BRIEF COMMUNICATIONS

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On the self-similarity assumption in dynamic models for large eddy simulations

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The consistency between dynamic models for large eddy simulations and their underlying self-similarity assumption is discussed. The interpretation of the resolved field is shown to be fully determined by the choice of the test-filter. Consequences for comparison to direct numerical simulations and experimental results are presented. © *1997 American Institute of Physics.* [S1070-6631(97)02207-1]

The description of turbulent flows requires the resolution of a range of scales which increases rapidly with the Reynolds number. Direct numerical simulations (DNS) are thus restricted to moderately turbulent flows. This has prompted the development of large eddy simulations (LES) based on the application of a spatial *grid filter* G_1 to the Navier–Stokes equation.[1–3] This filter damps the fluctuations with a characteristic length shorter than the filter width Δ_1. It is defined by a convolution:

$$\overline{u_i}(\mathbf{x})=\int_V d\mathbf{y} G_1\left(\frac{|\mathbf{x}-\mathbf{y}|}{\Delta_1}\right) u_i(\mathbf{y}), = G_1(\Delta_1)\star u_i, \tag{1}$$

where the operator $\star$ denotes the spatial convolution. The resulting LES equation is given by

$$\partial_t \overline{u_i} = \nu_0 \nabla^2 \overline{u_i} - \partial_j \overline{u_j u_i} - \partial_j \tau_{ij} - \partial_i \overline{p}, \tag{2}$$

where ν_0 is the molecular viscosity. This equation can be simulated on a coarser grid but it contains an unknown subgrid scale stress that needs to be modeled ($\tau_{ij}=\overline{u_i u_j}-\overline{u_i}\,\overline{u_j}$). This term accounts for the large scale effects of the unresolved small scales. We only consider here incompressible flows and the pressure p is chosen to satisfy the incompressibility condition.

We discuss some difficulties related to the definition of the grid filter G_1. Indeed, this filter is needed when LES are compared to DNS or to experiments. However, the grid filter is usually not specified by the LES practitioners, leaving some uncertainties regarding how the comparison must be done. We will show that the dynamic procedure (DP), a recently developed method[4–8] that allows one to compute the subgrid scale model parameters, can be used to define the grid filter G_1. However, it will be shown that the present formulation of the DP is usually incompatible with its underlying self-similarity assumption (SSA). A new formulation of the DP is thus proposed. It has the advantages of being compatible with the SSA and defining the resolved field without any ambiguity. Remarkably, this new interpretation does not change the implementation of the DP. The basic ingredient of the DP is an identity between subgrid stresses generated by different filters:[5]

$$L_{ij}+\widehat{\tau_{ij}}-T_{ij}=0, \tag{3}$$

where $L_{ij}=\widehat{\overline{u_i}\,\overline{u_j}}-\widehat{\overline{u}}_i\,\widehat{\overline{u}}_j$ and $T_{ij}=\widehat{\overline{u_i u_j}}-\widehat{\overline{u}}_i\,\widehat{\overline{u}}_j$ is the subgrid scale stress that appears in the LES equation when another filter G_2 is applied to the velocity $G_2\star u=\widehat{\overline{u}}$. This filter is supposed to be obtained by the successive application of the grid filter G_1 and a so-called test filter G_t: $\widehat{\overline{u}}=G_t\star G_1\star u$. Of course, when approximate models are used for $\tau_{ij}\approx\tau_{ij}^M$ and $T_{ij}\approx T_{ij}^M$, the identity (3) is violated. However, the quantity $E_{ij}\equiv L_{ij}+\hat{\tau}_{ij}^M-T_{ij}^M\neq 0$ may be used to calibrate the models. The Smagorinsky model[9] is used in what follows but the discussion is valid independently of the model

$$\tau_{ij}^M-\tfrac{1}{3}\tau_{kk}^M\delta_{ij}=-2C\Delta_1^2\sqrt{2\overline{S}_{kl}\overline{S}_{kl}}\,\overline{S}_{ij}, \tag{4}$$

where $\overline{S}_{ij}=(\partial_i\overline{u}_j+\partial_j\overline{u}_i)/2$ is the resolved strain tensor. A similar model is used for T_{ij} in terms of grid+test filtered quantities. In the DP, the coefficient C is estimated by minimizing $E_{ij}(C)$. It is expressed as a function of (a) the resolved field, (b) the test filter and (c) the ratio $\delta=\Delta_2/\Delta_1$,

