

# On the statistical solution of the Riemann equation and its implications for Burgers turbulence\*

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The statistics of the multivalued solutions of the forced Riemann equation,  $u_t + uu_x = f$ , is considered. An exact equation for the signed probability density function of these solutions and their gradient  $\xi = u_x$  is derived, and some properties of this equation are analyzed. It is shown in particular that the tails of the signed probability density function generally decay as  $|\xi|^{-3}$  for large  $|\xi|$ . Further considerations give bounds on the cumulative probability density function for the velocity gradient of the solution of Burgers equation. © 1999 American Institute of Physics.  
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In recent years, there has been a resurgence of interest in the statistical behavior of solutions of the forced Burgers equation<sup>1-22</sup>

$$u_t + uu_x = \nu u_{xx} + f, \quad (1)$$

where the force  $f$  is assumed to be a zero-mean, Gaussian, statistically homogeneous, and white-in-time random process with covariance

$$\langle f(x,t)f(y,s) \rangle = 2B(x-y)\delta(t-s), \quad (2)$$

where  $B(x)$  is smooth. Differentiation of (1) results in an equation for the velocity gradient,  $\xi = u_x$ ,

$$\xi_t + u\xi_x = -\xi^2 + \nu\xi_{xx} + f_x. \quad (3)$$

Let  $P(u, \xi, x, t)$  be the joint probability density function of  $u$  and  $\xi$ . An exact equation for  $P$  may be derived (for completeness we recall the derivation in the Appendix),

$$P_t = -uP_x + \xi P + (\xi^2 P)_\xi + B_0 P_{uu} + B_1 P_{\xi\xi} - \nu(\langle u_{xx}|u, \xi \rangle P)_u - \nu(\langle \xi_{xx}|u, \xi \rangle P)_\xi, \quad (4)$$

where  $B_0 = B(0)$ ,  $B_1 = -B''(0)$ , and  $\langle \cdot |u, \xi \rangle$  denotes the average conditional on  $u$  and  $\xi$ . The explicit form of  $\langle u_{xx}|u, \xi \rangle$  and  $\langle \xi_{xx}|u, \xi \rangle$  is not known, leaving (4) unclosed. There have been several proposals on how to approximate these terms in the inviscid limit  $\nu \rightarrow 0$ , leading to contradictory predictions.

In this paper, we shall adopt a different approach. Assume we drop the viscous term in (1) and (3). This operation, not to be confused with taking the inviscid limit  $\nu \rightarrow 0$  in (1) and (3) [see the remark after (11) below], results in the forced Riemann equation

$$u_t + uu_x = f, \quad (5)$$

and the equation for the gradient,  $\xi = u_x$ ,

$$\xi_t + u\xi_x = -\xi^2 + f_x. \quad (6)$$

Applying to (5) and (6) the same operation that leads to (4) starting from (1) and (3), we get the closed equation

$$P_t = -uP_x + \xi P + (\xi^2 P)_\xi + B_0 P_{uu} + B_1 P_{\xi\xi}. \quad (7)$$

In this paper we shall focus on the study of (7). Generally the solution of this equation cannot be interpreted as a probability density function. The origin of this problem lies in the multivalued nature of the solutions of the Riemann equation. As we will show this will force us to reinterpret the solution of (7) as a *signed* probability density function, the precise meaning of which will be given below. Further considerations will allow us to obtain the true probability density function associated with some branches of the multivalued solution of (5). This will immediately give us bounds on the cumulative probability density function of the gradient of the solutions of Burgers equation.

To begin with let us recall some well-known facts about the forced Riemann equation. The solution of (5) and (6) can be obtained by the method of characteristics, i.e., considering

$$\frac{dx^t}{dt} = u^t, \quad \frac{du^t}{dt} = f(x^t, t), \quad (8)$$

for the initial condition  $x^0 = y$ ,  $u^0 = u_0(y)$  where  $u_0(x) = u(x, 0)$  is the initial condition for (5). Then  $u(x, t) = u^t$  for that characteristic satisfying  $x^t = x$  and, of course,  $\xi(x, t) = u_x(x, t)$ . Note that  $x^t$ ,  $u^t$  depend parametrically on  $y$ , the actual value of which has to be determined to get the explicit expression for  $u(x, t)$ . In general, the resulting set of equations will have more than one solution, i.e.,  $u(x, t)$  will be multivalued (a typical solution is represented in Fig. 1). For example, for the unforced case ( $f=0$ ), the solution of (8) leads to the set

$$x = y + u_0(y)t, \quad u = u_0(y), \quad (9)$$

\*Dedicated with admiration to Bob Kraichnan on the occasion of his seventieth birthday.

