

Some remarks on the quasilinear treatment of the stochastic acceleration problem

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The range of applicability of the quasilinear theory is discussed through the example of the stochastic acceleration of a test particle in a random electric field. The scheme is able to reproduce the non-standard diffusive behaviors observed in direct numerical simulations. © 1997 American Institute of Physics. [S1070-664X(97)01705-9]

The quasilinear theory, developed thirty years ago, is the first attempt to understand plasma turbulence.^{1,2} Since then, it has been extensively used, discussed and, hopefully, improved. The quasilinear theory is successful for weak turbulence but irrelevant when the fluctuations reach high levels of intensity, the description of which requires more sophisticated tools. In this Brief Communication, we restrict ourselves to the strict quasilinear scheme. Our purpose is to rediscuss, through the example of the stochastic acceleration problem, the meaning of the theory. Indeed, in recent publications,^{3,4} the quasilinear scheme was claimed to be inefficient for describing situations where it should be relevant. It appears, however, that this conclusion has been drawn from a misinterpretation of the theory, which we intend to prove in this Brief Communication. It is subdivided in two parts: in the first, we summarize the derivation and the range of applicability of the quasilinear Fokker–Planck equation when applied to the problem of the stochastic acceleration of a test particle in a specified random electric field. This is a well-known matter, but we emphasize the possible pitfalls encountered when expressing the moments of the profile within the formalism. In the second part of the Brief Communication, an explicit example of a random electric field is studied. We show that the quasilinear scheme reproduces the non-standard “diffusive” behaviors (i.e., such that the mean square displacement of a test particle *does not* grow linearly with time) which are observed in the direct numerical simulations.^{3,5}

We investigate the one-dimensional motion of a charged particle in an external electric field $E(x, t)$:

$$\frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = \frac{q}{m} E[x(t), t] \equiv \mathcal{E}[x(t), t]. \quad (1)$$

The electric field \mathcal{E} ($\propto E$) is supposed to be a stationary and homogeneous random function with zero mean and with the following second-order correlation:

$$C(x, t) = \langle \mathcal{E}(x+x', t+t') \mathcal{E}(x', t') \rangle. \quad (2)$$

One also assumes that a decorrelation time τ_c exists, defined by the condition $|C(x, t)| \ll |C(x, 0)|$ if $t > \tau_c$: τ_c measures the time scale on which the random nature of \mathcal{E} is appreciable. Together with an initial condition $x(0) = x_0$, $v(0) = v_0$, Eqs. (1)–(2) define the stochastic process $\{x(t), v(t)\}$. The first problem to cope with when studying Eq. (1) is its non-linear character. Indeed, direct averages of

expressions derived from Eq. (1) involve correlations of \mathcal{E} along the particle orbit (Lagrangian correlations), while the statistics of \mathcal{E} is defined by Eq. (2) for fixed points in space (Eulerian correlations). The solution of Eqs. (1)–(2) requires connecting both of them, which, in general, cannot be made exactly. To put it differently, the average of Eq. (1) generally involves an infinite number of moments of $x(t)$ due to the non-linear dependence of \mathcal{E} on x . This also implies that, *a priori*, one cannot expect to approximate this equation by a finite set of equations for the first few moments of $\{x(t), v(t)\}$. The non-linear equation (1) can however be translated into a linear one by introducing the density $f(x, v; t)$ in the phase space $\{x, v\}$ and going over to the associated Liouville’s equation:

$$\partial_t f + v \partial_x f + \partial_v \mathcal{E} f = 0, \quad (3)$$

where $\partial_t f = \partial f / \partial t$ and so on. For the initial condition $f(x, v; 0) = \delta(x - x_0) \delta(v - v_0)$, the solution of Eq. (3) is $f(x, v; t) = \delta[x - x(t)] \delta[v - v(t)]$, where $\{x(t), v(t)\}$ is the solution of Eq. (1) for the initial condition $\{x_0, v_0\}$. As a result, the ensemble average $\langle f \rangle$ is the probability distribution function of $\{x, v\}$ which specifies the one-time statistics of the phase-space trajectory of the particle. Of course, the introduction of Eq. (3) does not solve the problem because this equation still leads to a hierarchy. Notice however that all the correlations are now Eulerian ones. For obtaining a closed equation for $\langle f \rangle$, we start from the average of Eq. (3) and the equation for the fluctuation $\delta f = f - \langle f \rangle$:

