Statistical description and transport in stochastic magnetic fields

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The statistical description of particle motion in a stochastic magnetic field is presented. Starting form the stochastic Liouville equation (or, hybrid kinetic equation) associated with the equations of motion of a test particle, the probability distribution function of the system is obtained for various magnetic fields and collisional processes. The influence of these two ingredients on the statistics of the particle dynamics is stressed. In all cases, transport properties of the system are discussed. © *1996 American Institute of Physics.* [S1070-664X(96)02303-8]

I. INTRODUCTION

The prediction of anomalous transport in tokamak-like or stellarator-like magnetized plasmas is a central issue in the plasma physics field.[1] Yet, because of its complexity, this problem has not received a final answer. Both classical and neoclassical transport theories,[2] accounting only for binary collisions, predict fluxes of particles and energy which are much lower than the observed ones. Among other candidates, the magnetic turbulence is likely to explain—at least partially—such a discrepancy.[3–28] Indeed, even very small magnetic fluctuations can locally destroy the nested magnetic surfaces configuration which prevents radial excursion of the particles (essentially electrons) and, hence, ensures confinement. This paper is devoted to the study of such phenomena.

In order to clarify the problem, we first introduce the formalism used in this work. We consider a magnetic field given in a Cartesian reference frame $\{\mathbf{r}\}=\{\mathbf{r}_\perp,z\}=\{x,y,z\}$ by

$$\mathbf{B}(\mathbf{r})=B_0\{\hat{\mathbf{z}}+b_x(\mathbf{r})\hat{\mathbf{x}}+b_y(\mathbf{r})\hat{\mathbf{y}}\}. \tag{1}$$

Here, $B_0\mathbf{b}=B_0\{b_x(\mathbf{r})\hat{\mathbf{x}}+b_y(\mathbf{r})\hat{\mathbf{y}}\}$ is the random magnetic perturbation added to the large homogeneous averaged part of the field $B_0\hat{\mathbf{z}}$ corresponding to the shearless slab geometry.[13,26,27] The statistics of $\mathbf{b}$ is specified below. The statistical description of the fluctuations is a classical tool used in turbulence theory for representing quantities with an apparent random variation in space.[29] The hope is then to derive some averaged behaviors even if the detailed description of this variation is impossible. For describing the dynamics of the plasma, we adopt the model discussed in Ref. 24. We first single out a test particle in the plasma. For a sufficiently strong main magnetic field B_0, the position of this particle can be identified with the position of its guiding center. The guiding center motion is influenced simultaneously by the action of the magnetic field (1) and by the effect of collisions. Here, we model these collisions by assuming that they induce a random variation $\boldsymbol{\eta}$ in the test particle velocity. It is convenient to introduce the components $\eta_\parallel$ and $\boldsymbol{\eta}_\perp=\{\eta_x,\eta_y\}$ in the directions, respectively, parallel and perpendicular to the reference magnetic field $B_0\hat{\mathbf{z}}$. Such a description of the collisions is thus analogous with the one used in the Brownian motion theory[29,30] and the equations of motion of the test particle are the *Langevin equations*. We note, however, that, in contrast to classical Brownian motion, the action of the collisions is introduced at the level of the velocities instead of the accelerations ("V-Langevin" equations, see Ref. 24). We thus deal with first-order differential equations for the motion of the test particle. They are obtained from the equations of the magnetic field lines, $dx/B_0b_x=dy/B_0b_y=dz/B_0$, in which random collisional velocities are added:

$$\frac{d\mathbf{r}_\perp(t)}{dt}=\mathbf{b}[\mathbf{r}(t)]\,\eta_\parallel(t)+\boldsymbol{\eta}_\perp(t), \tag{2a}$$

$$\frac{dz(t)}{dt}=\eta_\parallel(t), \tag{2b}$$

where the initial conditions are $\mathbf{r}(0)=\mathbf{r}_0$. Thus, the test particle moves on the magnetic lines due to the "parallel collisions" and leaves the lines due to the "perpendicular collisions." Equations (2) have to be completed by the statistical properties of the random velocities $\boldsymbol{\eta}$. The Langevin equations (2) are equivalent to the following Liouville-like equation[29,31]:

$$\partial_t f+\nabla_z\eta_\parallel f+\nabla_\perp\cdot(\eta_\parallel\mathbf{b}+\boldsymbol{\eta}_\perp)f=0, \tag{3}$$

with initial condition $f(\mathbf{r};0)=\delta(\mathbf{r}-\mathbf{r}_0)$. In order to stress the stochastic modeling of the collisions, Eq. (3) has been called the *hybrid kinetic equation* (HKE) in Ref. 24. Equations (2) and (3) are related in the same way as the Newton equations in classical mechanics and the Liouville equation in statistical physics. In particular, the solution of the specified initial-value problem for Eq. (3) is given by[29]

$$f(\mathbf{r};t)=\delta[\mathbf{r}-\mathbf{r}(t)], \tag{4}$$

where $\mathbf{r}(t)$ is the solution of Eqs. (2). Equation (4) expresses the uniqueness of the test particle trajectory in a given realization of the magnetic field $\mathbf{b}$ and of the velocities $\boldsymbol{\eta}$. Due to the stochastic nature of the Langevin equations (2), the test particle position may only be known in a statistical sense described by the average probability distribution function $F(\mathbf{r};t)$ of the particle on its phase space. Of course, the distribution function depends itself on the statistical properties of $\mathbf{b}$ and $\boldsymbol{\eta}$. For complex phenomena, the complete knowledge of F requires the determination of an infinite number of

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moments from the Langevin equation. In that case, the Liouville equation (3) may be more appropriate since F is given by the triple ensemble average of the function f over $\mathbf{b}$, $\eta_\parallel$ and $\boldsymbol{\eta}_\perp$, i.e.

$$F=\langle f\rangle_{\mathbf{b},\parallel,\perp}\,. \tag{5}$$

All observable quantities, like the moments of the test particle position, are obtained from F by the single phase-space average of statistical mechanics. Consequently, the equation for F may be referred to as the ''true'' kinetic equation of the model. This equation is obtained by averaging the hybrid equation (3) over $\mathbf{b}$ and $\boldsymbol{\eta}$. This operation yields the *continuity equation*

$$\partial_t F=-\nabla\cdot\boldsymbol{\Gamma}, \tag{6}$$

where the particle flux $\boldsymbol{\Gamma}$ is given by

$$\boldsymbol{\Gamma}_\perp=\langle(\eta_\parallel\mathbf{b}+\boldsymbol{\eta}_\perp)f\rangle_{\mathbf{b},\parallel,\perp}\,,\quad \Gamma_z=\langle\eta_\parallel f\rangle_{\mathbf{b},\parallel,\perp}\,. \tag{7}$$

Because the averages in Eq. (7) cannot be factorized, the expression (7) for the flux does not yield a closed equation for F. We face a closure problem, as usual in nonequilibrium statistical mechanics. Expressing the flux as a functional of F could actually summarize the purpose of the present paper. Here, it should also be noted that Eq. (6) is not *stricto sensu* a kinetic equation of the nonequilibrium statistical mechanics.[31] Within the latter framework, a kinetic equation like, e.g., the Boltzmann equation, is derived from the analysis of the dynamics of the entire set of interacting particles which constitute the plasma. Rather, Eq. (6) should thus be compared with equations derived within the framework of the theory of stochastic processes and whose paradigm is the Fokker–Planck equation.[29,30] Moreover, we remark that most of the kinetic descriptions adopted in the literature[8,10,12,13,21] lie at an intermediate step between Eqs. (3) and (6). They contain indeed the stochastic magnetic field, like Eq. (3), but account from the start for the collisions by a (non-stochastic) Fokker–Planck term[8,13] or a relaxation term of the Krook type.[21] The latter acts on velocity-space, which means that, in contrast to our description, the phase-space of the particle is then extended to its velocity. Reference 24 warns against the Krook-like descriptions, which may yield serious differences in the results. On the other hand, the Fokker–Planck equation should be equivalent to ours: the comparison may, however, be nontrivial because the stochastic velocities in Eq. (3) cannot be averaged immediately, contrary to the white-noise accelerations yielding the Fokker–Planck term.

The probability distribution function F provides us with the complete statistical description of the process. In particular, the transport properties of the system are obtained from F by considering a given initial density profile ρ_0 of the plasma instead of a single test particle. Equivalently, one could say that ρ_0 is the probability distribution of the initial position of the test particle. In real systems, the density profile usually varies with the radial direction x only; thus we take $\rho_0=\rho_0(x)$. The time-evolution of this density profile is then given by

$$\rho(x;t)=\int d\mathbf{r}'\,F(\mathbf{r}-\mathbf{r}';t)\rho_0(x'), \tag{8}$$

and its transport properties are characterized by the total flux $\boldsymbol{\Gamma}^{\mathrm{T}}=\Gamma_x^{\mathrm{T}}\hat{\mathbf{x}}$ induced by the inhomogeneous profile $\rho(x;t)$:

$$\Gamma_x^{\mathrm{T}}(x;t)=\int d\mathbf{r}'\,\Gamma_x(\mathbf{r}-\mathbf{r}';t)\rho_0(x'). \tag{9}$$

The general framework presented in this Introduction is used in this paper for studying more and more realistic situations. The purpose is to point out successively the effect and the importance of the various ingredients (parallel and perpendicular collisions, nonlinearity introduced by the magnetic field,...) entering the complete model. In Sec. II, we first consider a situation where the fluctuating part of the magnetic field has only an x-component depending on z. We also assume that the test particle does not experience perpendicular collisions. This yields a well-known subdiffusive behavior.[18,24,33] Moreover, the study of the density profile shows that the particle dynamics is a non-Gaussian process. These results are equivalent to those obtained from a continuous time random walk theory.[33] In Sec. III, we include perpendicular collisions. Asymptotically, the latter dominate the particle dynamics, producing true diffusion. In this asymptotic stage, the density profile is a Gaussian packet. There is no anomalous diffusion in this model. Anomalous diffusive effects are only observed when the magnetic field induces higher correlations between the parallel and perpendicular motions of the particle. This situation is studied in Sec. IV where we consider the general situation for the magnetic field ($b_x\neq 0$, $b_y\neq 0$) and for the collisions, as defined by Eqs. (1) and (2). The magnetic fluctuations are treated within the usual framework of the direct interaction approximation (DIA).[27,34–40] An additional statistical assumption concerning the collisions yields then the density profile together with an ordinary differential equation for its second moment. This equation directly provides the diffusion coefficient of the process. Here again, these results show how diffusion finally dominates the statistical evolution of the particle. The equation for the second moment is studied in Sec. V. All the usual regimes found in the literature are here recovered, including the well-known log-scaled Rechester–Rosenbluth[7,26] diffusion coefficient. The latter corresponds to a situation where the diffusion follows an exponential spreading of the mean square deviation of the particle position into the plasma. This is similar to the clumping process observed in the electrostatic turbulence problem.[27,41–45] Concluding remarks are given in Sec. VI.