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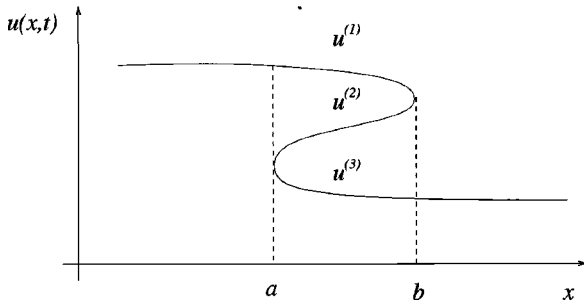


FIG. 1. Schematic representation of the multivalued solution of the Riemann equation obtained by the solution of (10).

which may be rewritten as the single equation

$$u = u_0(x - ut), \tag{10}$$

in  $u$ . In this case, implicit differentiation of (10) over  $u$  leads to the following equation for  $\xi$ :

$$\xi = (1 - \xi t)\xi_0(x - ut), \tag{11}$$

where  $\xi_0(x) = u_x(x, 0)$ . The solution of Burgers equation in the inviscid limit  $\nu \rightarrow 0$  (which is single-valued) can be obtained from the solution of the corresponding Riemann equation by introducing appropriate discontinuities. For example, for the unforced case, for the situation depicted in Fig. 1 this operation amounts to jumping from the upper-most branch to the lower-most one at some value of  $x$  in between  $a$  and  $b$  in such a way that the lobes at the right and at the left of such a cut have the same area (Maxwell's rule).

From the above considerations it should be clear that a full statistical description of the multivalued solutions of the forced Riemann equation is a complicated problem which is beyond the scope of the present paper. Indeed a complete statistical description would amount to introducing the probability that at point  $x$  and time  $t$ ,  $u(x, t)$  has  $N$  branches, and the probability density function that  $u(x, t)$  and  $u_x(x, t)$  have value  $u$  and  $\xi$  on the  $n$ -th branch conditional on the existence of  $N$  branches at  $\{x, t\}$ . Let us, however, show now that the solution of (7) gives some statistical information on  $u(x, t)$ .

We first look at the unforced case,  $B_0 = B_1 = 0$ . Then (7) can be solved by the method of characteristics. This yields for the initial condition  $P(u, \xi, x, 0) = P_0(u, \xi, x)$ ,

$$P(u, \xi, x, t) = (1 - \xi t)^{-3} P_0\left(u, \frac{\xi}{1 - \xi t}, x - ut\right). \tag{12}$$

It follows immediately from (12) that  $P \sim -\xi^{-3}$ , as  $|\xi| \rightarrow \infty$ , assuming that  $P_0$  is not concentrated on positive values of  $\xi$ . Since  $P$  may become negative for  $\xi > 1/t$ , it can no longer be interpreted as a true probability density function. To better understand the meaning of  $P$ , consider first the case with deterministic initial data  $P_0(u, \xi, x) = \delta(u - u_0(x))\delta(\xi - \xi_0(x))$ . Then (12) reads explicitly

$$P(u, \xi, x, t) = (1 - \xi t)^{-3} \delta(u - u_0(x - ut)) \times \delta\left(\frac{\xi}{1 - \xi t} - \xi_0(x - ut)\right). \tag{13}$$

Using the properties of the delta distribution this can be re-organized into

$$P(u, \xi, x, t) = \sum_k (-1)^{k+1} \delta(u - u^{(k)}(x, t)) \times \delta(\xi - u_x^{(k)}(x, t)), \tag{14}$$

where  $u^{(k)}(x, t)$  is the  $k$ -th branch (ordered as  $u^{(1)} \geq u^{(2)} \geq \dots$ ) of the solution of (10).<sup>23</sup> For example, for the situation depicted in Fig. 1, we have

$$P(u, \xi, x, t) = \delta(u - u^{(1)}(x, t))\delta(\xi - u_x^{(1)}(x, t)) - \delta(u - u^{(2)}(x, t))\delta(\xi - u_x^{(2)}(x, t)) + \delta(u - u^{(3)}(x, t))\delta(\xi - u_x^{(3)}(x, t)), \tag{15}$$

for  $a \leq x \leq b$ .

