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Stochastic modelling of turbulence and anomalous transport in plasmas

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Stochastic modelling of turbulence is considered. Exact and approximate solution procedures based on projection techniques are briefly reviewed. Procedures leading to equations that are local in time are discussed. Two different approximations of this type are presented, whose performances are tested and compared on a simple model case.

1. Introduction

Transport processes in magnetized plasmas are of theoretical as well as practical interest. In general, the observed rates of transport are several orders of magnitude larger than those expected from classical or neoclassical theories (Balescu 1988). The underlying mechanisms of such 'anomalous' transport have not yet been clearly identified. Nevertheless, it is generally assumed that plasma turbulence (i.e. fluctuations of the electromagnetic field, taken to be random) plays an important role.

A complete theory of anomalous transport should be formulated from first principles, deriving appropriate kinetic equations (see e.g. Balescu 1975). However, the extreme complexity of turbulent processes in a plasma and the necessity to obtain eventually concrete results force us to be less ambitious and to adopt a phenomenological approach based on stochastic modelling (see e.g. the discussion in Chapter 16 of Balescu (1997) and the PhD thesis of Vanden Eijnden (1997*b*)). In such an approach, some 'randomness' is introduced from the start in the equations of motion in order to account for turbulence effects. This paper is devoted to a short review of stochastic modelling. Basic concepts are recalled in Sec. 2. In Secs 3–5, exact or approximate procedures for the solution of this problem are considered. We focus mainly on those procedures leading to equations that are local in time, or convolutionless, in contrast with the time-convolutive equations most frequently used in the literature. Two different approximations are presented: the strict quasilinear approximation in Sec. 4 and an approximation that goes beyond in Sec. 5. In Sec. 6, these approximations are tested and compared using a simple model case proposed by Balescu (1994). The results are summarized in Sec. 7.

2. Stochastic modelling

Stochastic modelling amounts to adopting a set of random differential equations for the evolution of the dynamical system. Generally, this set of random differential equations may be written as

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{p}[\mathbf{q}(t), t], \qquad \mathbf{q}(t_0) = \mathbf{q}_0, \tag{2.1}$$

where $\mathbf{q}$ is the variable of an appropriate phase space, $\mathbf{p}(\mathbf{q}, t)$ is some time-dependent random function defined on this phase space, and $\mathbf{q}_0$ is a (possibly random) initial condition. For instance, for the motion of a charged particle of mass m and charge q subject to electromagnetic turbulence and collisions, (2.1) reads:

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{v}(t), \tag{2.2a}$$

$$\frac{d}{dt}\mathbf{v}(t) = \frac{q}{m}\{\mathbf{E}[\mathbf{r}(t), t] + \mathbf{v}(t) \times \mathbf{B}[\mathbf{r}(t), t]\} + \mathbf{a}_c[\mathbf{v}(t), t]. \tag{2.2b}$$

Here the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ and the acceleration $\mathbf{a}_c[\mathbf{v}(t), t]$ representing the effect of collisions are random quantities (for an illustration of this problem, see e.g. Coronado *et al.* 1992; Hannibal 1993; Vanden Eijnden 1997 *a*).

In the context of stochastic dynamics, the statistics of $\mathbf{p}(\mathbf{q}, t)$ is specified, which amounts to giving the characteristic functional defined as (Papoulis 1991)

$$K[\mathbf{z}] \equiv \mathscr{P} \exp\left[i \int dt\, d\mathbf{q}\, \mathbf{z}(\mathbf{q}, t) \cdot \mathbf{p}(\mathbf{q}, t)\right], \tag{2.3}$$

where $\mathscr{P}$ denotes the averaging (or expectation) over the statistics of $\mathbf{p}(\mathbf{q}, t)$, and $\mathbf{z}(\mathbf{q}, t)$ is a non-random test function (i.e. a smooth function with compact support). Note that the averaging operation is a projection:

$$\mathscr{P}^2 = \mathscr{P}. \tag{2.4}$$

For simplicity the statistics of $\mathbf{p}(\mathbf{q}, t)$ is usually assumed to be Gaussian, which is true if for all $\mathbf{z}(\mathbf{q}, t)$ the variable $\int dt\, d\mathbf{q}\, \mathbf{z}(\mathbf{q}, t) \cdot \mathbf{p}(\mathbf{q}, t)$ is a Gaussian random variable. The characteristic functional of a Gaussian random function is

$$K_{\mathrm{G}}[\mathbf{z}] = \exp\left[i \int dt\, d\mathbf{q}\; \mathbf{z}(\mathbf{q}, t) \cdot \mathscr{P}\mathbf{p}(\mathbf{q}, t) \right.$$
$$\left. -\frac{1}{2} \int dt\, dt'\, d\mathbf{q}\, d\mathbf{q}'\; \mathbf{z}(\mathbf{q}, t) \cdot \mathscr{P}\left\{\tilde{\mathbf{p}}(\mathbf{q}, t) \otimes \tilde{\mathbf{p}}(\mathbf{q}', t')\right\} \cdot \mathbf{z}(\mathbf{q}', t')\right], \tag{2.5}$$

where $\tilde{\mathbf{p}}(\mathbf{q}, t) = \mathbf{p}(\mathbf{q}, t) - \mathscr{P}\mathbf{p}(\mathbf{q}, t)$. The tensor $\mathscr{P}\{\tilde{\mathbf{p}}(\mathbf{q}, t) \otimes \tilde{\mathbf{p}}(\mathbf{q}', t')\}$ is referred to as the covariance of the random function $\mathbf{p}(\mathbf{q}, t)$.