II. THE SUBDIFFUSIVE SITUATION

We first study a very simple model.[24,33] We consider that the fluctuating part of the magnetic field (1) has only a component along x depending on the single coordinate z

$$\mathbf{B}(z)=B_0\{\hat{\mathbf{z}}+b(z)\hat{\mathbf{x}}\}. \tag{10}$$

The fluctuating field is assumed to be a Gaussian random process, determined by its second-order correlation,

$$\mathscr{B}(z)=\langle b(z+z')b(z')\rangle_b=\beta^2\exp\left(-\frac{z^2}{2\lambda_\parallel^2}\right), \tag{11}$$

where $\langle \cdots \rangle_b$ denotes the ensemble average over b. We also assume that particles only experience collisions in the direction of the magnetic field. Those are modeled by the noise $\eta_\parallel(t)$ whose statistics are defined by

$$\langle \eta_\parallel(t)\rangle_\parallel = 0, \quad \langle \eta_\parallel(t)\,\eta_\parallel(t')\rangle_\parallel = \chi_\parallel \nu R[\nu|t-t'|], \tag{12}$$

in which $\chi_\parallel = V_T^2/2\nu$ is the parallel collisional diffusion coefficient, V_T is the thermal velocity and ν is the collision frequency. In Eq. (12), $\langle \cdots \rangle_\parallel$ denotes an ensemble average over the parallel collisions and $R(u)$ is a rapidly decreasing function normalized on $[0,\infty[$, for which we define

$$\varphi(u) = \int_0^u R(v)\,dv, \quad \psi(u) = \int_0^u \varphi(v)\,dv. \tag{13}$$

The assumptions on R then imply $\varphi(u) \sim 1$ and $\psi(u) \sim u$ for $u \gg 1$. Finally, we assume that there is no correlation between the magnetic field and the collisions, i.e.

$$\langle b(z)\,\eta_\parallel(t)\rangle_{b,\parallel} = 0. \tag{14}$$

Due to the simple form of the magnetic field (10), there is no coupling between the x and y motions of the test particle. Hence, we can restrict our study to only one of these coordinates, and Eqs. (2) reduce to

$$\frac{dx(t)}{dt} = b[z(t)]\,\eta_\parallel(t), \tag{15a}$$

$$\frac{dz(t)}{dt} = \eta_\parallel(t), \tag{15b}$$

with initial conditions $x(0) = x_0$ and $z(0) = z_0$. The solution of Eqs. (15) is straightforward:

$$x(t) = x_0 + \int_0^t dt_1 b\left[z_0 + \int_0^{t_1} \eta_\parallel(t_2)dt_2\right]\eta_\parallel(t_1), \tag{16a}$$

$$z(t) = z_0 + \int_0^t dt_1\,\eta_\parallel(t_1). \tag{16b}$$

The hybrid kinetic equation associated with Eqs. (15) is

$$\partial_t f + \nabla_z \eta_\parallel f + \nabla_x \eta_\parallel b f = 0, \tag{17}$$

with initial condition $f(x,z;0) = \delta(x-x_0)\,\delta(z-z_0)$. Here, $f(x,z;t)$ is the probability of finding a particle at position $\{x,z\}$ and time t in a given realization of b and $\eta_\parallel$. The solution of the HKE (17) is

$$f(x,z;t) = \delta[x - x(t)]\,\delta[z - z(t)], \tag{18}$$

where $x(t)$ and $z(t)$ are given by Eqs. (16). We next consider the ensemble average of f on both $\eta_\parallel$ and b,

$$F(x,z;t) = \langle f(x,z;t)\rangle_{b,\parallel}. \tag{19}$$

In this "two-noise" model, F is the really relevant quantity. Performing explicitly the averages in Eq. (19) may however be a nontrivial operation. A straightforward calculation using the Fourier transform in x and z yields indeed

$$\begin{aligned} F(x,z;t) = (2\pi)^{-2}\int\int dk_x dk_z\; e^{ik_x(x-x_0)+ik_z(z-z_0)} \\ \times\left\langle \exp\left(-ik_x\int_0^t dt_1 b\left[z_0 + \int_0^{t_1}\eta_\parallel(t_2)dt_2\right]\eta_\parallel(t_1)\right.\right. \\ \left.\left. -ik_z\int_0^t \eta_\parallel(t_1)dt_1\right)\right\rangle_{b,\parallel}. \end{aligned} \tag{20}$$

By performing the average over b, one obtains

$$\begin{aligned} F(x,z;t) = (2\pi)^{-2}\int\int dk_x dk_z\; e^{ik_x(x-x_0)+ik_z(z-z_0)} \\ \times\left\langle \exp\left(-\frac{1}{2}k_x^2\int_0^t dt_1\int_0^t dt_2\right.\right. \\ \times \mathscr{B}\left[\int_{t_2}^{t_1}\eta_\parallel(t_3)dt_3\right]\eta_\parallel(t_1)\,\eta_\parallel(t_2) \\ \left.\left. -ik_z\int_0^t \eta_\parallel(t_1)dt_1\right)\right\rangle_\parallel. \end{aligned} \tag{21}$$

In this way, however, performing the average over $\eta_\parallel$ becomes a complex operation. Let us then formulate the problem differently. In the absence of any perpendicular collision, the ratio of Eqs. (15) is the (closed) equation for the magnetic line,

$$\frac{dx(z)}{dz} = b(z), \tag{22}$$

with initial condition $x(z_0) = x_0$. Here the z-coordinate plays the role of an effective time and Eq. (22) has the solution

$$x(z) = x_0 + \int_{z_0}^z b(z_1)dz_1. \tag{23}$$

On the other hand, we define a distribution function $g(x;z)$ which yields the probability that a line passes in x at "time" z. This distribution function obeys the Liouville equation,

$$\partial_z g + \nabla_x b g = 0, \tag{24}$$

with initial condition $g(x;z_0) = \delta(x-x_0)$. It implies

$$g(x;z) = \delta[x - x(z)]. \tag{25}$$

By comparing Eqs. (18) and (25), we thus conclude

$$\begin{aligned} f(x,z;t) &= g[x;z(t)]\,\delta[z - z(t)], \\ &\equiv g(x;z)\,\delta[z - z(t)]. \end{aligned} \tag{26}$$

This expression indicates that the time dependence of x comes only through z, i.e. $x(t) = x[z(t)]$. It should also be stressed that the factorization (26) holds independently of the assumptions about the magnetic field b, which only influences the Liouville equation (24). So, the factorization (26) will also be used in Sec. IV B where the complete magnetic field (1) is considered. We now average Eq. (26) on b and $\eta_\parallel$. Because the stochastic $z(t)$ only appears in the second delta function of the left hand side, the two averages can be performed independently, yielding

$$F(x,z;t) \equiv G(x;z)\mathscr{P}(z;t), \tag{27}$$

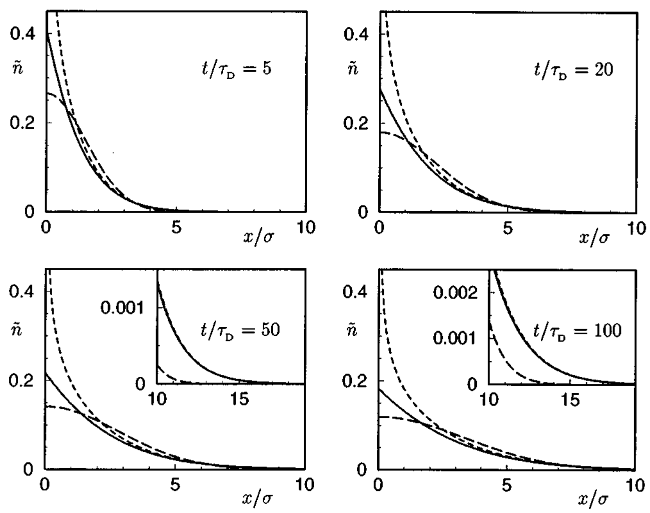

FIG. 1. The density profile defined by Eq. (32) (full line) is compared to the Gaussian packet with same standard deviation (dashed line) for successive times. The dotted line represents the asymptotic profile (49).

where

$$G(x;z)=\langle g(x;z)\rangle_b\,,\quad \mathscr{P}(z;t)=\langle \delta[z-z(t)]\rangle_\parallel\,. \tag{28}$$

The function $\mathscr{P}$ is calculated using the Fourier transform and second cumulant expansion:

$$\mathscr{P}(z;t)=(2\pi\langle \delta z^2(t)\rangle_\parallel)^{-1/2}\exp\left(-\frac{1}{2}\,\frac{(z-z_0)^2}{\langle \delta z^2(t)\rangle_\parallel}\right), \tag{29}$$

where $\langle \delta z^2(t)\rangle_\parallel=\langle z^2(t)\rangle_\parallel-z_0^2$. From Eqs. (13) and (16b), this quantity is given by

$$\langle \delta z^2(t)\rangle_\parallel=\chi_\parallel\nu\int_0^t dt_1\int_0^t dt_2\ R[\nu|t_1-t_2|]$$
$$\equiv 2\ \chi_\parallel\nu^{-1}\psi(\nu t). \tag{30}$$

On the other hand, the calculation for G yields

$$G(x;z)=(2\pi)^{-1}\int_{-\infty}^{\infty} dk_x\ e^{ik_x(x-x_0)}$$
$$\times\exp\left(-\frac{1}{2}k_x^2\int_{z_0}^{z} dz_1\int_{z_0}^{z} dz_2\,\mathscr{B}[z_1-z_2]\right). \tag{31}$$

Eqs. (27)–(31) yield the average of Eq. (21) which seemed impossible to perform.[46] The next step is the calculation of the reduced distribution function (or density profile) $n(x;t)$,

$$n(x;t)=\int_{-\infty}^{\infty} dz\ F(x,z;t)=\int_{-\infty}^{\infty} dz\ G(x;z)\mathscr{P}(z;t). \tag{32}$$

The integration over z in Eq. (32) with the complete G (31) can only be performed numerically (see Fig. 1). Analytical results can, however, be obtained as follows: We first note that

$$\int_{z_0}^{z} dz_1\int_{z_0}^{z} dz_2\,\mathscr{B}[z_1-z_2]$$
$$=\beta^2\lambda_\parallel\left\{\sqrt{2\pi}|z-z_0|\operatorname{erf}\left(\frac{|z-z_0|}{\sqrt{2}\lambda_\parallel}\right)\right.$$
$$\left.+2\lambda_\parallel(e^{-(1/2)\lambda_\parallel^{-2}(z-z_0)^2}-1)\right\}$$
$$\sim\sqrt{2\pi}\beta^2\lambda_\parallel|z-z_0|,\quad |z-z_0|\gg\lambda_\parallel. \tag{33}$$

Hence, the Fourier transform of G with respect to x has the asymptotic form [here $k\equiv k_x$]

$$G_k(z)\sim(2\pi)^{-1}\exp(-ikx_0-k^2D_m|z-z_0|),$$
$$|z-z_0|\gg\lambda_\parallel, \tag{34}$$

where D_m is the quasilinear magnetic field line diffusion coefficient,

$$D_m=\sqrt{\frac{\pi}{2}}\ \beta^2\lambda_\parallel\,. \tag{35}$$

Substituting Eq. (34) into Eq. (32) and integrating over z yields the result

$$n_k(t)=(2\pi)^{-1}\exp\left(-ikx_0+\frac{1}{2}k^4D_m^2\langle \delta z^2(t)\rangle_\parallel\right)$$
$$\times\operatorname{erfc}\left(\frac{1}{\sqrt{2}}k^2D_m\langle \delta z^2(t)\rangle_\parallel^{1/2}\right), \tag{36}$$

where $\operatorname{erfc}(x)$ is the complementary error function. Because of Eq. (34), Eq. (36) is the asymptotic form (i.e., for large t) of the density profile. Hence, we must use

$$\langle \delta z^2(t)\rangle_\parallel\sim 2\chi_\parallel t,\qquad \nu t\gg 1. \tag{37}$$

All the moments of n can now be obtained using the well-known formula

$$\langle x^p(t)\rangle_{b,\parallel}=2\pi(-i)^{-p}\frac{\partial^p}{\partial k^p}n_k(t)_{|k=0}\,, \tag{38}$$

combined with the series expansion of (36) (for simplicity, we take now $x_0=0$),

$$n_k(t)=(2\pi)^{-1}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\ a^{2n}k^{4n}\right.$$
$$\left.-\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!!}\ a^{2n+1}k^{4n+2}\right\}. \tag{39}$$