For random initial data, (14) must be averaged over the statistics of  $u_0(x)$ . In this case  $P$  has to be interpreted as a *signed probability density function*. Empirically, we can understand this object as follows. Consider a sample of  $N$  realizations of the multivalued function  $u(x, t)$ . Fix  $x, t, v$  and  $\zeta$ . Assume there are  $n^+(v, \zeta, x, t)$  values in the set  $\{u(x, t), \xi(x, t)\}$  on the positive branches [i.e.,  $k$  in (14) is odd] with  $u(x, t)$  less than  $v$  and  $\xi(x, t)$  less than  $\zeta$ , and  $n^-(v, \zeta, x, t)$  on the negative branches. Then we have

$$\lim_{N \rightarrow \infty} \frac{n^+(u, \xi, x, t) - n^-(u, \xi, x, t)}{N} = \int_{-\infty}^u du' \int_{-\infty}^{\xi} d\xi' P(u', \xi', x, t). \tag{16}$$

For the forced case, we have a similar situation. First, if the force and the initial data are deterministic, we can write down a similar formula as (14). For the random case, we can average this formula over the distribution of forces and initial data. This means that the solution of (7) can still be interpreted as a signed probability density function, now associated with the multivalued solutions of (5) and (6). Notice that within this interpretation the operation,

$$\langle b(u, \xi) \rangle = \int dud\xi b(u, \xi) P, \tag{17}$$

still defines an average, even though it is not a statistical average. Consequently we have

$$\int d\xi P_x = - \int d\xi \xi P_u, \tag{18}$$

since  $\langle a(u) \rangle_x = \langle a_u(u) \xi \rangle$  as a result of the constraint  $\xi = u_x$ . This property can also be derived directly from (7). From (18), it follows that (7) also preserves normalization [consistent with (16)] since

$$\frac{d}{dt} \int dud\xi P = \int dud\xi (-u P_x + \xi P) = 0. \tag{19}$$

The last equality is just  $-\langle u \rangle_x + \langle \xi \rangle = 0$ .

Coming back to the scaling properties, assume that  $P \sim \xi^{-\alpha}$  for  $|\xi| \rightarrow \infty$ . Balancing the terms at large values of  $\xi$  in

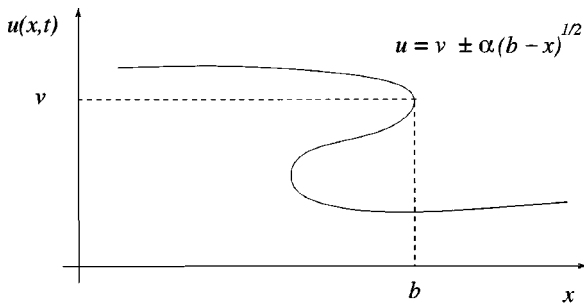


FIG. 2. Local behavior of multivalued solution of the Riemann equation near the local right-most position  $b$ .

(7) gives  $\alpha=3$ . This means that, generally, the solutions of this equation behave as  $P \sim \xi^{-3}$ . Moreover, (7) leads to the following equation for  $\langle \xi \rangle$ :

$$0 = \langle \xi \rangle_t + \langle u \xi \rangle_x = \lim_{\xi \rightarrow +\infty} \xi^3 Q - \lim_{\xi \rightarrow -\infty} \xi^3 Q, \quad (20)$$

where the first equality follows directly from (6) and we defined the reduced density

$$Q(\xi, x, t) = \int du P(u, \xi, x, t). \quad (21)$$

Since, generally,  $\lim_{\xi \rightarrow \pm\infty} \xi^3 Q \neq 0$ ,  $Q$  (and hence  $P$ ) must have different signs at  $\xi = -\infty$  and  $\xi = +\infty$ . This can be confirmed explicitly upon assuming statistical homogeneity. Then  $P_x = 0$  and an equation for  $Q$  can be derived from (7),