In the context of stochastic dynamics, the main problem concerns the derivation of the relationship between the (unknown) statistics of $\mathbf{q}(t)$ and the (known) statistics of $\mathbf{p}(\mathbf{q}, t)$. The statistics of $\mathbf{q}(t)$ can be specified by a time-dependent probability density defined on the phase space $\mathbf{q}$. This probability density is given by the average (Van Kampen 1974*a, b*)

$$\mathscr{P}\rho(\mathbf{q}, t), \tag{2.6}$$

where $\rho(t)$ is a random density satisfying the stochastic Liouville equation associated with (2.2):

$$\frac{\partial}{\partial t}\rho(t) = \mathbb{L}(t)\,\rho(t) \equiv -\frac{\partial}{\partial \mathbf{q}} \cdot [\mathbf{p}(\mathbf{q},t)\,\rho(t)]\,. \tag{2.7}$$

Note that, in general, the system is neither autonomous nor conservative. For deterministic $\mathbf{q}_0$, the initial condition for (2.7) is $\rho(\mathbf{q},t_0) = \delta(\mathbf{q} - \mathbf{q}_0)$. However, it is straightforward to also consider situations where the initial condition $\mathbf{q}_0$ is random and statistically independent of $\mathbf{p}[\mathbf{q},t]$. Indeed, this simply amounts to studying (2.7) with the initial condition $\rho(\mathbf{q},t_0) = \rho_0(\mathbf{q})$, where $\rho_0(\mathbf{q}_0)$ is the probability density of $\mathbf{q}_0$. Note that the assumption of statistical independence between $\mathbf{p}[\mathbf{q},t]$ and $\mathbf{q}_0$ implies $\mathscr{P}\{\rho(\mathbf{q},t_0)\} \equiv \rho(\mathbf{q},t_0)$.

Knowledge of $\mathscr{P}\rho(t)$ allows us to determine the behaviour of expectations of observables, that is, functions of the dynamical variables, and hence to describe transport. Indeed, let $A(\mathbf{q})$ be arbitrary observable, whose time evolution is defined by

$$A(\mathbf{q},t) \equiv A[\mathbf{q}(t)]. \tag{2.8}$$

Then the expectation of $A(\mathbf{q},t)$ is given by

$$\mathscr{P}\{A[\mathbf{q},t]\} = \int d\mathbf{q}\; A(\mathbf{q})\,\mathscr{P}\rho(\mathbf{q},t), \tag{2.9}$$

The average of (2.7) over $\mathbf{p}(\mathbf{q},t)$ is a complicated operation, because it amounts to evaluating $\mathscr{P}\{\mathbb{L}(t)\rho(t)\}$. In this paper, we examine questions arising in establishing exact or approximate equations for the evolution of the probability density $\mathscr{P}\rho(t)$.

3. Formal aspects

In this section, we derive exact equations for the evolution of the probability density $\mathscr{P}\rho(t)$ using operator techniques introduced by Weinstock (1969) and further developed for example by Terwiel (1974), Balescu (1975), Misguich and Balescu (1975) and Chaturvedi and Shibata (1979). Two classes of equations are obtained, being either convolutive in time or convolutionless.

3.1. Generalized master equations

It is convenient to introduce first a notation for the projection operator $\mathscr{Q}$ complementary to $\mathscr{P}$, namely

$$\mathscr{Q} \equiv \mathscr{I} - \mathscr{P} \quad \Leftrightarrow \quad \mathscr{P} + \mathscr{Q} = \mathscr{I}, \qquad \mathscr{P}\mathscr{Q} = \mathscr{Q}\mathscr{P} = 0, \tag{3.1}$$

where $\mathscr{I}$ denotes the identity. It follows that (2.7) may be decomposed as

$$\frac{\partial}{\partial t}\mathscr{P}\rho(t) = \mathscr{P}\mathbb{L}(t)\,\mathscr{P}\rho(t) + \mathscr{P}\mathbb{L}(t)\,\mathscr{Q}\rho(t), \tag{3.2a}$$

$$\frac{\partial}{\partial t}\mathscr{Q}\rho(t) = \mathscr{Q}\mathbb{L}(t)\,\mathscr{P}\rho(t) + \mathscr{Q}\mathbb{L}(t)\,\mathscr{Q}\rho(t). \tag{3.2b}$$

The (formal) solution of (3.2) may be expressed as (using $\mathscr{Q}\rho(t_0) = 0$, which follows from $\mathscr{P}\rho(t_0) = \rho(t_0)$)

$$\mathscr{Q}\rho(t) = \int_{t_0}^{t} ds\; \mathrm{Exp}_+\left\{\int_s^t du\; \mathscr{Q}\mathbb{L}(u)\mathscr{Q}\right\} \mathscr{Q}\mathbb{L}(s)\,\mathscr{P}\rho(s). \tag{3.3}$$

Here $\mathrm{Exp}_+\{\cdots\}$ denotes the time-ordered exponential defined for any operator $\mathbb{A}(t)$ as

$$\mathrm{Exp}_+\left\{\int_s^t du\ \mathbb{A}(u)\right\} \equiv \mathbb{I} + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \cdots \int_{t_0}^{t_2} dt_1\ \mathbb{A}(t_n)\mathbb{A}(t_{n-1}) \cdots \mathbb{A}(t_1). \tag{3.4}$$

