the Kadomtsev–Pogutse diffusion coefficient. This is of course consistent with the fact that the shear slows down the subdiffusive stage of the particle dynamics and, hence, reduces the step length of the asymptotic random walk.

(2) For a higher level of stochasticity of the magnetic field or a smaller level of perpendicular collisions, a decorrelation time faster than (37) may exist, as first noted by Rechester and Rosenbluth.[3] They argued that, in a stochastic field, two magnetic lines, however close to each other at some point, diverge exponentially with z. A particle pushed onto a neighboring line and moving along that new line can thus travel the distance $\lambda_\perp$ from its original line in a time t_d which, for a very small level of perpendicular collisions, is presumably faster than (37). More precisely, t_d must be obtained from the equation

$$2\chi_\perp t_d e^{\sqrt{2\chi_\| t_d}/L_{\mathrm{K}}} = \lambda_\perp^2. \quad (40)$$

The exponential factor accounts for the exponentiation with z of the lines; it involves $2\chi_\| t_d \equiv \langle z^2(t_d) \rangle$ and the Kolmogorov length L_{K} which is a measure of the exponentiation rate of the magnetic lines induced by the stochastic nature of the field. We analyze the relative motion of two magnetic lines and evaluate L_{K} in Appendix B. Clearly, Eq. (37) is the solution of Eq. (40) only if the exponential factor in this equation is approximately one. In this case, indeed, the classical perpendicular collisions are faster than the exponentiation of the lines for decorrelating the particle from its original magnetic line. On the contrary, when the inequality

$$\lambda_\perp^2 \chi_\| \gg L_{\mathrm{K}}^2 \chi_\perp, \quad (41)$$

is satisfied, i.e., for a small level of perpendicular collisions, the exponentiation mechanism becomes essential and strongly amplifies the effect of the perpendicular collisions alone. In this case, the solution of Eq. (40) can be obtained by iteration, which yields to the lowest order

$$t_d = (L_{\mathrm{K}}^2 / 2\chi_\|) \ln^2[(\chi_\| / \chi_\perp)(\lambda_\perp / L_{\mathrm{K}})^2]. \quad (42)$$

We stress that $\lim_{\chi_\perp \to 0} t_d = \infty$ due to the fact that no decorrelation between the particle and the magnetic line exists without perpendicular collisions. Consequently, for small values of $\chi_\perp$, t_d is always greater than the characteristic time (26) of the shear-induced subdiffusive stage of the particle motion, which implies using Eq. (30) in Eq. (36). Taking the decorrelation time (42) in formula (36) yields for the anomalous diffusion coefficient

$$\chi_\perp^{\mathrm{AN}} \propto \chi_\| (D_m \lambda_s^\gamma L_{\mathrm{K}}^{-(1+\gamma)})^{2/(2+\gamma)} \times (\ln[(\chi_\| / \chi_\perp)(\lambda_\perp / L_{\mathrm{K}})^2])^{-(2+2\gamma)/(2+\gamma)}. \quad (43)$$

The Rechester–Rosenbluth diffusion coefficient is obtained for $\gamma=0$, but the correct accounting for the effect of the shear leads to the smaller coefficient (43).

VI. CONCLUDING REMARKS

In the present paper, the statistical properties of the particle motion in a sheared configuration of the magnetic field have been analyzed within the framework of a test particle description. Our main observation is that the presence of shear leads to a kinetic equation for the particles, whose coefficients depend on the radial coordinate x. Though usually assumed in the literature, this space-dependence cannot be neglected even for large shear length. Indeed, the characteristic length entering the equation is not L_s but $\lambda_s = L_s \lambda_\perp / \lambda_\|$, which for realistic situations scales like $\lambda_s \ll L_s$. As a result, the test particle may experience radial excursions where both inequalities $|x| \lambda_s^{-1} > 1$ and $|x| L_s^{-1} \ll 1$ are simultaneously fulfilled: this prohibits any local approximation of the space-dependent coefficients entering the equations. We have shown that the correct consideration of this point leads to a slowing down of the subdiffusive behavior of the test particle during the intermediate stage of its dynamics, where the perpendicular collisions are negligible. During the true asymptotic stage of the particle motion, the perpendicular collision cannot be neglected anymore; their effect on the particle motion can be described as a random walk. Here also, we show that the presence of shear reduces the basic step length of the random walk and, thus, reduces the value of the anomalous diffusion coefficient as well.