Here

$$a=\frac{\sigma^2}{2}\left(\frac{t}{\tau_D}\right)^{1/2}, \tag{40}$$

where $\sigma^2=\sqrt{\pi}\beta^2\lambda_\parallel^2$ and $\tau_D=\lambda_\parallel^2/2\chi_\parallel$. Equations (38) and (39) yield for $p\geq 1$,

$$\langle x^{2p-1}(t)\rangle_{b,\parallel}=0,\quad \langle x^{2p}(t)\rangle_{b,\parallel}=M_p\sigma^{2p}\left(\frac{t}{\tau_D}\right)^{p/2}, \tag{41}$$

with

$$M_p = \frac{(2p-1)!}{2^{p-2}} \frac{1}{\Gamma(p/2)}. \tag{42}$$

Clearly the relations (41) and (42) correspond to a *subdiffusive* behavior of the process in absence of perpendicular collisions.[7,8,13,18,24,33] These expressions also show that all the moments of the density profile (36) are *identical* to those obtained in Ref. 33 from the standard long tail continuous time random walk (SLT-CTRW) theory. This is also corroborated by noting that the Laplace transform of Eq. (36) is

$$\tilde{n}_k(s) = (2\pi)^{-1}\{\sqrt{s}(\sqrt{s} + \tfrac{1}{2}\sigma^2 \tau_D^{-1/2} k^2)\}^{-1}, \tag{43}$$

obeying the non-Markovian continuity equation

$$s\tilde{n}_k(s) - n_k(0) = -(4\pi)^{-1}\sigma^2 \tau_D^{-1/2} k^2 \sqrt{s}\tilde{n}_k(s). \tag{44}$$

Equations (43) and (44) are typical results of SLT-CTRW with a waiting time distribution function whose series expansion is

$$\psi(s) = 1 - (\tau_D s)^{1/2} + \dots, \quad s \ll 1. \tag{45}$$

We thus recover the asymptotic equivalence between a specific SLT-CTRW and the hybrid kinetic equation. Expressions for the physical space form of the density profile (32) can finally be obtained in some specific ranges of values of x and t. We use the real form of the asymptotic G (34) and introduce the reduced variables

$$\tau = \frac{t}{\tau_D}, \quad \tilde{x} = \frac{x}{\sigma}, \quad \tilde{n} = \sigma n. \tag{46}$$

Equation (32) then yields

$$\tilde{n}(\tilde{x};\tau) = 2\int_0^\infty d\zeta\ (2\pi\tau)^{-1/2} \exp\left(-\frac{\zeta^2}{2\tau}\right) \times (2\sqrt{2}\pi\zeta)^{-1/2} \exp\left(-\frac{\sqrt{2}(\tilde{x}-\tilde{x}_0)^2}{4\zeta}\right), \tag{47}$$

or equivalently,

$$\tilde{n}(\tilde{x};\tau) = \frac{1}{\pi}(2\tau)^{-1/4} q^{1/3} \int_0^\infty d\xi\ \exp\left(-q^{4/3}\left(\frac{\xi^4}{32} + \frac{\sqrt{2}}{\xi^2}\right)\right), \tag{48}$$

where $q = |\tilde{x} - \tilde{x}_0|\tau^{-1/4}$. Now, for $q \gg 1$, Eq. (48) can be evaluated by the steepest descent method:

$$\tilde{n}(\tilde{x};\tau) = \sqrt{\frac{2}{3\pi}}\ \tau^{-1/4} q^{-1/3}\ \exp\left(-\frac{3}{4}q^{4/3}\right), \quad q \gg 1. \tag{49}$$

Equation (49) is plotted in Fig. 1 and compared with the Gaussian packet, whose mean square deviation (or second-order moment) is given by Eq. (41). Due to the asymptotic condition $q \gg 1$, Eq. (49) is not valid close to the origin $\tilde{x} = 0$. The norm on $\tilde{x} \in]-\infty, \infty[$ of Eq. (49), which is greater than 1, is thus meaningless. Hence, the information carried by Eq. (41) essentially is its long tail, which is due to the stretching exponent 4/3 in the exponential. Stretched exponentials were already found in Ref. 18 and 33. It implies that the probability for a particle starting from $\tilde{x}_0$ to reach a point $\tilde{x}$, with $|\tilde{x} - \tilde{x}_0|$ large enough, is in any case larger than the probability predicted by the Gaussian packet. This effect, which cannot be guessed from the second moments alone, shows the importance of knowing the entire distribution function for describing the statistics of the process.

III. THE ROLE OF PERPENDICULAR COLLISIONS

We now modify the model of Sec. II by including perpendicular collisions $\boldsymbol{\eta}_\perp$. We chose for $\boldsymbol{\eta}_\perp$ a Gaussian noise with the following correlations:

$$\langle \eta_i(t) \rangle_\perp = 0, \quad \langle \eta_i(t)\eta_j(t') \rangle_\perp = \chi_\perp \nu R[\nu|t-t'|]\delta_{ij}, \quad \{i,j\} = \{x,y\}, \tag{50}$$

in which $\chi_\perp = V_T^2 \nu / 2\Omega^2$ is the perpendicular collisional diffusion coefficient and Ω is the Larmor frequency. We assume that there is no correlation between the magnetic field and the collisions and between parallel and perpendicular collisions,

$$\langle b(z)\eta_\parallel(t)\rangle_{b,\parallel} = 0, \quad \langle b(z)\ \boldsymbol{\eta}_\perp(t)\rangle_{b,\perp} = 0, \quad \langle \eta_\parallel(t)\ \boldsymbol{\eta}_\perp(t)\rangle_{\parallel,\perp} = 0. \tag{51}$$

In this second model, we keep the simple form (10) for the fluctuating magnetic field b. This allows us to consider again only the x coordinate in the perpendicular plane. The equations of motion of the test particle are thus

$$\frac{dx(t)}{dt} = b[z(t)]\eta_\parallel(t) + \eta_x(t), \tag{52a}$$

$$\frac{dz(t)}{dt} = \eta_\parallel(t). \tag{52b}$$

The solution of Eqs. (52) is straightforward

$$x(t) = x_0 + \int_0^t dt_1\ b\left[z_0 + \int_0^{t_1} \eta_\parallel(t_2)dt_2\right]\eta_\parallel(t_1) + \int_0^t dt_1\,\eta_x(t_1), \tag{53a}$$

$$z(t) = z_0 + \int_0^t dt_1\,\eta_\parallel(t_1). \tag{53b}$$

The HKE associated with Eqs. (52) is

$$\partial_t f + \nabla_z \eta_\parallel f + \nabla_x(\eta_\parallel b + \eta_x) f = 0, \tag{54}$$

with the solution $f(x,z;t) = \delta[x - x(t)]\delta[z - z(t)]$, where $x(t)$ and $z(t)$ are given by Eqs. (53). We now note that the solution (53a) splits into

$$x(t) = x_1[z(t)] + x_2(t), \tag{55}$$

where

$$x_1[z(t)] = x_0 + \int_{z_0}^{z(t)} b(z_1)dz_1, \tag{56a}$$

$$x_2(t)=\int_0^t \eta_x(t_1)dt_1. \tag{56b}$$

Hence, the solution of the HKE (54) can be written as

$$f(x,z;t)=\delta[x-x_1[z(t)]-x_2(t)]\delta[z-z(t)],$$
$$\equiv\delta[x-x_1(z)-x_2(t)]\delta[z-z(t)]. \tag{57}$$

It implies that the "three-noise" average $F=\langle f\rangle_{b,\|,\perp}$ is again factorizable:

$$F(x,z;t)=H(x;z,t)\mathscr{P}(z;t), \tag{58}$$

where the distribution $\mathscr{P}$ is given by Eq. (29) and

$$H(x;z,t)=\langle\delta[x-x_1(z)-x_2(t)]\rangle_{b,\perp}. \tag{59}$$

This function can be calculated exactly:

$$H(x;z,t)=(2\pi)^{-1}\int_{-\infty}^{\infty} dk\ \mathrm{e}^{ik(x-x_0)}$$
$$\times\exp\left(-\frac{1}{2}k^2\int_{z_0}^{z}dz_1\int_{z_0}^{z}dz_2\mathscr{B}[z_1-z_2]\right.$$
$$\left.-k^2\chi_\perp\nu^{-1}\psi(\nu t)\right). \tag{60}$$

Equations (29), (58) and (60) give the expression of the probability distribution function F. We now use these results for calculating the density profile (32). Here again, analytical calculations are performed asymptotically. The asymptotic form of the Fourier transform of H with respect to x is (for simplicity, we take $x_0=0$)

$$H_k(z,t)\sim(2\pi)^{-1}\exp(-k^2D_m|z-z_0|-k^2\chi_\perp t),$$
$$|z-z_0|\gg\lambda_\|,\quad t\gg 1. \tag{61}$$

The density profile $n(x;t)$ can then be obtained from Eq. (32) with H substituted for G. The integration over z yields

$$n_k(t)=(2\pi)^{-1}\exp\left(-k^2\chi_\perp t+\frac{1}{2}k^4D_m^2\langle\delta z^2(t)\rangle_\|\right)$$
$$\times\mathrm{erfc}\left(\frac{1}{\sqrt{2}}k^2D_m\langle\delta z^2(t)\rangle_\|^{1/2}\right). \tag{62}$$

This density profile differs from (36) by the Gaussian factor $\exp(-k^2\chi_\perp t)$ coming from the perpendicular collisions. The moments of $n_k(t)$ are obtained from Eq. (38) and (62):

$$\langle x^{2p+1}(t)\rangle_{b,\|,\perp}=0,\quad\langle x^{2p}(t)\rangle_{b,\|,\perp}=(2p)!Q_p, \tag{63}$$

where $Q_0=1$ and

$$Q_{2p}=\sum_{m=0}^{p}\frac{1}{(2m)!(p-m)!}a^{2p-2m}(\chi_\perp t)^{2m}$$
$$+\frac{1}{\sqrt{\pi}}\sum_{m=0}^{p-1}\frac{2^{p-m}}{(2m+1)!(2p-2m-1)!!}a^{2p-2m-1}$$
$$\times(\chi_\perp t)^{2m+1},\quad[p>0], \tag{64a}$$

$$Q_{2p+1}=\sum_{m=0}^{p}\frac{1}{(2m+1)!(p-m)!}a^{2p-2m}(\chi_\perp t)^{2m+1}$$
$$+2\sqrt{\pi}\sum_{m=0}^{p}\frac{2^{p-m}}{(2m)!(2p-2m+1)!!}a^{2p-2m+1}(\chi_\perp t)^{2m},$$
$$[p\geq 0]. \tag{64b}$$

For large t, Eqs. (63) and (64) reduce to

$$\langle x^{2p}(t)\rangle_{b,\|,\perp}\sim\frac{(2p)!}{p!}(\chi_\perp t)^p,\quad t\gg D_m^2\frac{\chi_\|}{\chi_\perp^2}. \tag{65}$$

Here, the validity range in t has been obtained by comparing the shapes of the Gaussian packet $\exp(-k^2\chi_\perp t)$ and the non-Gaussian function (36), whose product is the density profile (62). The remaining term in Eq. (65) only comes from the Gaussian factor $\exp(-k^2\chi_\perp t)$. It shows the *diffusive* character of the process induced by the perpendicular collisions which asymptotically overcome the particle motion with the lines. The latter yields the subdiffusive terms in Eq. (63), which are thus similar to those obtained in Sec. II [see Eq. (41)]. Equation (65) also implies that the form of the density profile (62) in physical space is asymptotically the Gaussian packet,

$$n(x;t)=(4\pi\chi_\perp t)^{-1/2}\exp\left(-\frac{(x-x_0)^2}{4\chi_\perp t}\right). \tag{66}$$

Consequently, for a fluctuating magnetic field depending on the single z-coordinate, the two kinds of motions (diffusion from perpendicular collisions and subdiffusion along the lines) are asymptotically uncorrelated. The "most diffusive" behavior (which is here strict diffusion) dominates. We also stress that such a model does not produce (diffusive) anomalous transport, for which higher correlations (through b) between parallel and perpendicular motions are needed. This situation will be considered in the forthcoming sections.