Inserting (3.3) into (3.2) yields the so-called generalized master equation

$$\frac{\partial}{\partial t}\mathscr{P}\rho(t) = \mathscr{P}\mathbb{L}(t)\mathscr{P}\rho(t) + \int_{t_0}^{t} \mathrm{d}s\ \mathbb{G}(t,s)\,\mathscr{P}\rho(s), \tag{3.5}$$

where we have defined

$$\mathbb{G}(t,s) = \mathscr{P}\,\mathbb{L}(t)\mathscr{Q}\,\mathrm{Exp}_+\left\{\int_s^t du\ \mathscr{Q}\mathbb{L}(u)\mathscr{Q}\right\}\mathscr{Q}\mathbb{L}(s)\,\mathscr{P}. \tag{3.6}$$

3.2. Convolutionless equations

In terms of the (formal) solution of (2.7) we have (using $\mathscr{Q}\rho(t_0) = 0$)

$$\begin{aligned} &\rho(t) = \mathbb{U}(t|t_0)\,\rho(t_0) \\ \Rightarrow\quad &\mathscr{P}\rho(t) = \mathbb{V}(t|t_0)\mathscr{P}\rho(t_0), \quad \mathbb{V}(t|t_0) \equiv \mathscr{P}\mathbb{U}(t|t_0)\mathscr{P}, \end{aligned} \tag{3.7}$$

where

$$\mathbb{U}(t|t_0) = \mathrm{Exp}_+\left\{\int_{t_0}^{t} ds\ \mathbb{L}(s)\right\}. \tag{3.8}$$

So, in the case where $\rho(t_0)$ belongs to the $\mathscr{P}$ subspace, $\mathbb{V}(t|t_0)$ maps an initial state $\mathscr{P}\rho(t_0)$ to a state $\mathscr{P}\rho(t)$ at time t. However, in contrast with $\mathbb{U}(t|t_0)$, $\mathbb{V}(t|t_0)$ does not satisfy the flow property of a group (or a semigroup), i.e.

$$\mathbb{V}(t|s)\mathbb{V}(s|u) \neq \mathbb{V}(t|u), \tag{3.9}$$

which implies that the dynamics in the $\mathscr{P}$ subspace is non-Markovian. For $\mathbb{V}(t|t_0)$, we have the equation

$$\frac{\partial}{\partial t}\mathbb{V}(t|t_0) = \mathscr{P}\mathbb{L}(t)\mathscr{P}\mathbb{V}(t|t_0) + \int_{t_0}^{t} \mathrm{d}s\ \mathbb{G}(t,s)\,\mathbb{V}(s|t_0), \tag{3.10}$$

which results from (3.5). It is possible to write an equivalent equation for $\mathbb{V}(t|t_0)$ (and hence for $\mathscr{P}\rho(t)$; see below) without a convolution term on the right-hand side, that is, an equation of the form

$$\frac{\partial}{\partial t}\mathbb{V}(t|t_0) = \mathscr{P}\mathbb{L}(t)\mathscr{P}\mathbb{V}(t|t_0) + \mathbb{Z}(t,t_0)\mathbb{V}(t|t_0), \tag{3.11}$$

with an appropriate definition of $\mathbb{Z}(t,t_0)$. One way to proceed is as follows. Assuming that the inverse operator $\mathbb{V}^{-1}(t|t_0)$ exists (in the $\mathscr{P}$ subspace), we have

$$\begin{aligned} &\mathscr{P}\mathbb{L}(t)\mathscr{P} + \mathbb{Z}(t,t_0) = \left[\frac{\partial}{\partial t}\mathbb{V}(t|t_0)\right]\mathbb{V}^{-1}(t|t_0) = \mathscr{P}\mathbb{L}(t)\mathbb{U}(t|t_0)\mathscr{P}\left[\mathscr{P}\mathbb{U}(t|t_0)\mathscr{P}\right]^{-1} \\ &\Rightarrow\quad \mathbb{Z}(t,t_0) = \mathscr{P}\mathbb{L}(t)\mathscr{Q}\mathbb{U}(t|t_0)\mathscr{P}\left[\mathscr{P}\mathbb{U}(t|t_0)\mathscr{P}\right]^{-1}. \end{aligned} \tag{3.12}$$

We further transform this expression using

$$
\begin{aligned}
& \mathbb{U}^{-1}(t|t_0)\mathbb{U}(t|t_0) = \mathbb{I} \\
\Leftrightarrow\quad & \mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{Q}\mathbb{U}(t|t_0)\mathscr{P} + \mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{P}\mathbb{U}(t|t_0)\mathscr{P} = 0 \\
\Leftrightarrow\quad & \mathscr{Q}\mathbb{U}(t|t_0)\mathscr{P} = -[\mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{Q}]^{-1}\mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{P}\mathbb{U}(t|t_0)\mathscr{P},
\end{aligned} \tag{3.13}
$$

which yields

$$
\mathbb{Z}(t,t_0) = -\mathscr{P}\,\mathbb{L}(t)\mathscr{Q}[\mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{Q}]^{-1}\mathscr{Q}\mathbb{U}^{-1}(t|t_0)\mathscr{P}. \tag{3.14}
$$

Equation (3.11) may be (formally) integrated as

$$
\mathbb{V}(t|t_0) = \mathrm{Exp}_+\left\{\int_{t_0}^{t} ds\ [\mathscr{P}\mathbb{L}(s)\mathscr{P} + \mathbb{Z}(s,t_0)]\right\}. \tag{3.15}
$$

Also, the corresponding equation for $\mathscr{P}\rho(t)$ is

$$
\frac{\partial}{\partial t}\mathscr{P}\rho(t) = \mathscr{P}\mathbb{L}(t)\mathscr{P}\rho(t) + \mathbb{Z}(t,t_0)\mathscr{P}\rho(t). \tag{3.16}
$$

It is worth stressing that, in accordance with (3.9), the evolution of $\mathscr{P}\rho(t)$ predicted by (3.16) is non-Markovian owing to the dependence of $\mathbb{Z}(t,t_0)$ on t_0 (this point has led to some confusion in the literature; it is discussed in detail in Vanden Eijnden 1997*b*).