Finally, we stress that, due to the shear, all quantities depend not only on the characteristic lengths and the intensity of the magnetic fluctuations spectrum, but also on its shape. This yields the dependence on the exponent γ of all our results. Of course, within a non-self-consistent treatment, neither the exact form of the spectrum nor the explicit value of γ can be obtained. The latter should thus be put into the model from the outside.

ACKNOWLEDGMENTS

We are indebted to D. Carati, A. Grecos, J. H. Misguich, and H.-D. Wang for stimulating discussions. One of us (E.V.E.) has been partly supported by the ''Fonds pour la Formation a la Recherche dans l'Industrie et l'Agriculture'' (FRIA), Belgium.

APPENDIX A: DERIVATION OF THE EVOLUTION EQUATION FOR THE DENSITY PROFILE

For deriving the evolution equation of the density profile, we start from the Laplace transform of Eq. (18):

$$\overline{n(x;s)}=\int_{-\infty}^{\infty}dz\ \overline{\mathscr{P}}(z;s)H(x;z), \tag{A1}$$

where the "bar" denotes Laplace-transformed functions and s and t are conjugated variables. The function $\overline{\mathscr{P}}$ is

$$\overline{\mathscr{P}}=\frac{1}{\sqrt{4\chi_{\|}s}}\ \exp(-|z-z_0|\sqrt{s/\chi_{\|}}). \tag{A2}$$

We apply the operator $\partial_x D(x)\partial_x$ to Eq. (18) and use Eq. (20):

$$\partial_x(D(x)\partial_x\overline{n(x;s)})=\int_{-\infty}^{\infty}dz\ \overline{\mathscr{P}}(z;s)\partial_z H(x;z). \tag{A3}$$

Integrating by parts the right-hand side of this expression and using Eqs. (A2) and (20), we obtain

$$\partial_x(D(x)\partial_x\overline{n(x;s)})=-\frac{1}{\sqrt{\chi_{\|}s}}\delta(x-x_0)-\int_{-\infty}^{\infty}dz[\partial_z\overline{\mathscr{P}}(z;s)]H(x;\zeta). \tag{A4}$$

Finally, we use $\partial_z\overline{\mathscr{P}}=-\sqrt{s/\chi_{\|}}\overline{\mathscr{P}}$ and Eq. (A1) and multiply both sides of Eq. (A4) by $\sqrt{\chi_{\|}s}$:

$$\sqrt{\chi_{\|}s}\,\partial_x(D(x)\partial_x\overline{n(x;s)})=-\delta(x-x_0)+s\overline{n(x;s)}. \tag{A5}$$

The inverse Laplace transform of Eq. (A5) is Eq. (27), combined with the initial condition.[17]

APPENDIX B: RELATIVE MOTION OF THE MAGNETIC LINES

We study here the relative motion of two magnetic lines whose positions are specified by $x_1\equiv x$, $y_1\equiv y$ and $x_2\equiv x+\delta x$, $y_2\equiv y+\delta y$. Introducing the distribution function $g(x,y,\delta x,\delta y;z)$, it obeys the stochastic Liouville equation

$$\partial_z g+xL_s^{-1}\partial_y g+b(y)\partial_x g+\delta xL_s^{-1}\partial_{\delta y}g+[b(y+\delta y)-b(y)]\partial_{\delta x}g=0, \tag{B1}$$

where z plays the role of the time and the initial condition is $g(x,y,\delta x,\delta y;z_0)=\delta(x-x_0)\delta(y-y_0)\delta(\delta x-\delta x_0)\delta(\delta y-\delta y_0)$. The statistics of the two trajectories is given by the ensemble average of g, i.e., $G=\langle g\rangle_b$. Within the quasilinear regime, the derivation of the evolution equation for G is a standard matter (see Refs. 11 and 15) which yields the Fokker–Planck equation