$$Q_t = \xi Q + (\xi^2 Q)_\xi + B_1 Q_{\xi\xi}. \quad (22)$$

Even though the solutions of the Riemann equation cannot reach statistical steady state, (22) admits an integrable steady solution. The reason is that there are many cancellations between the contributions of the positive and the negative branches that eventually lead to a stationary value for the signed probability density function  $Q$ . The only integrable solution of (22) with  $Q_t = 0$  is

$$Q(\xi) = \frac{C}{B_1^{1/3}} \left( 1 - \frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} d\xi' \xi'^t e^{\Lambda'} \right), \quad (23)$$

where  $\Lambda = \xi^3 / (3B_1)$  and  $C$  is a numerical constant fixed by the normalization constraint,  $\int d\xi Q = 1$ . From (23) it follows that

$$Q(\xi) \sim -\frac{CB_1^{2/3}}{\xi^3} \quad \text{as } |\xi| \rightarrow \infty, \quad (24)$$

i.e.,  $Q$  has different signs for large positive and negative  $\xi$  (this function is plotted in Fig. 4 of Ref. 21).

There is a simple geometrical explanation for the  $|\xi|^{-3}$ -scaling. Let  $b$  be a local right-most position reached by the multivalued function  $u(x, t)$  at time  $t$  (see Fig. 2). Generically, the two branches of  $u(x, t)$  may be expanded around  $b$  as  $u(x, t) = v \pm \alpha(b-x)^{1/2}$  for some (random)  $\alpha$ . Hence

$$u_x(x, t) = \mp \frac{1}{2} \alpha (b-x)^{-1/2}. \quad (25)$$

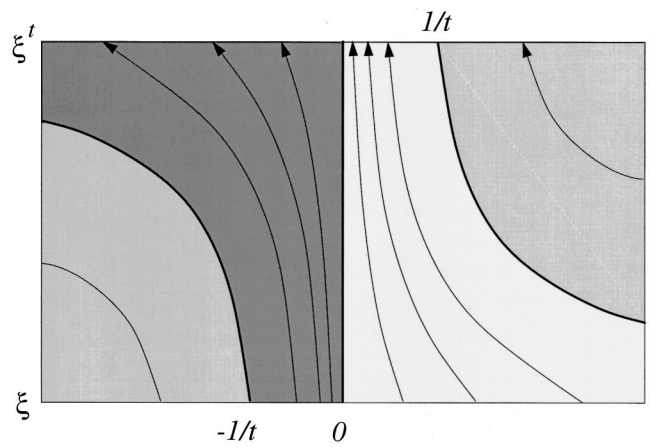


FIG. 3. Schematic representation of the flow (27). The three ranges defined in (28) are shown with different gray levels.

From (25) it follows that  $u_x$  is bigger than some  $\xi \gg 1$ , or smaller than  $-\xi$  when  $\alpha(b-x)^{-1/2} > \xi$ . Thus, for  $|\xi| \rightarrow \infty$ , the signed probability of such events is given asymptotically by

$$\begin{aligned} \text{sgn-prob}(u_x > \xi) &\sim +C|\xi|^{-2}, \\ \text{sgn-prob}(u_x < -\xi) &\sim -C|\xi|^{-2}, \end{aligned} \quad (26)$$

where  $C$  is some positive constant whose value is related to the statistics of  $\alpha$  and the spatial distribution of overturning points. The signed probability density associated with (26) behaves as  $|\xi|^{-3}$  for  $\xi \rightarrow -\infty$  and as  $-|\xi|^{-3}$  for  $\xi \rightarrow +\infty$ .

So far our results about the Riemann equation have no direct implications for Burgers turbulence. However, further manipulations give bounds on the probability density functions of the solutions of Burgers equation. For simplicity, we shall assume statistical homogeneity and consider (22) for  $Q$ . Moreover, we focus again on the unforced case first [i.e., we set temporarily  $B_1 = 0$  in (22)] and comment on the loss of positivity of the solution of this equation from a mathematical point of view. Equation (22) is in a form that standard maximum principle applies. This would imply that  $Q$  stays non-negative if it is initially so. However, in the present case, due to the unboundedness of the coefficients in (22), maximum principle is violated and negative values of  $Q$  creep in from infinity. Consider indeed the flow of characteristics in  $\xi$  solution of  $d\xi^t/dt = -(\xi^t)^2$ . This gives (see Fig. 3)