The exact (though formal) equations (3.5) and (3.16) are of course equivalent as long as no approximations are introduced. From a theoretical point of view, these equations are interesting, since they may permit systematic consideration of various approximation schemes.

4. The quasilinear approximation

In order to consider approximations of (3.5) and (3.16), it is convenient to decompose the Liouville operator into deterministic and purely random parts:

$$
\mathbb{L}(t) = \widehat{\mathbb{L}}(t) + \widetilde{\mathbb{L}}(t), \quad \widehat{\mathbb{L}}(t) = \mathscr{P}\mathbb{L}(t)\mathscr{P} \Rightarrow \mathscr{P}\,\widetilde{\mathbb{L}}(t)\,\mathscr{P} = 0. \tag{4.1}
$$

The first non-trivial approximation then results when $\mathbb{Z}(t,t_0)$ and $\mathbb{G}(t,s)$ are evaluated to second order in the random part of $\mathbb{L}(t)$. We get

$$
\mathbb{G}_{(2)}(t,s) = \mathscr{P}\,\widetilde{\mathbb{L}}(t)\widehat{\mathbb{U}}(t|s)\widetilde{\mathbb{L}}(s)\mathscr{P}, \tag{4.2}
$$

$$
\mathbb{Z}_{(2)}(t,t_0) = \int_{t_0}^{t} ds\ \mathscr{P}\,\widetilde{\mathbb{L}}(t)\widehat{\mathbb{U}}(t|s)\widetilde{\mathbb{L}}(s)\widehat{\mathbb{U}}^{-1}(t|s)\mathscr{P}, \tag{4.3}
$$

where the operator $\widehat{\mathbb{U}}(t|s)$ is the propagator corresponding to the unperturbed motion:

$$
\widehat{\mathbb{U}}(t|s) = \mathrm{Exp}_+\left\{\int_{s}^{t} du\ \widehat{\mathbb{L}}(u)\right\}. \tag{4.4}
$$

It should be noted that

$$
\mathbb{Z}_{(2)}(t,t_0) = \int_{t_0}^{t} dt'\ \mathbb{G}_{(2)}(t,s)\widehat{\mathbb{U}}^{-1}(t|s)\mathscr{P}, \tag{4.5}
$$

which implies that, in this approximation, (3.5) reduces to (3.16) when in the last term on the right-hand side $\mathscr{P}\rho(s)$ is approximated by $\widehat{\mathbb{U}}^{-1}(t|s)\mathscr{P}\rho(t)$. Of course, since this relation is not satisfied by the actual dynamics of $\mathscr{P}\rho(t)$, (4.2) and (4.3) yield approximate equations for $\mathscr{P}\rho(t)$ that are no longer equivalent.

Estimates for the validity of these approximations are difficult to establish, and in certain cases they may lead to inconsistencies, namely the positivity of the distribution not being preserved. Roughly speaking, these approximations are generally valid if $\mathscr{P}\rho(t)$ evolves on a time scale that is much longer than the time scale associated with the random part of $\mathbb{L}(t)$ (this point is discussed in detail in Vanden Eijnden 1997*b*). It should also be noted that, even though approximations yielding convolutive equations are most frequently used in the literature, no definitive criterion proves that (4.2) yields a better approximation than (4.3). On the contrary, the argument of simplicity speaks for the convolutionless (4.3) which, in contrast with (4.2), always reduces to a second-order differential operator. Direct calculation from (4.3) shows that the explicit form of this operator is

$$\mathbb{Z}_{(2)}(t,t_0) = \mathscr{P}\int_{t_0}^{t} ds\, \frac{\partial}{\partial \mathbf{q}} \cdot \tilde{\mathbf{p}}(\mathbf{q},t)\, \det[\mathbf{J}(\mathbf{q},t,s)] \times \mathbf{J}^{-1}(\mathbf{q},t,s) \cdot \frac{\partial}{\partial \mathbf{q}} \cdot \tilde{\mathbf{p}}[\mathbf{r}(s|\mathbf{q},t),s]\, \det[\mathbf{J}^{-1}(\mathbf{q},t,s)]\mathscr{P}, \tag{4.6}$$

where $\tilde{\mathbf{p}}(\mathbf{q},t) = \mathbf{p}(\mathbf{q},t) - \mathscr{P}\mathbf{p}(\mathbf{q},t)$ and we have defined the matrix

$$\mathbf{J}(\mathbf{q},t,s) = \frac{\partial}{\partial \mathbf{q}} \otimes \mathbf{r}(s|\mathbf{q},t). \tag{4.7}$$

In (4.6) and (4.7), $\mathbf{r}(s|\mathbf{q},t)$ satisfies

$$\frac{d}{ds}\mathbf{r}(s|\mathbf{q},t) = \mathscr{P}\mathbf{p}[\mathbf{r}(s|\mathbf{q},t),s], \quad \mathbf{r}(t|\mathbf{q},t) = \mathbf{q}, \tag{4.8}$$

arising from the unperturbed motion. Equation (4.6) will be referred to as the strict quasilinear approximation, some consequences of which will be analysed in Sec. 7 for a model test case.