$$\begin{aligned}\partial_z G+xL_s^{-1}\partial_y G+\delta xL_s^{-1}\partial_{\delta y}G&=\partial_x\int_0^{\infty}du\{\mathscr{C}_1(u)(\partial_x+uL_s^{-1}\partial_y)G\\&\quad+\mathscr{C}_2(u)(\partial_{\delta x}+uL_s^{-1}\partial_{\delta y})G\}\\&\quad+\partial_{\delta x}\int_0^{\infty}du\{\mathscr{C}_3(u)(\partial_x+uL_s^{-1}\partial_y)G\\&\quad+\mathscr{C}_4(u)(\partial_{\delta x}+uL_s^{-1}\partial_{\delta y})G\},\end{aligned} \tag{B2}$$

where

$$\begin{aligned}&\mathscr{C}_1(z)=\mathscr{B}(xL_s^{-1}z,z),\\&\mathscr{C}_2(z)=\mathscr{B}(xL_s^{-1}z-\delta y+\delta xL_s^{-1}z,z)-\mathscr{C}_1(z),\\&\mathscr{C}_3(z)=\mathscr{B}(xL_s^{-1}z+\delta y,z)-\mathscr{C}_1(z),\\&\mathscr{C}_4(z)=\mathscr{B}(xL_s^{-1}z+\delta xL_s^{-1}z,z)-\mathscr{C}_1(z)-\mathscr{C}_2(z)\\&\qquad-\mathscr{C}_3(z).\end{aligned} \tag{B3}$$

We look for the solution of Eq. (B2) for the case where the two magnetic lines are initially very close, i.e., for very small values of δx_0 and δy_0. During the initial stage of the magnetic lines motion, the coefficients (B3) can thus be approximated by the first term of their series expansion in δx and δy:

$$\begin{aligned}&\mathscr{C}_1(z)=\mathscr{B}(xL_s^{-1}z,z),\\&\mathscr{C}_2(z)\simeq-(\delta y-\delta xL_s^{-1}z)\mathscr{B}'(xL_s^{-1}z,z),\\&\mathscr{C}_3(z)\simeq\delta y\mathscr{B}'(xL_s^{-1}z,z),\\&\mathscr{C}_4(z)\simeq-\delta y(\delta y-\delta xL_s^{-1}z)\mathscr{B}''(xL_s^{-1}z,z),\end{aligned} \tag{B4}$$

where $\mathscr{B}'$ and $\mathscr{B}''$ are the first and second derivatives of $\mathscr{B}$ over its first argument. The relative motion of the magnetic lines can be characterized by their mean square relative distances:

$$\left\{\begin{matrix}\langle\delta x^2(z)\rangle_b\\ \langle\delta x(z)\delta y(z)\rangle_b\\ \langle\delta y^2(z)\rangle_b\end{matrix}\right\}=\int dx\,dy\,d\delta x\,d\delta y\left\{\begin{matrix}\delta x^2\\ \delta x\,\delta y\\ \delta y^2\end{matrix}\right\}G.$$

The equations for these quantities are obtained by multiplying Eq. (B2) by δx^2, $\delta x\,\delta y$, or δy^2 and integrating it over x, y, δx, and δy. Due to the x dependence of the coefficients $\mathscr{C}_j$, this operation cannot be performed exactly. We have to approximate the $\mathscr{C}_j$ by some local values in x; we here take the center of the slab $x=0$. As far as the mean values $\langle\delta x^2(z)\rangle_b$, $\langle\delta x(z)\delta y(z)\rangle_b$, and $\langle\delta y^2(z)\rangle_b$ are concerned [and not a quantity describing the spreading in x space, like for instance $\langle x^2(z)\delta x^2(z)\rangle_b$], this approximation is justified. It yields

$$\begin{aligned}&\frac{d}{dz}\langle\delta x^2(z)\rangle_b=\lambda_0^{-1}\langle\delta y^2(z)\rangle_b-\lambda_2^{-1}\langle\delta x^2(z)\rangle_b,\\&\frac{d}{dz}\langle\delta x(z)\delta y(z)\rangle_b-L_s^{-1}\langle\delta x^2(z)\rangle_b\\&\qquad=\lambda_1^{-1}\langle\delta y^2(z)\rangle_b-\lambda_2^{-1}\langle\delta x(z)\delta y(z)\rangle_b,\\&\frac{d}{dz}\langle\delta y^2(z)\rangle_b-2L_s^{-1}\langle\delta x(z)\delta y(z)\rangle_b=0,\end{aligned} \tag{B5}$$

where the lengths λ_p are defined by

$$\lambda_p^{-1} = -2\int_0^\infty du(uL_s^{-1})^p \mathscr{B}''(0,u) > 0. \tag{B6}$$