$$\xi^t = \frac{\xi}{1 + \xi t}. \quad (27)$$

Thus  $\xi^t = 0$  if  $\xi = 0$  (no crossing of the  $\xi$ -axis) and

$$\begin{aligned} \xi^t \in [-\infty, 0], & \quad \text{if } \xi \in [-1/t, 0], \\ \xi^t \in [0, 1/t], & \quad \text{if } \xi \in [0, \infty], \\ \xi^t \in [1/t, \infty], & \quad \text{if } \xi \in [-\infty, -1/t]. \end{aligned} \quad (28)$$

The third range corresponds to the characteristics which start from a negative  $\xi$ , cross infinity at time  $t = -1/\xi$ , and end up at a positive  $\xi^t$  at time  $t$ . These characteristics are the origin of the multivaluedness of the solutions of the Riemann equation and of the creeping of negative values of  $Q$ .

This scenario suggests the following procedure. Solve (22) with  $B_1 = 0$  by the method of characteristics but exclude those characteristics which cross infinity. Denote by  $Q^+(\xi, t)$  the solution obtained this way: it is given by [compare (12)]

$$Q^+(\xi, t) = \begin{cases} (1 - \xi t)^{-3} Q_0\left(\frac{\xi}{1 - \xi t}\right) & \text{if } \xi < 1/t, \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

It is not difficult to realize that  $Q^+$  is proportional to the probability density function of the positive branches of the solution of the Riemann equation since, by construction, we exclude all the negative ones. In particular [compare (16)],

$$\lim_{N \rightarrow \infty} \frac{n^+(\xi, t)}{N} = \int_{-\infty}^{\xi} d\xi' Q^+(\xi', t), \quad (30)$$

if there are  $n^+(\zeta, t)$  values in the set  $\{\xi(x, t)\}$  on the positive branches with  $\xi(x, t)$  less than  $\zeta$  ( $n^+$  is independent of  $x$  because of the statistical homogeneity). Note that by construction  $Q^+$  is non-negative, but, of course, it is not normalized to unity: due to the multiplication of the number of branches  $N^+(t) = \lim_{\xi \rightarrow \infty} n^+(\xi, t) \geq N$ , i.e., the effective number of realizations,  $N^+(t)$ , is bigger than the size of original sampling set,  $N$ . The interest of  $Q^+$  is clear if one recalls that the solution of Burgers equation is actually contained in the solution of the corresponding Riemann equation (i.e., solved for the same initial condition). This means in particular that if in a sample of  $N$  solutions of the unforced Burgers equation there are  $n^B(\zeta, t)$  values in the set  $\{\xi(x, t)\}$  with  $\xi(x, t)$  less than  $\zeta$ , then  $n^B(\zeta, t) \leq n^+(\zeta, t)$  where  $n^+$  is associated with the corresponding sample of solutions of the unforced Riemann equation. As a direct consequence we have

$$\int_{-\infty}^{\xi} d\xi' Q^B(\xi', t) \leq \int_{-\infty}^{\xi} d\xi' Q^+(\xi', t), \quad (31)$$

where  $Q^B(\xi, t)$  denotes the probability density function of the gradient of the solution of Burgers Eq. (1) with  $f = 0$  [i.e., the solution of (3) with  $f_x = 0$ ]. Inserting (29) into (31) provides us with an explicit bound on the cumulative probability distribution of the gradients for the unforced Burgers equation. It implies in particular that

$$\int_{-\infty}^{\xi} d\xi' Q^B(\xi', t) \leq C \xi^{-2} \quad \text{as } \xi \rightarrow -\infty, \quad (32)$$

for some (time-dependent) constant  $C$ . For monotonous  $Q^B$  this gives  $Q^B \leq 2C|\xi|^{-3}$  as  $\xi \rightarrow -\infty$ .