5. Beyond the strict quasilinear approximation

In this section, we study special classes of $\mathbf{p}(\mathbf{q},t)$ that allow us to go beyond the strict quasilinear approximation as defined in Sec. 4. To be more specific, we assume that $\mathbf{p}(\mathbf{q},t)$ may be decomposed into two parts:

$$\mathbf{p}(\mathbf{q},t) = \mathbf{p}_a(\mathbf{q},t) + \mathbf{p}_b(\mathbf{q},t), \tag{5.1}$$

$\mathbf{p}_a$ and $\mathbf{p}_b$ being stochastically independent processes (i.e. $\mathscr{P} \equiv \mathscr{P}_a\mathscr{P}_b = \mathscr{P}_b\mathscr{P}_a$, the subscripts referring to the corresponding processes). Moreover, we assume that $\mathbf{p}_b(\mathbf{q},t)$ is a white-noise process, that is, a Gaussian process, whose covariance is delta-correlated in time:

$$\mathbf{p}_b(\mathbf{q},t) = \mathbf{B}(\mathbf{q},t)\cdot\boldsymbol{\alpha}(t), \quad \text{with} \quad \mathscr{P}_b\{\boldsymbol{\alpha}(t+s)\otimes\boldsymbol{\alpha}(s)\} = \delta(t)\mathbf{I}, \tag{5.2}$$

where $\mathbf{B}(\mathbf{q},t)$ is a non-singular deterministic matrix and $\mathbf{I}$ is the unit matrix.

Owing to the statistical independence between $\mathbf{p}_a$ and $\mathbf{p}_b$, averaging over the two processes can be performed independently. Assume that we consider first only the stochasticity of $\mathbf{p}_b$ and average the Liouville equation over this process using the approximation of Sec. 4. Both (4.2) and (4.3) then yield the following Fokker–Planck equation for the probability density $\mathscr{P}_b\rho(t)$:

$$\frac{\partial}{\partial t}\mathscr{P}_b\rho(t) = -\frac{\partial}{\partial \mathbf{q}}\cdot[\mathbf{p}_a(\mathbf{q},t) + \boldsymbol{\imath}(\mathbf{q},t)]\mathscr{P}_b\rho(t) + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}}\cdot\boldsymbol{\chi}(\mathbf{q},t)\cdot\frac{\partial}{\partial \mathbf{q}}\mathscr{P}_b\rho(t), \qquad (5.3)$$

where

$$\boldsymbol{\imath}(\mathbf{q},t) \equiv -\tfrac{1}{2}\mathbf{B}(\mathbf{q},t)\cdot\left(\frac{\partial}{\partial \mathbf{q}}\cdot\mathbf{B}(\mathbf{q},t)\right), \quad \boldsymbol{\chi}(\mathbf{q},t) \equiv \mathbf{B}(\mathbf{q},t)\cdot\mathbf{B}^{\mathrm{T}}(\mathbf{q},t). \qquad (5.4)$$

It is worth noting that the results of stochastic analysis show that (5.3) is exact provided that Stratanovitch's interpretation† is used for the white noise (see e.g. Gard 1987). Thus no approximation has been made so far.

Consider next the possibility of deriving from (5.3) an equation for the complete average $\mathscr{P}\rho(t) = \mathscr{P}_a\mathscr{P}_b\rho(t)$. Straightforward adaptation of the calculations in Secs 3 and 4 shows that to second order in the random field $\mathbf{p}_a(\mathbf{q},t)$, we obtain a kinetic equation whose $\mathbb{G}_{(2)}(t,s)$ and $\mathbb{Z}_{(2)}(t,t_0)$ are given by (4.2) and (4.3) respectively, where $\widehat{\mathbb{L}}(t)$ is replaced by $\widehat{\mathbb{F}}(t)$ with

$$\widehat{\mathbb{F}}(t) = -\frac{\partial}{\partial \mathbf{q}}\cdot[\mathscr{P}_a\mathbf{p}_a(\mathbf{q},t)\mathscr{P}_a + \boldsymbol{\imath}(\mathbf{q},t)] + \frac{1}{2}\frac{\partial}{\partial \mathbf{q}}\cdot\boldsymbol{\chi}(\mathbf{q},t)\cdot\frac{\partial}{\partial \mathbf{q}}, \qquad (5.5)$$

arising from the Fokker–Planck equation (5.3).

It is worth stressing that the above two-step procedure of approximation is *not* equivalent to strict quasilinear approximation, which would amounts to averaging over both the processes $\mathbf{p}_a$ and $\mathbf{p}_b$ in a single step. This is because the expansion procedure derived in Secs 3 and 4 is not a linear operation. As for the approximations in Sec. 4, estimates for the validity of the present approximations are difficult to establish and no definitive criterion favours either the convolutive or the convolutionless equation. In the following section, we shall study the convolutionless equation using a model test case. We note that generally the possibility of computing $\mathbb{Z}_{(2)}(t,t_0)$ explicitly depends on $\widehat{\mathbb{F}}(t)$, and in any case the resulting expression may be rather complicated. In particular, a general expression like (4.6) is not available. These difficulties are related to the necessity of computing the inverse propagator $\widehat{\mathbb{U}}^{-1}(t|t_0)$ associated with $\widehat{\mathbb{F}}(t)$, whose domain of definition is smaller than that of the $\widehat{\mathbb{U}}^{-1}(t|t_0)$ associated with $\widehat{\mathbb{L}}(t)$.

6. Application to a model case

As an illustration, we apply the approximations of Secs 4 and 5 to a simple model introduced by Balescu (1994) and studied further by Balescu *et al.* (1994), and which may be viewed as a caricature of the motion of the particle in the guiding-centre approximation subject to a turbulent magnetic field and collisions. The system of equations for the model is

$$\frac{d}{dt}x(t) = v(t)b(z), \qquad (6.1a)$$

† Mathematicians prefer Ito's interpretation. However, only Stratanovitch's interpretation is compatible with physical intuition, which views a white noise as a random process whose correlation time goes to zero.

$$\frac{d}{dt} z(t) = v(t), \tag{6.1b}$$

$$\frac{d}{dt} v(t) = -\nu v(t) + \alpha(t), \tag{6.1c}$$

with the initial condition $x(t_0) = x_0$, $z(t_0) = z_0$ and $v(t_0) = v_0$. Here z is the coordinate along the main component of the magnetic field, B_0 say, x is the coordinate along the component of the magnetic perturbation $B_0 b(z)$, depending only on z, and v models the velocity of the particle along the magnetic field line.