$$C=C[\overline{u}_i, G_t, \delta], \tag{5}$$

and does not depend explicitly on G_1 and G_2 (details may be found in Refs. 5–8). For simplifying the numerical scheme, both τ_{ij} and T_{ij} are modeled in the same way with the same C (although the DP might be implemented with different models at grid and test levels[7]). This choice is justified by the SSA, itself motivated by the fact that the flow is supposed to exhibit a well developed inertial range in which self-similarity arguments are valid. However, this justification holds only if the filters G_1 and G_2 are themselves self-similar, i.e., if

$$G_1(x)=\delta^{-d}G_2\left(\frac{x}{\delta}\right). \tag{6}$$

Hence, G_1 and G_2 must have identical shapes and may only differ by their characteristic width. Indeed, the model coefficient depends on the filter shape.[10] Unfortunately, in the DP nothing ensures that the filters G_1 and G_2 have the same shape. Another difficulty comes from the definition of δ. Indeed, Δ_1 and Δ_2 are usually not known and must be guessed. For example, Δ_1 is often identified with the mesh size which implies that the LES are under-resolved[11] while Δ_2 is approximated by Δ_t, which is only true for the sharp Fourier cutoff.

We now present a simple re-interpretation of the filters that solves the aforementioned difficulties of the present formulation of the DP *independently of the type of the test filter*. The discussion is presented for a one-dimensional filter but is valid in d-dimensions. Let us introduce an infinite set of self-similar filters in the sense of the definition (6) $\{\mathcal{G}_n\equiv\mathcal{G}(\ell_n)\}$ defined by

$$\mathcal{G}_n(x)=\alpha^{-n}K\left(\frac{x}{\alpha^n\ell_0}\right), \tag{7}$$

where n is an integer and $K(x)=K(-x)$ is the filter kernel. The characteristic width of $\mathcal{G}_n$ is $\ell_n=\alpha^n\ell_0$. Here, ℓ_0 is an arbitrary length and $\alpha>1$ is a parameter. Clearly, the filters $\mathcal{G}_n$ defined by (7) are all self-similar. Let us now consider a second set $\{\mathcal{G}_n^\star\equiv\mathcal{G}^\star(\ell_n^\star)\}$ defined by

$$\mathcal{G}_n^\star\equiv\mathcal{G}_n\star\mathcal{G}_{n-1}\star\cdots\star\mathcal{G}_{-\infty}. \tag{8}$$

For positive kernel K, the relation between $\ell_n^\star$ and ℓ_n can be derived by analogy with probability distribution functions (PDF). Indeed, the filters are normalized in order to ensure $\mathcal{G}_n\star w=w$ for w constant. Let us denote x_n a stochastic process for which the PDF is $\mathcal{G}_n$. The first moment $\langle x_n\rangle$ vanishes because $\mathcal{G}_n(x)=\mathcal{G}_n(-x)$. The second moment may then be related to the filter width ($\ell_n^2\propto\sigma_n^2$) where

$$\sigma_n^2=\int dx x^2\mathcal{G}_n(x). \tag{9}$$

Remarkably, $\mathcal{G}_n^\star$ as defined by (8) corresponds to the PDF of $x_n^\star=x_n+x_{n-1}+\cdots+x_{-\infty}$ if all the x_i are independent stochastic processes.[12] In that case, the second moment of $x_n^\star$ is given by $(\sigma_n^\star)^2=\sigma_n^2+\sigma_{n-1}^2+\cdots+\sigma_{-\infty}^2$. Using the property that the σ_n, like the ℓ_n, follow a geometrical law, one obtains: $(\sigma_n^\star)^2=\sigma_n^2\alpha^2/(\alpha^2-1)$ and, consequently,

$$\ell_n^\star=\frac{\alpha}{\sqrt{\alpha^2-1}}\ell_n. \tag{10}$$

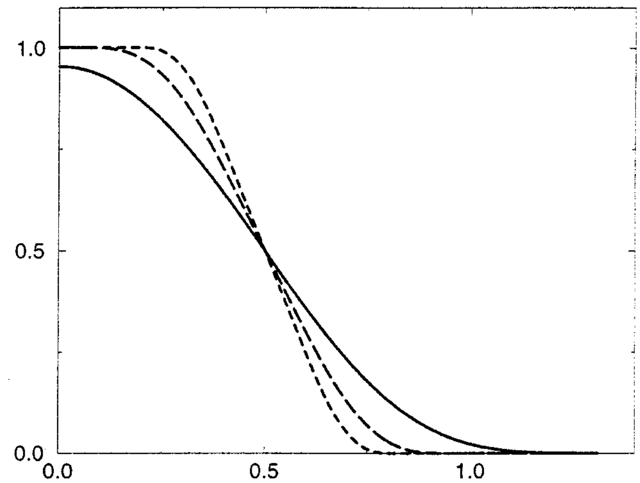

FIG. 1. Iterative filter $\mathcal{G}^\star(x)$ when $\mathcal{G}$ is the top-hat filter for different values of the width ratio: $\alpha=2.5$ (solid line), $\alpha=2.0$ (dashed line), and $\alpha=1.5$ (dotted line).