Similar bounds can be obtained for the forced case. Indeed, the solution of (22) is the average on  $\beta$  of  $\tilde{Q}$  solution of

$$\tilde{Q}_t = \xi \tilde{Q} + (\xi^2 \tilde{Q})_{\xi} - \beta \tilde{Q}_{\xi}, \quad (33)$$

where  $\beta$  is a white-noise process independent of  $x$  and with covariance  $\langle \beta(t) \beta(s) \rangle = 2B_1 \delta(t - s)$ . This equation can be solved by the methods of characteristics [i.e., upon solving the equation  $d\xi'/dt = -(\xi')^2 + \beta$ ] and the same operation of excluding those characteristics that cross infinity can be applied. Denote by  $Q^+$  the average on  $\beta$  of the solution ob-

tained in this way: then  $Q^+$  is proportional to the probability density function of some of the positive branches of the solution of the forced Riemann equation. In particular the bound (31) still applies now with  $Q^B(\xi, t)$  being the probability density function of the gradient of the solution of the forced Burgers equation (1). No explicit expression can be given to  $Q^+$  but  $Q^+ \sim C|\xi|^{-3}$  as  $\xi \rightarrow -\infty$ . This scaling is suggested by the geometric argument involving the overturning points; it is confirmed upon taking moments of (33) and noting that for  $\int d\xi Q^+(\xi, t)$  to grow exponentially, as it should in the forced case due to the multiplication of the number of branches,  $\int d\xi \xi Q^+(\xi, t)$  has to grow as well, which implies  $-\lim_{\xi \rightarrow -\infty} \xi^3 Q^+(\xi, t) > 0$ . Thus relation (32) also applies for the forced case during the evolution. Unfortunately, no conclusion can be drawn for the statistically stationary solution of the forced Burgers equation since  $Q^+$  does not converge to a steady state (i.e., the constant  $C$  in the large negative  $\xi$ -scaling,  $Q^+ \sim C|\xi|^{-3}$ , grows unbounded as  $t \rightarrow \infty$ ).

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## APPENDIX: DERIVATION OF (4)

In this appendix we recall the derivation of (4). Let

$$\Theta(\lambda, \mu, x, t) = e^{-i\lambda u(x, t) - i\mu \xi(x, t)}. \quad (A1)$$

Then,  $\langle \Theta \rangle$  is the characteristic function of the joint random process  $\{u(x, t), \xi(x, t)\}$ , whose probability distribution function is given by

$$P(u, \xi, x, t) = \int \frac{d\lambda d\xi}{(2\pi)^2} e^{i\lambda u + i\mu \xi} \langle \Theta(\lambda, \mu, x, t) \rangle. \quad (A2)$$

Time differentiation of (A1) gives

$$\begin{aligned} \Theta_t &= -i(\lambda u_t + \mu \xi_t) \Theta \\ &= -i\lambda(-uu_x + vu_{xx} + f) \Theta \\ &\quad - i\mu(-u\xi_x - \xi^2 + v\xi_{xx} + f_x) \Theta, \end{aligned} \quad (A3)$$

where we used (1) and (3). Due to the white-in-time character of the force, the terms involving  $f$  and  $f_x$  can be averaged immediately, yielding

$$-i\langle (\lambda f + \mu f_x) \Theta \rangle = -\lambda^2 B_0 \langle \Theta \rangle - \mu^2 B_1 \langle \Theta \rangle. \quad (A4)$$

For the convective terms in (A3), we use

$$\begin{aligned} &i\langle (\lambda uu_x + \mu(u\xi_x + \xi^2)) \Theta \rangle \\ &= -\langle u \Theta_x \rangle + i\mu \langle \xi^2 \Theta \rangle \\ &= -\langle u \Theta \rangle_x + \langle \xi \Theta \rangle + i\mu \langle \xi^2 \Theta \rangle \\ &= -i\langle \Theta \rangle_{x\lambda} + i\langle \Theta \rangle_{\mu} - i\mu \langle \Theta \rangle_{\mu\mu}. \end{aligned} \quad (A5)$$

The dissipative terms have to be expressed as conditional averages. Combining the above formulas and going back to the  $\{u, \xi\}$ -representation gives (4).