The model involves three random variables, namely $b(z)$, $\alpha(t)$ and v_0, which are assumed to be statistically independent (i.e. $\mathscr{P} = \mathscr{P}_b \mathscr{P}_\alpha \mathscr{P}_0$) and represent the effect of magnetic turbulence ($b(z)$) and collisions ($\alpha(t)$ and v_0) (more details of this representation may be found the Appendix of Balescu *et al.* 1994). The perturbed field is assumed to be a zero-mean Gaussian process, whose covariance is

$$\mathscr{P}_b\{b(z+z')b(z')\} = \beta^2 \mathscr{B}(z/\lambda_\parallel), \tag{6.2}$$

where β and $\lambda_\parallel$ are a characteristic strength and a characteristic length associated with the field. The following model covariance will be used in the sequel:

$$\mathscr{P}_b\{b(z+z')b(z')\} = \beta^2 e^{-z^2/[2\lambda_\parallel^2]}. \tag{6.3}$$

The collisions are modelled by the white-noise acceleration $\alpha(t)$ specified by

$$\mathscr{P}_\alpha\{\alpha(t+s)\alpha(s)\} = \nu V_t^2 \delta(t) \tag{6.4}$$

and the statistics over the initial velocity v_0 specified by

$$\mathscr{P}_0 f(v_0) = \int_{-\infty}^{\infty} dv_0 \, \frac{e^{-v_0^2/V_t^2}}{\pi^{1/2} V_t} f(v_0). \tag{6.5}$$

Here $V_t = (2T/m)^{1/2}$ is the thermal velocity and ν is the collision frequency.

The model is made dimensionless by rescaling the variables as $\nu t \to t$, $v/V_T \to v$, $\nu z/V_t \to z$ and $\nu x/(\beta V_t) \to x$. This amounts to setting $\nu = 1$ in (6.1) and replacing (6.2), (6.4) and (6.5) by

$$\mathscr{P}_b\{b(z+z')b(z')\} = \mathscr{B}\left(\gamma^{1/2} z\right), \tag{6.6a}$$

$$\mathscr{P}_\alpha\{\alpha(t+s)\alpha(s)\} = \delta(t), \tag{6.6b}$$

$$\mathscr{P}_0 f(v_0) = \int_{-\infty}^{\infty} dv_0 \, \frac{e^{-v_0^2}}{\pi^{1/2}} f(v_0), \tag{6.6c}$$

where $\gamma = V_t^2/\nu^2\lambda_\parallel^2$ is the only remaining parameter.

6.1. *Exact solution*

In the dimensionless variables, the Liouville equation associated with (6.1) is

$$\frac{\partial}{\partial t}\rho(t) = -vb(z)\frac{\partial}{\partial x}\rho(t) - v\frac{\partial}{\partial z}\rho(t) + \frac{\partial}{\partial v}\left[v - \alpha(t)\right]\rho(t). \tag{6.7}$$

Using the property that $x(t) = x[z(t)]$ which follows from (6.1), the solution of (6.7) for the initial condition $\rho(t_0) = \delta(x - x_0)\delta(z - z_0)\delta(v - v_0)$ can be obtained by the method of characteristics. This yields

$$\rho(t) = \delta[x - x(z|x_0, z_0)]\;\delta[z - z(t|z_0, v_0)]\;\delta[v - v(t|v_0)], \tag{6.8}$$

where

$$x(z|x_0, z_0) = x_0 + \int_{z_0}^{z} dz'\ b(z'), \tag{6.9a}$$

$$z(t|z_0, v_0) = z_0 + v_0(1 - e^{-t+t_0}) + \int_{t_0}^{t} ds\ (1 - e^{-t+s})\alpha(s), \tag{6.9b}$$

$$v(t|v_0) = v_0 e^{-t+t_0} + \int_{t_0}^{t} ds\ e^{-t+s}\alpha(s). \tag{6.9c}$$

It follows from the property that $\mathscr{P} = \mathscr{P}_b \mathscr{P}_\alpha \mathscr{P}_0$, that the average of (6.8) can be expressed as

$$\mathscr{P}\rho(t) = \mathscr{P}_b \mathscr{P}_\alpha \mathscr{P}_0 \rho(t) = \int_{-\infty}^{\infty} dv_0\ \frac{e^{-v_0^2}}{\pi^{1/2}} \mathscr{P}_b \mathscr{P}_\alpha \rho(t), \tag{6.10}$$

where

$$\mathscr{P}_b \mathscr{P}_\alpha \rho(t) = \mathscr{P}_b \left\{ \delta[x - x(z|x_0, z_0)] \right\} \mathscr{P}_\alpha \left\{ \delta[z - z(t|z_0, v_0)]\ \delta[v - v(t|v_0)] \right\}, \tag{6.11}$$