[These are positive quantities if $\mathscr{B}(y,z)$ is maximum in $y=0$, which is natural.] Recalling that $\mathscr{B}$ is a function of $y/\lambda_\perp$ and $z/\lambda_\parallel$, with $\int_0^\infty dz \mathscr{B}(0,z) = D_m$, we have from dimensional analysis

$$\begin{aligned} &\lambda_0 \propto \lambda_\perp^2/D_m, \\ &\lambda_1 \propto \lambda_\perp^2 L_s/D_m \lambda_\parallel, \\ &\lambda_2 \propto \lambda_\perp^2 L_s^2/D_m \lambda_\parallel^2. \end{aligned} \tag{B7}$$

The Kolmogorov length L_K is the inverse of the highest real part of the eigenvalues of the system (B5). For large shear, $L_s \gg \lambda_\parallel$, we obtain

$$L_K \propto (L_s^2 \lambda_\perp^2/D_m)^{1/3}, \tag{B8}$$

which is the value adopted in the literature (see Ref. 6).

[1] M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavsky, Nucl. Fusion **6**, 297 (1966).
[2] T. H. Stix, Phys. Rev. Lett. **30**, 833 (1973).
[3] A. B. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).
[4] B. B. Kadomtsev and O. P. Pogutse, in *Plasma Physics and Controlled Nuclear Fusion Research 1978*, Proceedings of the 7th International Conference, Innsbruck (International Atomic Energy Agency, Vienna, 1979), Vol. 1, p. 649.
[5] J. H. Krommes, C. Oberman, and R. G. Kleva, J. Plasma Phys. **30**, 11 (1983).
[6] M. B. Isichenko, Plasma Phys. Controlled Fusion **33**, 795 (1991).
[7] J. M. Rax and R. B. White, Phys. Rev. Lett. **68**, 1523 (1992).
[8] R. Balescu, H.-D. Wang, and J. H. Misguich, Phys. Plasmas **1**, 3826 (1994); H.-D. Wang, M. Vlad, E. Vanden Eijnden, F. Spineanu, J. H. Misguich, and R. Balescu, Phys. Rev. E **51**, 4844 (1995).
[9] R. Balescu, Phys. Rev. E **51**, 3826 (1995).
[10] J. H. Misguich, M. Vlad, F. Spineanu and R. Balescu, Comments Plasma Phys. Controlled Fusion **17**, 45 (1995).
[11] E. Vanden Eijnden and R. Balescu, Phys. Plasmas **3**, 815 (1996).
[12] E. Vanden Eijnden and R. Balescu, Phys. Plasmas **3**, 874 (1996).
[13] Delta-correlated collisional processes correspond to a regime of high collisionality for the plasma. The opposite, strictly collisionless limit, is defined by[8] $\langle \eta_\parallel(t)\eta_\parallel(t')\rangle_\eta = \frac{1}{2}V_T^2$, $\langle \eta_x(t)\eta_x(t')\rangle_\eta = \langle \eta_y(t)\eta_y(t')\rangle_\eta = 0$. For considering this case, we actually need to substitute $2\chi_\parallel t \to \frac{1}{2}V_T^2 t^2$ in the distribution (17) and hence in the moments (30). Because $\chi_\perp = 0$ in that case, no decorrelation mechanism exists and the particle cannot leave its magnetic line. The ''intermediate'' asymptotic stage thus goes on forever. In contrast with the shearless situation, this stage remains subdiffusive, with $\langle x^2(t)\rangle_{b,\eta} \propto (D_m \lambda_s^\gamma V_T t)^{2/(2+\gamma)}$, and no diffusion coefficient exists in strictly collisionless situations.
[14] A. Brissaud and U. Frisch, J. Math. Phys. **15**, 524 (1974).
[15] N. G. Van Kampen, Phys. Rep. **24**, 171 (1976).
[16] In Ref. 7, a sheared situation is considered, but with $\lambda_\perp = \infty$, which yields $D(x) \equiv D_m$. We do not consider the scaling $\lambda_\parallel \ll \lambda_\perp$, which this situation implies.
[17] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, NJ, 1941).