IV. MAGNETIC FIELD NONLINEARITY AND ANOMALOUS TRANSPORT

We now consider the complete model of Sec. I. The magnetic field is given by Eq. (1) and its fluctuating part is again assumed to be a Gaussian random process. In addition, the second-order correlations are given by[26]

$$\mathscr{B}_{ij}(\mathbf{r})=\left\{\delta_{ij}-\frac{r_\perp^2\delta_{ij}-r_ir_j}{\lambda_\perp^2}\right\}\mathscr{A}(\mathbf{r}),\quad\{i,j\}=\{x,y\}, \tag{67}$$

where

$$\mathscr{A}(\mathbf{r})=\beta^2\exp\left(-\frac{r_\perp^2}{2\lambda_\perp^2}-\frac{z^2}{2\lambda_\|^2}\right). \tag{68}$$

The equations of motion of the test particle are thus Eqs. (2) and the HKE associated with these equations is Eq. (3). The exact solutions of both these equations can no longer be obtained—even formally—and some approximations are now necessary.

A. The direct interaction approximation

We first Fourier transform Eq. (3),

$$\partial_t f_{\mathbf{k}} + i\eta_\parallel k_z f_{\mathbf{k}} + i\,\boldsymbol{\eta}_\perp\cdot\mathbf{k}_\perp f_{\mathbf{k}} + i\eta_\parallel \int d\mathbf{k}'\,\mathbf{b}_{\mathbf{k}-\mathbf{k}'}\cdot\mathbf{k}'_\perp f_{\mathbf{k}'} = 0, \tag{69}$$

with the initial condition $f_{\mathbf{k}}(0)=(2\pi)^{-3}\exp(-i\mathbf{k}\cdot\mathbf{r}_0)$. Next we focus on the "noise" $\mathbf{b}$ without considering the randomness of the velocities $\boldsymbol{\eta}$. This is justified since $\mathbf{b}$ and $\boldsymbol{\eta}$ are supposed to be uncorrelated. Then, we apply the well-known direct interaction approximation (DIA)[27,34–40] for averaging Eq. (69) on $\mathbf{b}$. This is justified at least for small Kubo number, i.e. for $\alpha=\beta\lambda_\parallel/\lambda_\perp \lesssim 1$[35,40] and leads to the following equation for $\langle f\rangle_{\mathbf{b}}$:

$$\begin{aligned}\partial_t\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}} &+ i\eta_\parallel(t)k_z\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}} + i\,\boldsymbol{\eta}_\perp(t)\cdot\mathbf{k}_\perp\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}}\\ &= -\int_0^t dt_1 \int d\mathbf{k}'(k_x k'_y - k_y k'_x)^2\,\eta_\parallel(t_1)\,\eta_\parallel(0)\\ &\quad\times \mathscr{A}_{\mathbf{k}'} g_{\mathbf{k}-\mathbf{k}'}(t_1)\langle f_{\mathbf{k}}(t-t_1)\rangle_{\mathbf{b}},\end{aligned} \tag{70}$$

with the initial condition $\langle f_{\mathbf{k}}(0)\rangle_{\mathbf{b}}=(2\pi)^{-3}\exp(-i\mathbf{k}\cdot\mathbf{r}_0)$. In Eq. (70), the propagator $g_{\mathbf{k}}(t)$ obeys the following equation:

$$\begin{aligned}\partial_t g_{\mathbf{k}}(t) &+ i\eta_\parallel(t)k_z g_{\mathbf{k}}(t) + i\,\boldsymbol{\eta}_\perp(t)\cdot\mathbf{k}_\perp g_{\mathbf{k}}(t)\\ &= -\int_0^t dt_1 \int d\mathbf{k}'(k_x k'_y - k_y k'_x)^2\,\eta_\parallel(t_1)\,\eta_\parallel(0)\\ &\quad\times \mathscr{A}_{\mathbf{k}'} g_{\mathbf{k}-\mathbf{k}'}(t_1) g_{\mathbf{k}}(t-t_1),\end{aligned} \tag{71}$$

with the initial condition $g_{\mathbf{k}}(0)=1$. We stress that both $\langle f\rangle_{\mathbf{b}}$ and g are still fluctuating quantities depending on the noise $\boldsymbol{\eta}$. For treating Eqs. (70) and (71), we assume here that the $\mathbf{k}$-dependence of the propagator on the right hand sides can be neglected, i.e. $g_{\mathbf{k}-\mathbf{k}'}(t_1)\simeq g_{-\mathbf{k}'}(t_1)$. We also take $\langle f_{\mathbf{k}}(t-t_1)\rangle_{\mathbf{b}}\simeq\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}}$ and $g_{\mathbf{k}}(t-t_1)\simeq g_{\mathbf{k}}(t)$. Within these two Markovian approximations,[40] Eqs. (70) and (71) become diffusion equations with solutions

$$\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}} = g_{\mathbf{k}}(t)\langle f_{\mathbf{k}}(0)\rangle_{\mathbf{b}}, \tag{72a}$$

$$g_{\mathbf{k}}(t)=\exp\left(-i\mathbf{k}\cdot\int_0^t \boldsymbol{\eta}(t_1)dt_1 - \frac{1}{2}\mathbf{k}_\perp\mathbf{k}_\perp : \mathscr{M}(t)\right), \tag{72b}$$

where the 2×2 diffusion tensor $\mathscr{M}$ is the solution of the second-order nonlinear stochastic differential equation,

$$\begin{aligned}\frac{d^2}{dt^2}\mathscr{M}_{ij}(t) &= 2\int d\mathbf{k}\; m^{ij}_{\mathbf{k}}\,\eta_\parallel(t)\,\eta_\parallel(0)\mathscr{A}_{\mathbf{k}}\\ &\quad\times\exp\left(i\mathbf{k}\cdot\int_0^t \boldsymbol{\eta}(t_1)dt_1 - \frac{1}{2}\mathbf{k}_\perp\mathbf{k}_\perp : \mathscr{M}(t)\right),\\ &\quad \{i,j\}=\{x,y\},\end{aligned} \tag{73}$$

in which $m^{ij}_{\mathbf{k}}=k_\perp^2\delta_{ij}-k_ik_j$. Here, $\mathscr{M}(0)=0$ and $(d/dt)\mathscr{M}_{|t=0}=0$.

B. Decomposition of the particle motion

In order to simplify the discussion, we decompose Eq. (73) into different terms which will be shown to correspond to different processes. We first identify the tensor $\mathscr{M}$. By definition, the second moment of $\langle f\rangle_{\mathbf{b}}$ is given by

$$\langle r_i(t)r_j(t)\rangle_{\mathbf{b}} = -(2\pi)^3\{\partial_{k_i}\partial_{k_j}\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b}}\}_{\mathbf{k}=0},\qquad \{i,j\}=\{x,y\}. \tag{74}$$

By inserting the solutions (72) into the time derivative of Eqs. (74), we thus obtain

$$\frac{d}{dt}\langle r_i(t)r_j(t)\rangle_{\mathbf{b}} = \int_0^t dt_1\{\eta_i(t_1)\,\eta_j(t)+\eta_j(t_1)\,\eta_i(t)\} + \frac{d}{dt}\mathscr{M}_{ij}(t). \tag{75}$$

After averaging over the collisions, the first term on the right hand side of Eq. (75) accounts for classical collisional effects in the perpendicular x-y plane. Hence, the average over $\boldsymbol{\eta}$ of the tensor $\mathscr{M}$ yields the anomalous part of the particle second moment, i.e.

$$\frac{d}{dt}\langle r_i(t)r_j(t)\rangle^{\mathrm{AN}}_{\mathbf{b},\parallel,\perp} = \frac{d}{dt}\,\langle\mathscr{M}_{ij}(t)\rangle_{\parallel,\perp}. \tag{76}$$

Next, it is useful to analyze Eq. (73) within the limits of zero perpendicular collisions, i.e. when $\boldsymbol{\eta}_\perp=0$. From Sec. II, we know that in this case the test particle sticks to a magnetic line. Hence, its evolution can be described as the superposition of the particle motion on the magnetic line and of the line motion itself. For $\boldsymbol{\eta}_\perp=0$, such a factorization can also be performed on Eq. (73) by introducing the (stochastic) variable,

$$z(t)=z_0+\int_0^t \eta_\parallel(t_1)dt_1. \tag{77}$$

Equation (77) is identical to Eq. (16b) and yields thus the z-coordinate of the particle on the magnetic line in a given realization of $\eta_\parallel$. Let us denote by $\mathscr{M}^{(0)}$ the solution of Eq. (73) when $\boldsymbol{\eta}_\perp=0$. Using the variable (77), Eq. (73) becomes

$$\begin{aligned}\frac{d^2}{dz^2}\mathscr{M}^{(0)}_{ij}(z) &= 2\int d\mathbf{k}\; m^{ij}_{\mathbf{k}}\mathscr{A}_{\mathbf{k}}\\ &\quad\times\exp\left(ik_z(z-z_0)-\frac{1}{2}\mathbf{k}_\perp\mathbf{k}_\perp : \mathscr{M}^{(0)}(z)\right),\end{aligned} \tag{78}$$

with initial conditions $\mathscr{M}^{(0)}(z_0)=0$ and $(d/dz)\mathscr{M}^{(0)}{}_{|z=z_0}=0$. Equation (78) has been previously derived in Ref. 27 within the DIA (see also Ref. 26) for describing the statistical evolution of the magnetic lines alone, i.e. without considering particles. Contrary to the time dependence in Eq. (73), the z-dependence is explicit in Eq. (78). Hence, this differential equation can be solved for a given realization of the parallel collisions $\eta_\parallel$. By performing the $\mathbf{k}$-integration in Eq. (78), we showed in Ref. 27 that this equation has a diagonal solution,

$$\mathscr{M}^{(0)}_{ij}=M_{(0)}\delta_{ij}, \tag{79}$$

which obeys

$$\frac{d^2}{dz^2} M_{(0)}(z) = 2\beta^2 \exp\left(-\frac{1}{2} \frac{z^2}{\lambda_\parallel^2} \right) \frac{\lambda_\perp^4}{[\lambda_\perp^2 + M_{(0)}(z)]^2}. \quad (80)$$

The solution of this equation can be obtained by usual perturbation techniques as a series expansion in the Kubo number α. Up to the leading order in α, this yields

$$M_{(0)}(z) \approx \beta^2 \lambda_\parallel \left\{ \sqrt{2\pi} |z - z_0| \mathrm{erf}\left(\frac{|z - z_0|}{\sqrt{2}\lambda_\parallel} \right) + 2\lambda_\parallel (\mathrm{e}^{-(1/2)\lambda_\parallel^{-2}(z-z_0)^2} - 1) \right\}. \quad (81)$$

We now add the particle motion along the magnetic line whose statistical evolution is given by Eq. (81); thus, we take $z = z(t)$ in this equation. Using the results of Sec. II, the average of Eq. (81) over $\eta_\parallel$ is obtained from

$$\langle M_{(0)}[z(t)] \rangle_\parallel = \int_{-\infty}^{\infty} dz M_{(0)}(z) \mathscr{P}(z;t). \quad (82)$$

Using Eqs. (29) and (81), this integral can be performed exactly, yielding

$$\langle M_{(0)}[z(t)] \rangle_\parallel = 2\beta^2 \lambda_\parallel [\lambda_\parallel^2 + 2\chi_\parallel \nu^{-1} \psi(\nu t)]^{1/2} \sim 2\beta^2 \lambda_\parallel \sqrt{2\chi_\parallel t}. \quad (83)$$

Up to the leading order in α, the function $\langle M_{(0)} \rangle_\parallel$ is identical to the particle mean square displacement obtained within the model of Sec. II and whose asymptotic value is given by Eq. (41).