Explicit evaluation shows that, owing to the statistical homogeneity in z of $b(z)$ and the statistical stationarity of $\alpha(t)$, $\mathscr{P}\rho(t)$ depends on t_0, x_0 and z_0 only through $t - t_0$, $x - x_0$ and $z - z_0$. Thus we can take $t_0 = 0$, $x_0 = 0$ and $z_0 = 0$ without loss of generality. The result is

$$\mathscr{P}\rho(t) = \frac{1}{[(2\pi)^3 g(z) h(t)]^{1/2}} \exp\left[-\frac{x^2}{2g(z)} - \frac{z^2 - 2\varphi(t) z v + 2\psi(t) v^2}{2h(t)} \right], \tag{6.12}$$

where we have defined

$$\varphi(t) = 1 - e^{-t}, \quad \psi(t) = t - 1 + e^{-t}, \quad h(t) = \psi(t) - \tfrac{1}{2}\varphi^2(t), \tag{6.13}$$

$$g(z) = \int_0^z dz_1 \int_0^z dz_2\ \mathscr{B}(\gamma^{1/2}(z_1 - z_2)). \tag{6.14}$$

Note that it follows from (6.12) that the average of $x^2(t)$ is

$$\mathscr{P}x^2(t) = \frac{1}{[2\pi\psi(t)]^{1/2}} \int_{-\infty}^{\infty} dz\ g(z) e^{-z^2/[2\psi(t)]}. \tag{6.15}$$

This yields in particular for the model covariance (6.3)

$$\mathscr{P}x^2(t) = 2\gamma^{-1}\{[1 + \gamma\psi(t)]^{1/2} - 1\}. \tag{6.16}$$

This results implies that $\mathscr{P}x^2(t) \sim 2(\gamma t)^{1/2}$ for $t \gg 1$, which shows the (generally) subdiffusive behaviour of the model. Note also that diffusive behaviour is observed in the limit of $\gamma = 0$, corresponding to a very strongly collisional regime:

$$\lim_{\gamma\to 0} \mathscr{P}x^2(t) = \psi(t) \quad \Rightarrow \quad \lim_{\gamma\to 0} \mathscr{P}x^2(t) \sim t \quad \text{for } t \gg 1. \tag{6.17}$$

6.2. Quasilinear treatment

Application of the strict quasilinear approximation (4.6) to (6.7) gives

$$\frac{\partial}{\partial t}\mathscr{P}\rho(t) = -v\frac{\partial}{\partial z}\mathscr{P}\rho(t) + \frac{\partial}{\partial v}\left(v + \frac{1}{2}\frac{\partial}{\partial v} \right)\mathscr{P}\rho(t) + D(v,t)\frac{\partial^2}{\partial x^2}\mathscr{P}\rho(t), \tag{6.18}$$

where

$$D(v,t) = v^2 \int_0^t ds\ e^s \mathscr{B}(\gamma^{1/2} v(e^s - 1)). \tag{6.19}$$

Owing to the v dependence of $D(v,t)$, the analytical solution of (6.18) is generally not available. However, the corresponding value of $\mathscr{P}x^2(t)$ can be obtained:

$$\mathscr{P}x^2(t) = 2\int_0^t ds \int_{-\infty}^{\infty} dv\ D(v,s)\frac{e^{-v^2}}{\pi^{1/2}}. \tag{6.20}$$

Thus, in contrast with the exact result (6.16), the strict quasilinear approximation predicts (generally) diffusion. Indeed, combining (6.19) and (6.20), we obtain

$$\lim_{t\to\infty}\frac{d}{dt}\mathscr{P}x^2(t) = \frac{1}{\pi^{1/2}}\int_0^{\infty} dr\ \mathscr{B}(\gamma^{1/2}r), \tag{6.21}$$

which gives, for instance for the model covariance (6.3),

$$\lim_{t\to\infty}\frac{d}{dt}\mathscr{P}x^2(t) = \left(\frac{2}{\gamma}\right)^{1/2}. \tag{6.22}$$

Also, in the limit of high collisionality, (6.18) yields superdiffusion:

$$\lim_{\gamma\to 0}\mathscr{P}x^2(t) = e^t - 1 - t, \tag{6.23}$$

which should be compared with the exact result (6.17). Equations (6.21) and (6.23) clearly imply the failure of the strict quasilinear approximation for this model.† In the present case, the difficulty can be explained upon noting that the magnetic fluctuations enter (6.19) via the factor $\mathscr{B}\left(\gamma^{1/2}v\,(e^s-1)\right)$. Hence the effective time-scale for the evolution of the magnetic fluctuations is velocity-dependent, and, for small v, it may be much longer that the effective time scale for the evolution of $\mathscr{P}\rho(t)$. As briefly stated in Sec. 4, this invalidates the strict quasilinear approximation.

6.3. Beyond the quasilinear treatment

Owing to the white-noise nature of $\alpha(t)$ and the property of statistical independence, $\mathscr{P} = \mathscr{P}_b\mathscr{P}_\alpha\mathscr{P}_0$, it follows that $\mathscr{P}_\alpha\rho(t)$ satisfies the following Fokker–Planck equation:

$$\frac{\partial}{\partial t}\mathscr{P}_\alpha\rho(t) = \widehat{\mathbb{F}}\,\mathscr{P}_\alpha\rho(t) + \widetilde{\mathbb{L}}\,\mathscr{P}_\alpha\rho(t), \tag{6.24}$$

where

$$\widehat{\mathbb{F}} = -v\frac{\partial}{\partial z} + \frac{\partial}{\partial v}\left(v + \frac{1}{2}\frac{\partial}{\partial v}\right), \quad \widetilde{\mathbb{L}} = -vb(z)\frac{\partial}{\partial x}. \tag{6.25}$$

For applying the approximation of Sec. 5 we thus need to evaluate (4.3) with $\widehat{\mathbb{U}}(t+s|s) = e^{\widehat{\mathbb{F}}t}$. This can be done by introducing the following Fourier representation for $\rho(t)$:

$$\bar\rho(k,p,q,t) = \int dx\,dz\,dv\ e^{-ikx-ipz-iqv}\rho(x,z,v,t). \tag{6.26}$$