Of course, this result is only valid if the kernel $K(x)$ is positive and has a finite second moment.[13] From the relation (10), we conclude that filter widths $\ell_n^\star$ follow the same geometrical law as the ℓ_n, $\ell_n^\star=\alpha\ell_{n-1}^\star$. Moreover, the shape of $\mathcal{G}_n^\star$ does not depend on n since $\mathcal{G}_n^\star$ is obtained by an infinite number of convolutions. This is easily seen in Fourier space, where the convolutions reduce to simple products. Hence, the $\mathcal{G}_n^\star$ also constitute a set of self-similar filters.

Starting from these two sets of self-similar filters, it is now easy to give a self-similar formulation of the DP: Let us suppose that the test-filter and the grid filter are defined by:

$$G_t(\Delta_t)=\mathcal{G}_n(\ell_n), \tag{11}$$

$$G_1(\Delta_1)=\mathcal{G}_{n-1}^\star(\ell_{n-1}^\star). \tag{12}$$

As a direct consequence of (7) and (8), the ''grid+test'' filter is given by $G_2(\Delta_2)=\mathcal{G}_n^\star(\ell_n^\star)$. Hence, with the definitions (11) and (12) for the test and grid filters, the filters G_1 and G_2 are automatically self-similar. Also, for any test-filter G_t and any value of δ in the DP, the grid filter can be constructed explicitly:

$$G_1=G_t(\Delta_t/\delta)\star G_t(\Delta_t/\delta^2)\star\cdots\star G_t(\Delta_t/\delta^\infty). \tag{13}$$

In some cases, G_1 may be computed analytically:

$$G_1=\begin{cases}\mathcal{G}(\ell_{n-1}) & \text{sharp Fourier filter,}\\ \mathcal{G}\left(\sqrt{\dfrac{\alpha^2}{\alpha^2-1}}\ell_{n-1}\right) & \text{Gaussian filter.}\end{cases} \tag{14}$$

For more complicated filters, G_1 can be evaluated numerically by iterating G_t. Typical profiles of G_1 for the top-hat filter G_t are shown in Fig. 1.

The formulation of the DP in terms of the sets (7 and 8) and of the definitions (11 and 12) has several major advantages. First, the dynamic model is compatible with the SSA. It is now fully justified to use the same model for τ_{ij} and T_{ij} *independently of the test filter* G_t *used to transform* $\overline{u}_i$ *into* $\widehat{\overline{u}}_i$. Second, this new interpretation of the grid and test level velocities does not modified the implementation of the

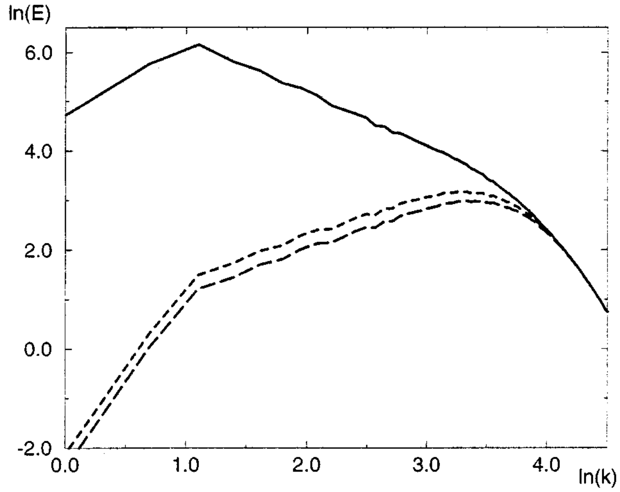

FIG. 2. Comparison between (1) 512^3 DNS energy spectrum ($k_{max}=256$) of decaying isotropic turbulence at $Re_\lambda=63.5$ (solid line); (2) the corresponding unresolved energy spectrum for a three dimensional filter-grid based on the one-dimensional filter expressed by relation (13) where G_t is a top-hat filter, $\delta=2$ and $\Delta_t k_{max}/\delta=0.5$ (dotted line); (3) the same quantity obtained with the usual interpretation of the one-dimensional grid filter given by the first term only in the relation (13) (dashed line).

DP. Indeed, the filters G_1 and G_2 are never used in the DP which only requires the knowledge of G_t. The formulation presented here thus improves the theoretical basis of the DP. It has also nontrivial consequenced in *a priori* tests or when LES results are compared to experiments or DNS. Indeed, the grid filter is now fully determined by the quantities entering the DP (G_i and δ) and its explicit form is given by the relation (13).