Let us now come back to the realistic situation where the particle experiences perpendicular collisions, i.e. when $\boldsymbol{\eta}_\perp \neq 0$. It is natural to describe the anomalous effects induced by nonzero perpendicular collisions as a correction added to $M_{(0)}$ and we introduce the decomposition

$$\mathscr{M}_{ij}(t) = M_{(0)}[z(t)] \delta_{ij} + \tilde{\mathscr{M}}_{ij}(t). \quad (84)$$

The evolution of the second term on the right hand side of Eq. (84) is considered in the next section. It will be shown to account for diffusive anomalous effects, in contrast to the subdiffusive ones described by $M_{(0)}$. At this point, it is worth summarizing our results as follows. The particle mean square displacement in the perpendicular plane may be seen as the superposition of three factors:

• A motion with the magnetic lines, which is thus modeled by the first term on the right hand side of Eq. (84).

• A motion due to the perpendicular collisions alone, which decorrelate the particle from the magnetic lines. This factor is represented by the first term on the right hand side of Eq. (75).

• A motion induced by the decorrelation of the particle from the lines due to anomalous effects alone, i.e. induced by the magnetic field nonlinearity. This third factor is modeled by the tensor $\tilde{\mathscr{M}}$ defined by Eq. (84).

C. Evaluation of the diffusive anomalous term

Here we introduce some further approximations which allow us to average Eqs. (73) and (84) over the collisions and to derive an explicit form for the anomalous term $\langle \tilde{\mathscr{M}} \rangle_{\parallel,\perp}$. We first assume that Eqs. (73) and (84) yield a growth for the averaged tensor $\langle \tilde{\mathscr{M}} \rangle_{\parallel,\perp}$ which is dominant when compared to the spreading of the random variable $\tilde{\mathscr{M}}$ around its mean value $\langle \tilde{\mathscr{M}} \rangle_{\parallel,\perp}$, i.e.

$$\langle \tilde{\mathscr{M}}_{ij}(t) \rangle_{\parallel,\perp}^2 \gg \langle (\tilde{\mathscr{M}}_{ij}(t) - \langle \tilde{\mathscr{M}}_{ij}(t) \rangle_{\parallel,\perp})^2 \rangle_{\parallel,\perp}, \quad \forall t. \quad (85)$$

The relation (85) amounts to assume the independence between the different parts of the particle motion introduced in Sec. IV B. It implies indeed that we can substitute $\langle \tilde{\mathscr{M}} \rangle_{\parallel,\perp}$ for $\tilde{\mathscr{M}}$ in Eqs. (73) and (84) averaged on $\boldsymbol{\eta}$. Noting also that, when the assumption (85) is used in Eq. (73), $\langle \tilde{\mathscr{M}} \rangle_{\parallel,\perp}$ reduces to a diagonal tensor, i.e.

$$\langle \tilde{\mathscr{M}}_{ij}(t) \rangle_{\parallel,\perp} = \langle \tilde{M}(t) \rangle_{\parallel,\perp} \delta_{ij}, \quad (86)$$

then Eqs. (73) and (84) yield

$$\frac{d^2}{dt^2} [\langle \tilde{M}(t) \rangle_{\parallel,\perp} + \langle M_{(0)}[z(t)] \rangle_\parallel] = \int d\mathbf{k}\, k_\perp^2 \mathscr{A}_{\mathbf{k}} \mathrm{e}^{(-1/2)k_\perp^2 \langle \tilde{M}(t) \rangle_{\parallel,\perp}} \left\langle \eta_\parallel(t) \eta_\parallel(0) \times \exp\left(i\mathbf{k} \cdot \int_0^t \boldsymbol{\eta}(t_1) dt_1 - \frac{1}{2} k_\perp^2 M_{(0)}[z(t)] \right) \right\rangle_{\parallel,\perp}, \quad (87)$$

with initial conditions $\langle \tilde{M}(0) \rangle_{\parallel,\perp} = 0$ and $(d/dt) \langle \tilde{M} \rangle_{\parallel,\perp|t=0} = 0$. In Eq. (87), $M_{(0)}$ is obtained from Eq. (80). We now take advantage of the discussion of Sec. IV B. The second time-derivative of $\langle M_{(0)} \rangle_\parallel$, appearing on the left hand side of Eq. (87), can be identified with the right hand side of Eq. (87) in which we let $\langle \tilde{M} \rangle_{\parallel,\perp} = 0$ and $\boldsymbol{\eta}_\perp = 0$. This follows directly from Eq. (78) written in function of t and averaged over $\eta_\parallel$. Hence, Eq. (87) can be written as

$$\frac{d^2}{dt^2} \langle \tilde{M}(t) \rangle_{\parallel,\perp} = \int d\mathbf{k}\, k_\perp^2 \mathscr{A}_{\mathbf{k}} \left\{ \mathrm{e}^{-(1/2)k_\perp^2 \langle \tilde{M}(t) \rangle_{\parallel,\perp}} \times \left\langle \exp\left(i\mathbf{k}_\perp \cdot \int_0^t \boldsymbol{\eta}_\perp(t_1) dt_1 \right) \right\rangle_\perp - 1 \right\} \times \left\langle \eta_\parallel(t) \eta_\parallel(0) \exp\left(ik_z \int_0^t \eta_\parallel(t_1) dt_1 - \frac{1}{2} k_\perp^2 M_{(0)}[z(t)] \right) \right\rangle_\parallel. \quad (88)$$

Equation (88) has the following interesting property. For zero perpendicular collisions, this equation admits the solution

$$\langle \tilde{M}(t) \rangle_{\parallel,\perp} = 0, \quad \forall t, \quad \text{if } \boldsymbol{\eta}_\perp = 0. \quad (89)$$

This implies the impossibility for the particle to leave its magnetic line when $\boldsymbol{\eta}_\perp = 0$. Such a property is the direct result of Eq. (84) and shows the importance of this decomposition. We also stress that the property (89) holds independently of the approximations that could be made on the solution $M_{(0)}$ used in Eq. (88) and on the averages over $\boldsymbol{\eta}$ of

Eq. (88). This will be useful in the forthcoming calculations. Let us now consider the two remaining averages of Eq. (88) over $\boldsymbol{\eta}$. We need first

$$\left\langle \exp\left(i\mathbf{k}_\perp \cdot \int_0^t \boldsymbol{\eta}_\perp(t_1)dt_1 \right) \right\rangle_\perp = \exp(-k_\perp^2 \chi_\perp \nu^{-1} \psi(\nu t)). \tag{90}$$

Next, we consider the term averaged on $\eta_\parallel$ in Eq. (88). It is approximated by using the asymptotic form $M_{(0)}(z) \sim 2D_m|z-z_0|$ of the solution (81), which combined with Eq. (77) yields

$$\left\langle \eta_\parallel(t)\eta_\parallel(0)\exp\left(ik_z\int_0^t \eta_\parallel(t_1)dt_1 - k_\perp^2 D_m \left| \int_0^t \eta_\parallel(t_1)dt_1 \right| \right) \right\rangle_\parallel$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dq[\chi_\parallel \nu R(\nu t) - q^2\chi_\parallel^2\varphi^2(\nu t)]\mathrm{e}^{-q^2\chi_\parallel \nu^{-1}\psi(\nu t)}$$

$$\times \frac{k_\perp^2 D_m}{k_\perp^4 D_m^2 + (q+k_z)^2}.$$

The magnetic line diffusion coefficient D_m is a small quantity compared to $\lambda_\perp^2/\lambda_\parallel$. This allows us to approximate the last factor in the integral by the Dirac distribution $\pi\delta(q+k_z)$, which gives

$$\left\langle \eta_\parallel(t)\eta_\parallel(0)\exp\left(ik_z\int_0^t \eta_\parallel(t_1)dt_1 - k_\perp^2 D_m \left| \int_0^t \eta_\parallel(t_1)dt_1 \right| \right) \right\rangle_\parallel$$

$$\simeq [\chi_\parallel \nu R(\nu t) - k_z^2\chi_\parallel^2\varphi^2(\nu t)]\exp[-k_z^2\chi_\parallel \nu^{-1}\psi(\nu t)]. \tag{91}$$

Finally, we introduce the dimensionless quantities,

$$\mu(\nu t) = 2\langle \tilde{M}(t)\rangle_{\parallel,\perp}\lambda_\perp^{-2}, \quad \tau = \nu t, \tag{92}$$

and by combining Eqs. (84)–(91), Eq. (88) becomes the following second order differential equation:

$$\frac{d^2\mu(\tau)}{d\tau^2} = 2\alpha^2\bar{\chi}_\parallel \mathscr{F}(\tau)\{\mathscr{G}[\mu(\tau);\tau]-1\}, \tag{93}$$

with initial condition $\mu(0)=0$ and $(d/d\tau)\mu_{|\tau=0} = 0$. Here, $\bar{\chi}_\parallel = 2\chi_\parallel/\lambda_\parallel^2\nu$. In Eq. (93), the functions $\mathscr{F}$ and $\mathscr{G}$ are obtained by inserting the functions (90) and (91) into Eq. (88) and by performing the integration over $\mathbf{k}$:

$$\mathscr{F}(\tau) = [1+\bar{\chi}_\parallel\psi(\tau)]^{-1/2}$$
$$\times\{R(\tau) - \tfrac{1}{2}\bar{\chi}_\parallel\varphi^2(\tau)[1+\bar{\chi}_\parallel\psi(\tau)]^{-1}\}, \tag{94}$$

and

$$\mathscr{G}[\mu(\tau);\tau] = \left[1+\bar{\chi}_\perp\psi(\tau)+\frac{1}{2}\mu(\tau)\right]^{-2}, \tag{95}$$

where $\bar{\chi}_\perp = 2\chi_\perp/\lambda_\perp^2\nu$. We now relate our results to transport. Using Eqs. (76), (83) and (84), we define a running diffusion coefficient by

$$\frac{1}{2}\frac{d}{d\tau}\langle \bar{x}^2(\tau)\rangle_{\mathbf{b},\parallel,\perp}^{\mathrm{AN}} = \frac{1}{2}\frac{d\mu(\tau)}{d\tau} + 2\alpha^2\frac{d}{d\tau}[1+\bar{\chi}_\parallel\psi(\tau)]^{1/2}$$
$$\sim \frac{1}{2}\frac{d\mu(\tau)}{d\tau} + \alpha^2\bar{\chi}_\parallel^{1/2}\tau^{-1/2}, \quad \tau \gg 1. \tag{96}$$

where $\bar{x} = \sqrt{2}x/\lambda_\perp$. The second term on the right hand side of Eq. (96) is the dimensionless time derivative of the function $\langle M_{(0)}\rangle_\parallel$ isolated in Sec. IV B. Hence, the two terms composing the anomalous part of the mean square displacement have completely different asymptotic behaviors: If $\bar{\chi}_\perp = 0$, only the second term on the right hand side of Eq. (96) remains, yielding subdiffusion as in Sec. II. However, when $\bar{\chi}_\perp \neq 0$, the first term on the right hand side of Eq. (96) becomes non-zero. From Eq. (93) and the properties $\mathscr{F}(\tau)<0$ for $\tau \gg 1$ and $\mathscr{G}[\mu(\tau);\tau] \leqslant 1$, this first term is expected to correspond to diffusive anomalous transport. The anomalous diffusion coefficient can thus be defined by

$$\bar{\chi}_\perp^{\mathrm{AN}} = \frac{1}{2}\lim_{\tau\to\infty}\frac{d}{d\tau}\langle \bar{x}^2(\tau)\rangle_{\mathbf{b},\parallel,\perp}^{\mathrm{AN}} = \frac{1}{2}\lim_{\tau\to\infty}\frac{d\mu(\tau)}{d\tau}. \tag{97}$$

Explicit values for this coefficient are obtained in Sec. V from the study of the differential equation (93).