- <sup>1</sup>J. D. Fournier and U. Frisch, "L'équation de Burgers déterministe et statistique," *J. Mec. Theor. Appl.* **2**, 699 (1983).
- <sup>2</sup>S. F. Shandarin and Ya. B. Zeldovich, "The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium," *Rev. Mod. Phys.* **61**, 185 (1989).
- <sup>3</sup>S. N. Gurbatov, A. N. Malakhov, and A. I. Saichev, *Nonlinear Random Waves and Turbulence in Nondispersive Media: Waves, Rays and Particles* (Manchester University Press, Manchester, 1991).
- <sup>4</sup>T. Gotoh and R. H. Kraichnan, "Statistics of decaying Burgers turbulence," *Phys. Fluids A* **5**, 445 (1993).
- <sup>5</sup>T. Gotoh, "Inertial range statistics of Burgers turbulence," *Phys. Fluids* **6**, 3985 (1994).
- <sup>6</sup>M. Avellaneda, R. Ryan, and W. E, "PDFs for velocity and velocity gradients in Burgers turbulence," *Phys. Fluids* **7**, 3067 (1995).
- <sup>7</sup>M. Avellaneda and W. E, "Statistical properties of shocks in Burgers turbulence," *Commun. Math. Phys.* **172**, 13 (1995).
- <sup>8</sup>M. Avellaneda, "Statistical properties of shocks in burgers turbulence, II: Tail probabilities for velocities, shock-strengths and rarefaction intervals," *Commun. Math. Phys.* **169**, 45 (1995).
- <sup>9</sup>J.-P. Bouchaud, M. Mézard, and G. Parisi, "Scaling and intermittency in Burgers turbulence," *Phys. Rev. E* **52**, 3656 (1995).
- <sup>10</sup>A. Chekhlov and V. Yakhot, "Kolmogorov turbulence in a random-force-driven Burgers equation" *Phys. Rev. E* **51**, R2739 (1995).
- <sup>11</sup>A. Chekhlov and V. Yakhot, "Kolmogorov turbulence in a random-force-driven Burgers equation: Anomalous scaling and probability density functions," *Phys. Rev. E* **52**, 5681 (1995).
- <sup>12</sup>A. M. Polyakov, "Turbulence without pressure," *Phys. Rev. E* **52**, 6183 (1995).
- <sup>13</sup>J.-P. Bouchaud and M. Mézard, "Velocity fluctuations in forced Burgers turbulence," *Phys. Rev. E* **54**, 5116 (1996).
- <sup>14</sup>V. Gurarie and A. Migdal, "Instantons in the Burgers equation," *Phys. Rev. E* **54**, 4908 (1996).
- <sup>15</sup>V. Yakhot and A. Chekhlov, "Algebraic tails of probability functions in the random-force-driven Burgers turbulence," *Phys. Rev. Lett.* **77**, 3118 (1996).
- <sup>16</sup>E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, "Intermittency of Burgers turbulence," *Phys. Rev. Lett.* **78**, 1452 (1997).
- <sup>17</sup>S. A. Boldyrev, "Velocity-difference probability density functions for Burgers turbulence," *Phys. Rev. E* **55**, 6907 (1997).
- <sup>18</sup>W. E, K. Khanin, A. Mazel, and Ya. Sinai, "Probability distribution functions for the random forced Burgers equation" *Phys. Rev. Lett.* **78**, 1904 (1997).
- <sup>19</sup>S. N. Gurbatov, S. I. Simdyankin, E. Aurell, U. Frisch, and G. Tóth, "On the decay of Burgers turbulence," *J. Fluid Mech.* **344**, 339 (1997).
- <sup>20</sup>S. A. Boldyrev, "Burgers turbulence, intermittency, and nonuniversality," *Phys. Plasmas* **5**, 1681 (1998).
- <sup>21</sup>T. Gotoh and R. H. Kraichnan, "Burgers turbulence with large-scale forcing," *Phys. Fluids* **10**, 2859 (1998).
- <sup>22</sup>W. E and E. Vanden Eijnden, "Asymptotic theory for the probability density functions in Burgers turbulence," submitted to *Phys. Rev. Lett.*
- <sup>23</sup>"Formal solutions"  $u_f(x,t) = \sum_k (-1)^{k+1} u^{(k)}(x,t)$  were already introduced in Ref. 1. No conclusion was drawn about the statistics of the Riemann equation in Ref. 1, but it was already acknowledged that approximations that do not account properly for shock creation may spuriously predict properties of Burgers solutions which are, in fact, only observed on solutions of the Riemann equation.