The application of G_1 to DNS data illustrated in Fig. 2 for isotropic turbulence where the unresolved energy clearly depends on the interpretation of the grid filter. In that case, the tri-dimensional filter expressed in Fourier space is usually built as the triple product of one-dimensional filter $G^{3d}(\vec{k})=G^{1d}(k_x)G^{1d}(k_y)G^{1d}(k_z)$. For isotropic turbulence, the energy spectra in the LES and DNS are related following $E^{LES}(k)=q(k)E^{DNS}(k)$ where the factor $q(k)$ is defined by

$$q(k)=\frac{1}{4\pi}\int d\Omega(G^{1d}(k_x)G^{1d}(k_y)G^{1d}(k_z))^2 \qquad (15)$$

and $\int d\Omega$ represents the integration over the sphere of radius k. The unresolved energy is given by $E^{DNS}-E^{LES}$.

Also, we remark that the self-similar formulation of the DP is similar to the application of the renormalization group to the Navier–Stokes equation.[14] Each filter $\mathscr{G}_i$ composing G_1 may be regarded as one step in the small scale elimination used in the renormalization group procedure and the test filter used in the DP corresponds to the last iteration. The existence of a fixed point in the renormalization equation is assumed by using the same C at level n (G_2) and $n-1$ (G_1). Also, the scaling exponent appearing in the renormalization group are anticipated by using the scaling derived by Smagorinsky using Kolmogorov-like arguments. Hence, the self-similar formulation of the DP plays in the context of LES the same role as the renormalization group in the context of statistical theories of turbulence.

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[1] A. Leonard, ''Energy cascade in large-eddy simulation of turbulent fluid flows,'' Adv. Geophys. A **18**, 237 (1974).
[2] D. Leslie and G. Quarini, ''The Application of Turbulence Theory to the Formulation of Subgrid Modelling Procedures,'' J. Fluid Mech. **91**, 65 (1979).
[3] R. Rogallo and P. Moin, ''Numerical simulation of turbulent flows,'' Annu. Rev. Fluid Mech. **16**, 99 (1984).
[4] M. Germano, U. Piomelli, P. Moin, and W. Cabot, ''A dynamic subgrid-scale eddy-viscosity model,'' Phys. Fluids A **3**, 1760 (1991).
[5] M. Germano, ''Turbulence: the filtering approach,'' J. Fluid Mech. **238**, 325 (1992).
[6] D. Lilly, ''A proposed modification of the Germano subgrid-scale closure method,'' Phys. Fluids **4**, 633 (1992).
[7] P. Moin and J. Jimenéz, in *AIAA 24th Fluid Dynamics Conference*, paper 93-3099, (AIAA, Orlando FL, 1993).
[8] S. Ghosal, T. S. Lund, P. Moin, and K. Akselvoll, ''A dynamic localization model for large-eddy simulation of turbulent flows,'' J. Fluid Mech. **286**, 229, (1995).
[9] J. Smagorinsky, ''General Circulation Experiments with the Primitive Equations,'' Mon. Weather Rev. **91**, 99 (1963).
[10] D. Lilly, *Proceedings IBM Scientific Computing Symposium on Environmental Sciences* (Yorktown Heights, N.Y., 1967), p. 195.
[11] S. Ghosal, ''An Analysis of Numerical Errors in Large-Eddy Simulations of Turbulence,'' J. Comput. Phys. **125**, 187 (1996).
[12] W. Feller, *An Introduction to Probability Theory and Its Applications, Volume II* (Wiley, New York, 1971). The analogy between the filters and the PDF should not lead to the conclusion that the $\mathscr{G}_n^\star$ are always Gaussian filters. Although they represent the PDF of a stochastic variable given by the sum of an infinite number of independent processes, the central limit theorem does not apply because the PDF of $x_n^\star$ is usually dominated by a small number of variables x_i corresponding to $i=n,n-1,\ldots$.
[13] Both these conditions are satisfied for most of the filters with the notable exception of the Fourier cutoff. In that case, however, it is easy to show that $\mathscr{C}_n^\star=\mathscr{C}_n$.
[14] D. Forster, D. Nelson, and M. Stephen, ''Large-distance and long-time properties of a randomly stirred fluid,'' Phys. Rev. A **16**, 732 (1977).
[15] A. Kolmogorov, ''Local Structure of Turbulence in an Incompressible Fluid at Very High Reynolds Numbers,'' Dokl. AN SSSR **30**, 299 (1941).