D. The probability distribution function and the density profile

Finally, let us consider the probability density function $\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b},\parallel,\perp}$ defined by the previous scheme. Equation (72), together with the decomposition (84) and the assumption (85), imply that $\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b},\parallel,\perp}$ can be obtained from

$$\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b},\parallel,\perp} = (2\pi)^{-3}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_0 - (1/2)k_\perp^2\langle \tilde{M}(t)\rangle_{\parallel,\perp}}$$
$$\times\left\langle \exp\left(i\mathbf{k}\cdot\int_0^t \boldsymbol{\eta}(t_1)dt_1 - \frac{1}{2}k_\perp^2 M_{(0)}[z(t)] \right) \right\rangle_{\parallel,\perp}. \tag{98}$$

We are interested in the density profile defined by

$$N_{k_x,k_y}(t) = 2\pi\langle f_{\mathbf{k}}(t)\rangle_{\mathbf{b},\parallel,\perp|k_z=0} \equiv n_{k_x}(t)n_{k_y}(t). \tag{99}$$

The functions $n_k(t)$ are evaluated by using the asymptotic form $M_{(0)}(z) \sim 2D_m|z-z_0|$ of the solution (81) in Eqs. (98) and (99). Except for the Gaussian factor $\exp(-k^2\langle \tilde{M}(t)\rangle_{\parallel,\perp}/2)$, the functions $n_k(t)$ are then identical to the expression (62). Hence, we obtain (for $x_0 = 0$)

$$n_k(t) = (2\pi)^{-1}\exp\left(-k^2\chi_\perp t - \frac{1}{2}k^2\langle \tilde{M}(t)\rangle_{\parallel,\perp} + \frac{1}{2}k^4D_m^2\langle \delta z^2(t)\rangle_\parallel \right)$$
$$\times \mathrm{erfc}\left(\frac{1}{\sqrt{2}}k^2D_m\langle \delta z^2(t)\rangle_\parallel^{1/2} \right). \tag{100}$$

All the moments of Eq. (100) can thus be obtained from Eqs. (63) and (64) with $\chi_\perp t + \langle \tilde{M}(t)\rangle_{\parallel,\perp}/2$ substituted for $\chi_\perp t$. Equation (100) shows that the statistics of the particle dynamics is non-Gaussian when $D_m\langle \delta z^2(t)\rangle_{\|}^{1/2} \gtrsim \max\{2\chi_\perp t, \langle \tilde{M}(t)\rangle_{\|,\perp}\}$. This condition corresponds to the initial stage of the motion and the non-Gaussian character is accounted for by the decomposition (84), which is essential. In particular, for $\chi_\perp=0$, the particle does not leave its magnetic line and Eq. (100) remains identical to the non-Gaussian profile (36) for all times t. Hence, a situation with nonzero perpendicular collisions is necessary for the emergence of Gaussian statistics. More precisely, for $\chi_\perp \neq 0$, we obtain, using the results of Sec. III, that the physical space form of Eq. (100) is asymptotically the Gaussian packet,

$$n(x;t)=(4\pi[\chi_\perp+\chi_\perp^{\mathrm{AN}}]t)^{-1/2} \times \exp\left(-\frac{(x-x_0)^2}{4[\chi_\perp+\chi_\perp^{\mathrm{AN}}]t}\right), \quad t\gg 1, \tag{101}$$

where $\chi_\perp^{\mathrm{AN}}=\bar{\chi}_\perp^{\mathrm{AN}}\nu\lambda_\perp^2/2$ and we have used $\langle \tilde{M}(t)\rangle_{\|,\perp}\sim 2\chi_\perp^{\mathrm{AN}}t$ for $t\gg 1$. We thus see that, due to the perpendicular collisions, diffusion finally dominates the particle motion. More information about the time necessary to get the density profile (101) will be obtained in the next sections.

V. THE VARIOUS REGIMES

In order to discuss the different regimes that are described by our model, we must obtain an explicit solution for the anomalous term given by the differential equation (93). The parameters in this equation belong to the following domains: $\bar{\chi}_\| \in [0,\infty[$, $\alpha \in [0,1[$ and $\bar{\chi}_\perp \in [0,1[$. The constraints on α and $\bar{\chi}_\perp$ ensure the validity of the DIA and justify the guiding center approximation. The discussion is simplified by considering the following simple choice for the function R appearing in $\mathscr{F}$:

$$R(\tau)=\begin{cases}1, & \text{if } 0\leqslant\tau<1,\\ 0, & \text{if } \tau\geqslant 1,\end{cases} \tag{102}$$

so that Eq. (94) reduces to

$$\mathscr{F}(\tau)=\begin{cases}\left(1+\frac{1}{2}\bar{\chi}_\|\tau^2\right)^{-3/2}, & \text{if } 0\leqslant\tau<1,\\ -\frac{1}{2}\bar{\chi}_\|\left(1-\frac{1}{2}\bar{\chi}_\|+\bar{\chi}_\|\tau\right)^{-3/2}, & \text{if } \tau\geqslant 1.\end{cases} \tag{103}$$

A. The classical situation

We first consider cases where

$$1+\bar{\chi}_\perp\psi(\tau)\gg\frac{1}{2}\mu(\tau), \quad \forall\tau\in[0,\infty[. \tag{104}$$

In particular, this inequality reduces to $\bar{\chi}_\perp\tau\gg\bar{\chi}_\perp^{\mathrm{AN}}\tau$ for $\tau\gg 1$. This corresponds to situations where the anomalous transport does not exceed the collisional one, which are rather academic cases. The condition (104) allows us to approximate $\mathscr{G}$ by

$$\mathscr{G}\simeq(1+\bar{\chi}_\perp\psi(\tau))^{-2}. \tag{105}$$

Combining Eqs. (93), (97) and (103) and considering only the asymptotic limit $\tau\to\infty$, we get, for the anomalous diffusion coefficient,

$$\begin{aligned}\bar{\chi}_\perp^{\mathrm{AN}}=\alpha^2\bar{\chi}_\|\bigg\{&\int_0^1 d\tau\left(1+\frac{1}{2}\bar{\chi}_\|\tau^2\right)^{-3/2}\left(1+\frac{1}{2}\bar{\chi}_\perp\tau^2\right)^{-2}\\ &-\frac{1}{2}\bar{\chi}_\|\int_1^\infty d\tau\left(1-\frac{1}{2}\bar{\chi}_\|+\bar{\chi}_\|\tau\right)^{-3/2}\\ &\times\left(1-\frac{1}{2}\bar{\chi}_\perp+\bar{\chi}_\perp\tau\right)^{-2}\bigg\}.\end{aligned} \tag{106}$$

These two integrals can be performed explicitly, but the discussion is simplified by considering two limiting regimes.

1. The fluid regime

First, we consider the limit $\bar{\chi}_\|\ll\bar{\chi}_\perp\leqslant 1$ in which collisional effects are very important. Let us stress that this inequality between the nondimensional collisional diffusion coefficients is not incompatible with the constraint $\chi_\|>\chi_\perp$. However, this limit requires very small values of the ratio $\lambda_\perp/\lambda_\|$. Indeed, from their definitions,

$$\frac{\bar{\chi}_\|}{\bar{\chi}_\perp}=\frac{\lambda_\perp^2}{\lambda_\|^2}\frac{\chi_\|}{\chi_\perp}. \tag{107}$$

Equation (106) then reduces to

$$\bar{\chi}_\perp^{\mathrm{AN}}=\alpha^2\bar{\chi}_\|\{1+O(\bar{\chi}_\perp)\}, \quad \bar{\chi}_\|\ll\bar{\chi}_\perp\leqslant 1. \tag{108}$$

Up to the leading order, this yields the dimensional diffusion coefficient,

$$\chi_\perp^{\mathrm{AN}}\approx\beta^2\chi_\|. \tag{109}$$

This corresponds to the so-called fluid regime introduced in Ref. 13. The diffusion coefficient (109) is independent of $\chi_\perp$. We stress that, due to the inequality $\bar{\chi}_\|\ll\bar{\chi}_\perp$, the limit $\bar{\chi}_\perp\to 0$ cannot be taken in Eqs. (108) and (108). For $\alpha \in [0,1[$, we also have $\bar{\chi}_\perp^{\mathrm{AN}}<\bar{\chi}_\|\ll\bar{\chi}_\perp$, which justifies the assumption (104).

2. The Kadomtsev–Pogutse regime

Next we take $\bar{\chi}_\perp\leqslant\bar{\chi}_\|$. Integration in Eq. (106) then yields

$$\bar{\chi}_\perp^{\mathrm{AN}}=\frac{3}{4}\pi\alpha^2\bar{\chi}_\|^{1/2}\{\bar{\chi}_\perp^{1/2}+O(\bar{\chi}_\perp)\}, \quad \bar{\chi}_\perp\leqslant\bar{\chi}_\|, \tag{110}$$

or dimensionally

$$\chi_\perp^{\mathrm{AN}}\approx\frac{3}{4}\pi\beta^2\lambda_\|\lambda_\perp^{-1}\sqrt{\chi_\|\chi_\perp}. \tag{111}$$

This is the Kadomtsev–Pogutse diffusion coefficient.[9] Due to the inequality $\bar{\chi}_\perp^{\mathrm{AN}}\ll\bar{\chi}_\perp$, needed for justifying Eq. (104), Eq. (111) holds if

$$\alpha^4\bar{\chi}_\|\ll\bar{\chi}_\perp. \tag{112}$$

Hence, Eq. (111) is no longer valid when $\bar{\chi}_\perp\to 0$. This limit will be considered in the next section.

B. The anomalous situation

We now consider situations where the scaling (104) does not hold for all τ. For analyzing Eq. (93), we first consider the range $\tau \in [0,1[$. Using Eqs. (103) and (105) and the condition $\bar{\chi}_\perp \leqslant 1$, Eq. (93) reduces to

$$\frac{d^2\mu(\tau)}{d\tau^2} \simeq -2\alpha^2 \bar{\chi}_\parallel \left(1+\frac{1}{2}\bar{\chi}_\parallel \tau^2\right)^{-3/2} \bar{\chi}_\perp \tau^2, \quad \tau \in [0,1[. \tag{113}$$

Both μ and $(d/d\tau)\mu$ are decreasing functions on the interval $[0,1]$. We stress that μ is only a part of the particle mean square displacement [see Eq. (96)]. Thus, the negative value of μ predicted by Eq. (113) is compatible with the constraint $\langle \bar{x}^2(\tau)\rangle_{\mathbf{b},\parallel,\perp} \geqslant 0$. This may be demonstrated from Eq. (96). For small values of $\bar{\chi}_\perp$, this first stage of the motion is, moreover, not very relevant and Eq. (113) yields $\mu(1) \approx 0$ and $(d/d\tau)\mu|_{\tau=1} \approx 0$. We take these values as new initial conditions and study Eq. (93) on $\tau \in [1,\infty[$. Using Eqs. (95), (102) and (103), this equation is explicitly

$$\begin{aligned}\frac{d^2\mu(\tau)}{d\tau^2} &= \alpha^2 \bar{\chi}_\parallel^2 \left(1-\frac{1}{2}\bar{\chi}_\parallel + \bar{\chi}_\parallel \tau\right)^{-3/2} \\ &\quad \times \left(1-\left[1+\bar{\chi}_\perp\left(\tau-\frac{1}{2}\right)+\frac{1}{2}\mu(\tau)\right]^{-2}\right) \\ &\simeq \theta\left(\tau-\frac{1}{2}\right)^{-3/2}\left(1-\left[1+\bar{\chi}_\perp\left(\tau-\frac{1}{2}\right)\right.\right. \\ &\quad \left.\left. +\frac{1}{2}\mu(\tau)\right]^{-2}\right), \quad \tau \in [1,\infty[. \end{aligned} \tag{114}$$

Here we introduced the parameter $\theta = \alpha^2 \bar{\chi}_\parallel^{1/2}$; the second equality holds if $\bar{\chi}_\parallel \gtrsim 1$, which we assume. A numerical integration of Eq. (114) is presented in Figs. 2 and 3. Analytical treatments of Eq. (114) can only be performed if the second—nonlinear—term entering the right hand side of this equation is approximated. This implies first defining two ranges of values of τ within $[1,\infty[$ where approximate solutions of Eq. (114) can be found; we denote these solutions by $\mu_<$ and $\mu_>$. Next, we match these solutions at some time τ^* where both are valid.