Explicit evaluation then yields the following equation for $\mathscr{P}\bar\rho(t)$:

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{P}\bar\rho(k,p,q,t) &= (p-q)\frac{\partial}{\partial q}\mathscr{P}\bar\rho(k,p,q,t) - \frac{q^2}{2}\mathscr{P}\bar\rho(k,p,q,t)\\ &\quad - k^2\int\frac{dp'}{2\pi}\frac{dq'}{2\pi}K(p,q,t|p',q')\mathscr{P}\bar\rho(k,p',q',t),\end{aligned} \tag{6.27}$$

† It should be noted that the convolutive approximation (4.2) also fails for this model (see the calculation in Balescu *et al.* 1994).

where K is the operator

$$K(p,q,t|p',q') = -\int_0^t ds\, e^s \int \frac{dp_1}{2\pi}\frac{dq_1}{2\pi}\frac{dp_2}{2\pi}\,\frac{\partial}{\partial q}\left[\frac{\partial}{\partial q} + \int_0^s du\, e^{-u} g(p-p_2,q,u)\right]$$

$$\times \bar{G}(p-p_2,q,s|p_1,q_1)\bar{\mathscr{B}}(p_2/\gamma^{1/2})\bar{G}(p_1+p_2,q_1,-s|p',q'), \qquad (6.28)$$

where

$$g(p,q,t) = p\left(1-e^{-t}\right) + qe^{-t}. \qquad (6.29)$$

In (6.28), $\bar{\mathscr{B}}(p)$ is the Fourier transform of the covariance of the magnetic field,

$$\bar{\mathscr{B}}(p) = \int dz\, e^{-ipz}\mathscr{B}(z), \qquad (6.30)$$

and $\bar{G}(p,q,t|p_0,q_0)$ is the Fourier transform,

$$\bar{G}(p,q,t|p_0,q_0) = \int dz\, dv\, dz_0\, dv_0\; e^{-ipz-iqv+ip_0z_0+iq_0v_0} G(z,v,t|z_0,v_0), \qquad (6.31)$$

of the Green function corresponding to the Fokker–Planck operator $\widehat{\mathbb{F}}$:

$$G(z,v,t|z_0,v_0) = e^{\widehat{\mathbb{F}}t}\delta(z-z_0)\delta(v-v_0). \qquad (6.32)$$

The explicit expression for $\bar{G}(p,q,t|p_0,q_0)$ is

$$\bar{G}(p,q,t|p_0,q_0) = (2\pi)^2 \exp\left[-\frac{1}{2}\int_0^t ds\; g^2(p,q,s)\right]\delta(p-p_0)\delta[g(p,q,t)-q_0]\,, \quad (6.33)$$

which, on using (6.28), gives the following explicit expression for the last term on the right-hand side of (6.27):

$$k^2 \int_0^t ds\, e^s \int \frac{dp'}{2\pi}\,\frac{\partial}{\partial q}\left[\frac{\partial}{\partial q} + \int_0^s du\, e^{-u} g(p-p',q,u)\right]$$

$$\times \exp\left(-\frac{1}{2}\int_0^s du\, \{g^2(p-p',q,u) - [g(p,q,u) - p'(e^{s-u}-e^{-u})]^2\}\right)$$

$$\times \bar{\mathscr{B}}(p'/\gamma^{1/2})\,\mathscr{P}\bar{\rho}(k,p,q-p'(e^s-1),t). \qquad (6.34)$$

Equation (6.34) does not admit any simple expression in the original representation $\{x,z,v\}$. Moreover, the analytical solution of (6.27) is generally not available. However, direct calculation shows that this equation predicts the exact expression (6.16) for $\mathscr{P}x^2(t)$, which should be contrasted with the wrong result predicted by the strict quasilinear approximation. Finally, we note that in the limit of high collisionality $\gamma = 0$, (6.27) reads, in the original representation,

$$\frac{\partial}{\partial t}\mathscr{P}\rho(t) = -v\frac{\partial}{\partial z}\mathscr{P}\rho(t) + \frac{\partial}{\partial v}\left(v + \frac{1}{2}\frac{\partial}{\partial v}\right)\mathscr{P}\rho(t)$$

$$+v\frac{\partial^2}{\partial x^2}\left[a(t)\frac{\partial}{\partial v} + b(t)\frac{\partial}{\partial z} + c(t)v\right]\mathscr{P}\rho(t). \qquad (6.35)$$

where $a(t) = \cosh t - 1$, $b(t) = \sinh t - t$ and $c(t) = e^t - 1$. Again, direct calculation shows that (6.35) predicts the exact expression (6.17) for $\mathscr{P}x^2(t)$ in the limit $\gamma = 0$.

7. Concluding remarks

Stochastic modelling of turbulence and anomalous transport has been considered. Methods of analysis based on projection techniques have been reviewed. These methods may lead to exact equations that are either convolutive in time or convolutionless. They permit the derivation of approximate equations as expansions in the purely random part of the Liouville operator. In this work, we have focused mainly on those approximations leading to convolutionless equations. Two approximations have been considered: the strict quasilinear approximation, which is strictly quadratic in the random part of $\mathbb{L}(t)$, and an approximation that goes beyond it by treating exactly a part of the random part of $\mathbb{L}(t)$. These two approximations have been tested and compared on a model case, showing indeed the superiority of the second approximation.

Dedication

It is a great pleasure for us to dedicate this paper to our teacher and friend Radu Balescu.

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