• The first range is defined by the inequality

$$\bar{\chi}_\perp\left(\tau-\frac{1}{2}\right)+\frac{1}{2}\mu_<(\tau) \ll 1. \tag{115}$$

From Eq. (114), $\mu(\tau)$ is a growing function of time on $[1,\infty[$; hence, Eq. (115) holds for τ sufficiently close to 1. Equation (115) allows us to approximate Eq. (114) by

$$\frac{d^2\mu_<(\tau)}{d\tau^2} = \theta\left(\tau-\frac{1}{2}\right)^{-3/2}\left(2\bar{\chi}_\perp\left(\tau-\frac{1}{2}\right)+\mu_<(\tau)\right). \tag{116}$$

The solution of Eq. (116) is a function of $\bar{\tau} = \tau - 1/2$. We obtain

$$\begin{aligned}\mu_<(\bar{\tau}) &= -2\bar{\chi}_\perp \bar{\tau} + \bar{\chi}_\perp \sqrt{\bar{\tau}}\,\{c_1 I_2[4\sqrt{\theta}\bar{\tau}^{1/4}] \\ &\quad + c_2 K_2[4\sqrt{\theta}\bar{\tau}^{1/4}]\},\end{aligned} \tag{117}$$

where $I_2(x)$ and $K_2(x)$ are modified Bessel functions and

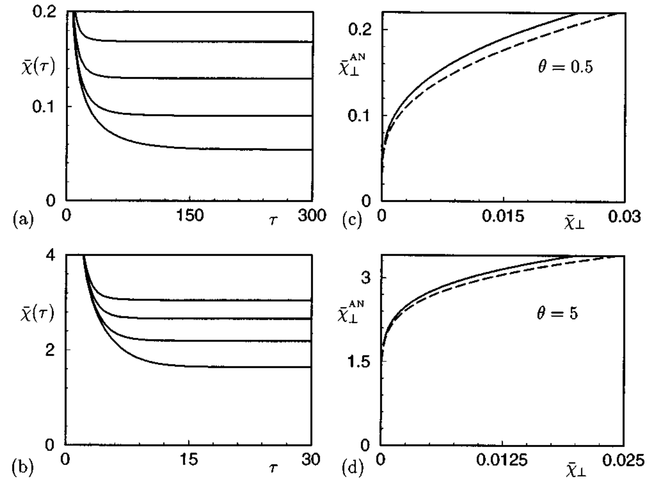

FIG. 2. Numerical solution of Eq. (114). The graphs (a) and (b) show the evolution with τ of $\bar{\chi}(\tau) \equiv (1/2)d\mu(\tau)/d\tau + \theta(\tau-(1/2))^{-1/2}$ for various values of $\bar{\chi}_\perp$. The curves from top to bottom correspond to $\bar{\chi}_\perp = 10^{-2}, 4\,10^{-3}, 10^{-3}, 10^{-4}$ [Graph (a) $\theta=0.5$; Graph (b) $\theta=5$]. For large times, $(1/2)d\mu(\tau)/d\tau + \theta(\tau-(1/2))^{-1/2}$ tends to $\bar{\chi}_\perp^{\mathrm{AN}}$, as expected from Eq. (123). In the graphs (c) and (d), the limiting value $\bar{\chi}_\perp^{\mathrm{AN}}$ is plotted against $\bar{\chi}_\perp$ (full line). The dashed line is the analytical approximation of $\bar{\chi}_\perp^{\mathrm{AN}}$ provided by Eqs. (125) and (128).

$$c_1 = \sqrt{2}[aK_1(a)+4K_2(a)], \tag{118a}$$

$$c_2 = \sqrt{2}[aI_1(a)-4I_2(a)], \tag{118b}$$

in which $a = 2^{7/4}\sqrt{\theta}$. The asymptotic expansions of the modified Bessel functions are

$$K_2(x) \sim \sqrt{\frac{\pi}{2x}}\,\mathrm{e}^{-x}, \quad I_2(x) \sim \frac{1}{\sqrt{2\pi x}}\,\mathrm{e}^{x}, \quad x \gg 1. \tag{119}$$

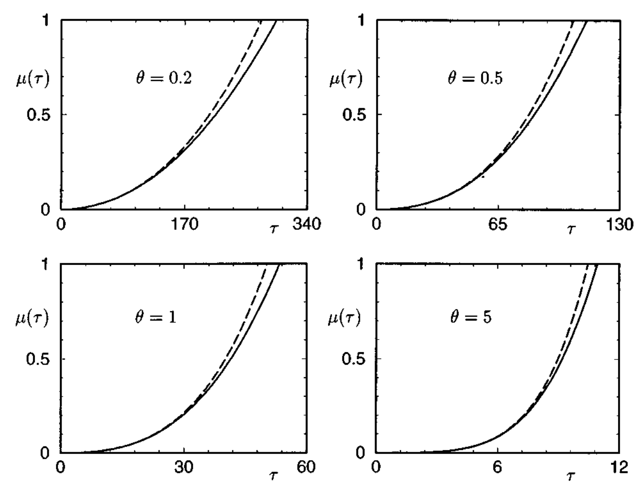

FIG. 3. Equation (117) (dashed line) is compared to the numerical solution of Eq. (93) (full line) for different values of θ. Equation (117) thus provides a rather good approximation of the clump lifetime τ^* —defined as the time at which the function $\mu(\tau)+\bar{\chi}_\perp(2\tau-1) \approx \mu(\tau)$ reaches the value 1 [see Eq. (125)]. Here, $\bar{\chi}_\perp = 10^{-4}$ and $\bar{\chi}_\parallel = 10$.

If both inequalities (115) and $\theta^2\bar{\tau}\gg 1$ hold, which implies considering sufficiently small values of $\bar{\chi}_\perp$, the solution (117) is thus asymptotically

$$\mu_<(\bar{\tau})+2\bar{\chi}_\perp\bar{\tau}\sim\frac{1}{2\sqrt{2\pi}}\bar{\chi}_\perp c_1\theta^{-1/4}\bar{\tau}^{3/8}\exp(4\sqrt{\theta}\bar{\tau}^{1/4}),$$

$$\theta^2\bar{\tau}\gg 1. \tag{120}$$

Of course, this exponential growth cannot go on forever since the inequality (115) must be satisfied.

• We next consider the range defined by

$$\bar{\chi}_\perp\left(\tau-\frac{1}{2}\right)+\frac{1}{2}\mu_>(\tau)>0. \tag{121}$$

This inequality allows to approximate Eq. (114) by (using $\bar{\tau}=\tau-1/2$)

$$\frac{d^2\mu_>(\bar{\tau})}{d\bar{\tau}^2}\simeq\theta\bar{\tau}^{-3/2}, \tag{122}$$

or equivalently

$$\frac{1}{2}\frac{d\mu_>(\bar{\tau})}{d\bar{\tau}}=\bar{\chi}_\perp^{\mathrm{AN}}-\theta\bar{\tau}^{-1/2}. \tag{123}$$

Here the integration constant has been identified with $\bar{\chi}_\perp^{\mathrm{AN}}$ using the relation (97). Of course, at this point, the explicit value of $\bar{\chi}_\perp^{\mathrm{AN}}$ remains unspecified.

In order to obtain an approximation of the solution μ of Eq. (114) valid over the entire range $\tau\in[1,\infty[$, we now match $\mu_<$ to $\mu_>$. This operation must be performed at some specific time $\tau=\tau^*$ at which both solutions $\mu_<$ and $\mu_>$ are relevant approximations of μ. In other words, τ^* must belong to the finite intersection between the ranges (115) and (121). Except for this constraint τ^* is arbitrary, and we choose the time at which the second order time-derivatives of $\mu_<$ and $\mu_>$ coincide. Using Eqs. (116) and (122), this leads to the following equation for $\bar{\tau}^*=\tau^*-1/2$:

$$2\bar{\chi}_\perp\bar{\tau}^*+\mu_<(\bar{\tau}^*)=1, \tag{124}$$

or equivalently, using Eq. (120),

$$1=\frac{1}{2\sqrt{2\pi}}\bar{\chi}_\perp c_1\theta^{-1/4}\bar{\tau}^{*3/8}\exp(4\sqrt{\theta}\bar{\tau}^{*1/4}). \tag{125}$$

Here it should be noted that the transition from the exponential growth (120) to the linear growth (123) is similar to a clumping process.[27,41–45] The time $\bar{\tau}^*$ can thus be identified with the clump lifetime. We also stress that the exponentiation described by Eq. (120) only appears when perpendicular collisions allow the particle to leave the magnetic line, i.e. if $\bar{\chi}_\perp\neq 0$. This is confirmed by the transcendental equation (125), which implies that the clump lifetime $\bar{\tau}^*$ goes to infinity as $\bar{\chi}_\perp$ goes to zero, i.e.

$$\lim_{\bar{\chi}_\perp\to 0}\bar{\tau}^*=\infty. \tag{126}$$

Finally, we may match $\mu_<$ to $\mu_>$ and $(d/d\tau)\mu_<$ to $(d/d\tau)\mu_>$ at $\tau=\tau^*$. For evaluating the anomalous diffusion coefficient only the second of these two matching conditions is necessary and, using Eqs. (120) and (123), it yields the following equation for $\bar{\chi}_\perp^{\mathrm{AN}}$:

$$\bar{\chi}_\perp^{\mathrm{AN}}-\theta\bar{\tau}^{*-1/2}=\frac{1}{4\sqrt{2\pi}}\bar{\chi}_\perp c_1\theta^{1/4}\bar{\tau}^{*-3/8}\exp(4\sqrt{\theta}\bar{\tau}^{*1/4})$$

$$=\frac{1}{2}\sqrt{\theta}\bar{\tau}^{*-3/4}, \tag{127}$$

where the second equality is obtained using Eq. (125). Due to Eq. (120), the clump lifetime is such that $\theta^2\bar{\tau}^*\gg 1$. The right hand side of Eq. (127) is thus a small correction and, to leading order, this equation reduces to

$$\bar{\chi}_\perp^{\mathrm{AN}}=\theta\bar{\tau}^{*-1/2}. \tag{128}$$

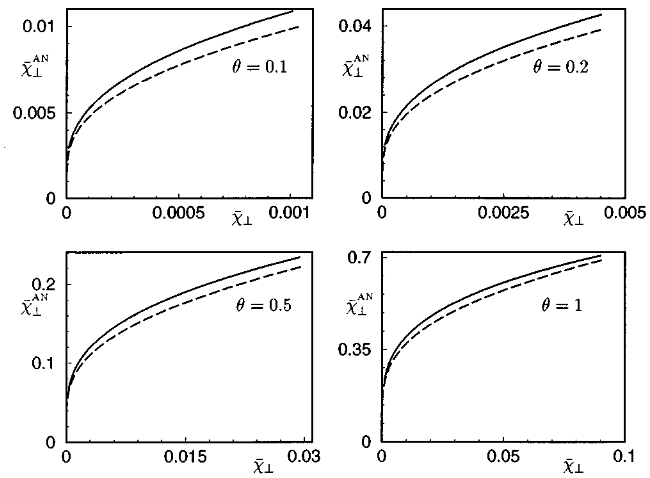

FIG. 4. The anomalous diffusion coefficient $\bar{\chi}_\perp^{\mathrm{AN}}$[Eq. (128)] is plotted against $\bar{\chi}_\perp$[Eq. (125)] as function of the parameter $\bar{\tau}^*$ (dashed lines). The values of θ correspond to strongly collisional situations. Here, the clump lifetime varies from θ^{-2} (corresponding to the upper right of the curves), to infinity (corresponding to the lower left of the curves). The full lines represent the $\bar{\chi}_\perp^{\mathrm{AN}}$ obtained from a numerical integration of Eq. (114).

In spite of our approximations, the expression (128) gives a rather accurate approximation of $\bar{\chi}_\perp^{\mathrm{AN}}$, as can be checked from a numerical evaluation of the anomalous diffusion coefficient using the complete equation (114) (see Figs. 2, 4 and 5). Equation (125), solved in $\bar{\chi}_\perp$, and Eq. (128) link $\bar{\chi}_\perp^{\mathrm{AN}}$ to $\bar{\chi}_\perp$ in function of the parameter $\bar{\tau}^*$. In particular, this allows us to plot $\bar{\chi}_\perp^{\mathrm{AN}}$ as a function of $\bar{\chi}_\perp$ for given values of θ (see Figs. 4 and 5). On the other hand, by eliminating $\bar{\tau}^*$ from Eqs. (125) and (128) we obtain the following equation:

$$1=\frac{1}{2\sqrt{2\pi}}\bar{\chi}_\perp c_1\sqrt{\theta}(\bar{\chi}_\perp^{\mathrm{AN}})^{-3/4}\exp[4\theta(\bar{\chi}_\perp^{\mathrm{AN}})^{-1/2}]. \tag{129}$$

We study this equation in two subranges according to the value of c_1.

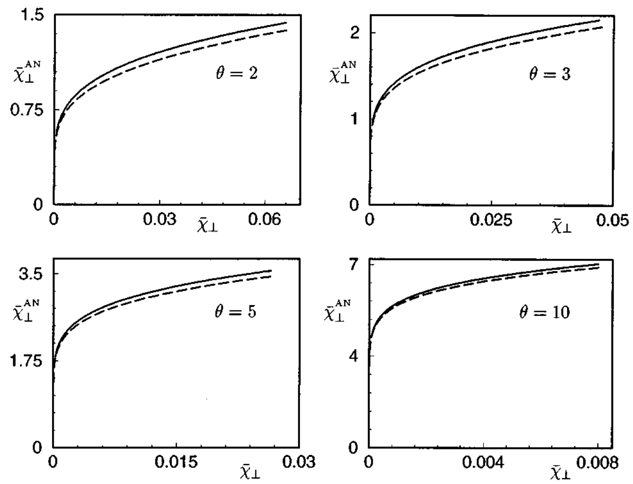

FIG. 5. The anomalous diffusion coefficient $\overline{\chi}_\perp^{AN}$[Eq. (128)] is plotted against $\overline{\chi}_\perp$[Eq. (125)] as function of the parameter $\overline{\tau}^*$ (dashed lines). The values of θ correspond to weakly collisional situations. Here, the clump lifetime varies from 1 (corresponding to the upper right of the curves), to infinity (corresponding to the lower left of the curves). The full lines represent the $\overline{\chi}_\perp^{AN}$ obtained from a numerical integration of Eq. (114).

1. The Rechester–Rosenbluth regime

First, we consider the case $\theta<1$, i.e. $\overline{\chi}_\parallel<\alpha^{-4}$, which is a strongly collisional situation. In this case, we have approximately $c_1\simeq 1/\theta$ in Eq. (129), and this equation reduces to

$$1=\frac{1}{2\sqrt{2\pi}}\,\overline{\chi}_\perp\,\theta^{-1/2}(\overline{\chi}_\perp^{AN})^{-3/4}\exp[4\,\theta(\overline{\chi}_\perp^{AN})^{-1/2}], \tag{130}$$

or equivalently,

$$\overline{\chi}_\perp^{AN}=16\theta^2\ln^{-2}[2\sqrt{2\pi\theta}(\overline{\chi}_\perp^{AN})^{3/4}\overline{\chi}_\perp^{-1}]. \tag{131}$$

We use the first iteration around $\overline{\chi}_\perp^{AN}=16\theta^2$ of Eq. (131) as an approximation for the solution of this equation:

$$\overline{\chi}_\perp^{AN}=16\theta^2\ln^{-2}[16\sqrt{2\pi}\theta^2\overline{\chi}_\perp^{-1}], \tag{132}$$

or in dimensional form,

$$\chi_\perp^{AN}=\frac{2\chi_\parallel\lambda_\perp^2}{\pi L_K^2}\ln^{-2}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\lambda_\perp}{L_K}\right)^2\frac{\chi_\parallel}{\chi_\perp}\right], \tag{133}$$

where L_K is the Kolmogorov length,

$$L_K=\frac{\lambda_\perp^2}{4D_m}=\sqrt{\frac{2}{\pi}}\,\frac{\lambda_\perp^2}{4\beta^2\lambda_\parallel}. \tag{134}$$

Except for the exponent 2 of the logarithm, the diffusion coefficient (133) scales like the Rechester–Rosenbluth diffusion coefficient.[7,13,16,21,26] Equation (133) [or Eq. (131)] can be used in the limit $\chi_\perp\to 0$. In particular, we have the property

$$\lim_{\chi_\perp\to 0}\frac{\chi_\perp^{AN}}{\chi_\perp}=\infty. \tag{135}$$

We stress, however, that, the smaller $\chi_\perp$ is, the higher is the clump life time $\overline{\tau}^*=(\theta/\overline{\chi}_\perp^{AN})^2$[see Eq. (126)]. Hence, for small values of $\chi_\perp$, the diffusive character of the evolution appears only after a time $t\gg\overline{\tau}^*/\nu$ which may be very long. This constraint yields the validity range in t for the density profile (101).

2. The weakly collisional regime

Next, we take $\theta>1$, i.e. $\overline{\chi}_\parallel>\alpha^{-4}$, which is a weakly collisional situation. We have then $c_1\simeq 2^{7/8}\sqrt{\pi}\theta^{1/4}\exp(-2^{7/4}\sqrt{\theta})$, and Eq. (129) yields

$$1=2^{-5/8}\overline{\chi}_\perp\,\theta^{3/4}(\overline{\chi}_\perp^{AN})^{-3/4}\exp[-2^{7/4}\sqrt{\theta}+4\,\theta(\overline{\chi}_\perp^{AN})^{-1/2}], \tag{136}$$

or equivalently

$$\overline{\chi}_\perp^{AN}=\sqrt{2}\,\theta\,\{1+2^{-7/4}\theta^{-1/2}\ln[2^{5/8}(\overline{\chi}_\perp^{AN})^{3/4}\overline{\chi}_\perp^{-1}\theta^{-3/4}]\}^{-2}. \tag{137}$$

First iteration around $\overline{\chi}_\perp^{AN}=\sqrt{2}\,\theta$ of Eq. (137) yields

$$\overline{\chi}_\perp^{AN}=\sqrt{2}\,\theta\,\{1+2^{-7/4}\theta^{-1/2}\ln[2\overline{\chi}_\perp^{-1}]\}^{-2}, \tag{138}$$

or in dimensional form

$$\chi_\perp^{AN}=\frac{1}{\sqrt{\pi}}D_mV_T\left\{1+\frac{1}{4}\pi^{1/4}\lambda_\perp\sqrt{\frac{\nu}{D_mV_T}}\ln\left[\frac{\nu\lambda_\perp}{\chi_\perp}\right]\right\}^{-2}. \tag{139}$$

Diffusion coefficients in the weakly collisional regime were previously obtained in Ref. 21 from an analytical treatment using an idealized model for the magnetic field which may be difficult to justify, and in Ref. 26 from a Rechester–Rosenbluth like argument. Although similar, these two results are not identical. Equation (139) presents scaling similarities with the results of Refs. 21 and 26. In particular, in all three cases, the dimensional factor is D_mV_T and we have a logarithmic dependence on $\chi_\perp$, with the properties $\lim_{\chi_\perp\to 0}\chi_\perp^{AN}=0$ and $\lim_{\chi_\perp\to 0}\chi_\perp^{AN}/\chi_\perp=\infty$. However, Eq. (139) cannot be linked exactly to the result of Ref. 21 or to the one of Ref. 26.

VI. CONCLUSION

This work intends to provide a statistical description of the test particle dynamics in a specified stochastic magnetic field. The starting point of our discussion is the stochastic Liouville equation (or, hybrid kinetic equation) associated with the equations of motion of the test particle. We have considered successive configurations for the magnetic field and the collisions. In the first situation, the stochastic part of the magnetic field is assumed to have only an x-component depending on z and perpendicular collisions are neglected. In the second situation, perpendicular collisions are included. Finally, we have considered the general model in which the particle experiences collisions in both the parallel and the perpendicular directions and the fluctuating part of the magnetic field has two components in the perpendicular plane depending on the three spatial coordinates $\{x,y,z\}$. Our results for the three cases can be summarized as follows.

• The first situation is known to be equivalent to a specific standard long tail continuous time random walk (SLT-CTRW) model[33] and yields a subdiffusive behavior for the particle motion. These results are recovered here, showing also the intrinsic non-Gaussian character of the process. The density profile, solution of the hybrid kinetic equation, has been obtained and appears to be identical to the density profile derived from the SLT-CTRW model.

• In the second situation, the perpendicular collisions yield diffusion for the test particle dynamics. The latter was shown to gradually dominate the statistics of the process. In the asymptotic stage, the density profile is a Gaussian packet. For the simple magnetic field of this second situation, however, the anomalous effects remain at the subdiffusive level which is asymptotically smoothed out.

• In the third situation, the nonlinearity introduced by the magnetic field yields nonlinear coupling between the parallel and the perpendicular motions of the particle and produces anomalous diffusive effects. In contrast with the first two situations, this model cannot be solved analytically without approximations. We have treated the magnetic stochasticity within the direct interaction approximation (DIA). Moreover, the collisions were averaged within an additional statistical assumption whose role and importance is stressed below. Both these approximations yield the density profile of the system and a second order nonlinear ordinary differential equation for its second moment. The solution of this equation allowed us to obtain all the diffusion regimes known in the literature, including the log-scaled diffusion coefficients of the Rechester–Rosenbluth type. We showed also that diffusion arises in this case after an exponentiation process, or clumping, between the particle and its starting magnetic line.

Regarding these last results, some remarks are necessary. Starting from a Fokker–Planck equation describing the particle dynamics in its velocity extended phase-space, Krommes *et al.* demonstrated in Ref. 13 that the DIA applied to their model fails to provide the Rechester–Rosenbluth diffusion coefficient. Krommes *et al.* actually showed that this problem arises because the DIA is unable to account for the departures from Gaussianity of statistics of the particle motion. We stress, however, that our use of the DIA is not equivalent to the one of Ref. 13. Indeed, we used the DIA for averaging the hybrid kinetic equation only over the magnetic field, and thus obtained an equation which still describes a fluctuating distribution function due to the collisions. The key step is then the decomposition (84), because it accounts for non-Gaussian correlations between the particle and its starting magnetic field. In particular, this decomposition ensures that the particle cannot leave its magnetic line in the absence of perpendicular collisions. The non-Gaussian character of the particle dynamics appears clearly from the density profile (100). The Gaussian density profile (101) only emerges after the exponentiation process between the particle and the line, which itself only occurs in the presence of perpendicular collisions and for times shorter than the clump lifetime. Thus, the non-Gaussianity of the exponentiation process proves that our scheme goes beyond strict DIA, as presented in Ref. 13. Hence, we confirm the conclusion of Ref. 13 that the DIA is insufficient for describing the intermediate exponentiation step of the particle motion. We believe moreover that our approximation is more transparent than the renormalization procedure developed in Ref. 13 for transcending the DIA.

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[32] It should be noted that the hybrid kinetic equation (3) can be coupled with the Maxwell equations, and, hence, allows a self-consistent treatment of the magnetic field. In contrast, the Langevin equations (2) are necessarily *non-self-consistent*. The self-consistency problem will not be considered here and the magnetic field properties will be specified *a priori*.
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[46]Two comments about this result: The distribution function F depends on z, contrary to the density profile $n(x;t)$ introduced in Ref. 33. The density profile $n(x;t)$ of Ref. 33, defined with a different choice of normalization, must thus be identified with our reduced distribution function (32).