$$\partial_t \langle f \rangle + v \partial_x \langle f \rangle + \partial_v \langle \mathcal{E} \delta f \rangle = 0, \quad (4a)$$

$$\partial_t \delta f + v \partial_x \delta f + \partial_v (\mathcal{E} \langle f \rangle + \mathcal{E} \delta f - \langle \mathcal{E} \delta f \rangle) = 0. \quad (4b)$$

By definition, in the quasilinear regime, the term $\mathcal{E} \delta f - \langle \mathcal{E} \delta f \rangle$ is negligible. The solution of Eq. (4b) with initial condition $\delta f(x, v; 0) = 0$ is then

$$\delta f(x, v; t) = - \int_0^t dt' \mathcal{E}(x - vt', t - t') (\partial_v + t' \partial_x) \times \langle f(x - vt', v; t - t') \rangle.$$

This expression is used for evaluating the term $\langle \mathcal{E} \delta f \rangle$ in Eq. (4a):

$$\begin{aligned}\langle \mathcal{E} \delta f \rangle &= - \int_0^t dt' C(vt', t') (\partial_v + t' \partial_x) \langle f(x - vt', v; t - t') \rangle \\ &\simeq - \int_0^t dt' C(vt', t') (\partial_v + t' \partial_x) \langle f(x, v; t) \rangle.\end{aligned}\quad (5)$$

The last step is justified if $C(vt, t)$ goes to zero before any significant variation of $\langle f \rangle$ occurs, which, roughly speaking, is true if the effect of the noise \mathcal{E} is weak on the time scale τ_c [see Eq. (15) below]. This condition also justifies the neglect of the non-linear term in Eq. (4b) and, hence, it gives the range of validity of the quasilinear theory (see Ref. 2). By injecting Eq. (5) in Eq. (4a), we obtain for $t > \tau_c$,

$$\partial_t \langle f \rangle + v \partial_x \langle f \rangle = \partial_v D_1(v) \partial_x \langle f \rangle + \partial_v D_0(v) \partial_v \langle f \rangle, \quad (6)$$

where

$$D_0(v) = \int_0^\infty dt C(vt, t), \quad D_1(v) = \int_0^\infty dt t C(vt, t). \quad (7)$$

The Fokker–Planck equation (6) is the fundamental result of the quasilinear scheme. It approximates the random dynamics given by Eqs. (1)–(2) by the Markovian process specified by Eq. (6). Again, this is justified since the noise \mathcal{E} “forgets its past” on the time scale τ_c , which allows defining a slower “coarse-grained” time scale for which the process is indeed quasi-Markovian. Let us now take Eq. (6) for granted for all times and analyze its consequences. Integration over x of this equation yields the equation for the velocity distribution $n(v; t) = \int dx \langle f(x, v; t) \rangle$:

$$\partial_t n = \partial_v D_0(v) \partial_v n, \quad (8)$$

with initial condition $n(v; 0) = \delta(v - v_0)$. The function $D_0(v)$ entering this Fokker–Planck equation is the usual quasilinear diffusion coefficient in velocity space. It is worth stressing that no connection between the moments of the density n and $D_0(v)$ is necessary in deriving Eq. (8). Such a connection can now be established as follows; defining the centered moments of the velocity by

$$\langle \Delta v^n(t) \rangle = \int_{-\infty}^{\infty} dv (v - v_0)^n n(v; t), \quad (9)$$

we have

$$\begin{aligned}\frac{d}{dt} \langle \Delta v(t) \rangle|_{t=0} &= \frac{d}{dv} D_0(v)|_{v=v_0}, \\ \frac{d}{dt} \langle \Delta v^2(t) \rangle|_{t=0} &= 2D_0(v_0), \\ \frac{d}{dt} \langle \Delta v^n(t) \rangle|_{t=0} &= 0, \quad \forall n > 2.\end{aligned}\quad (10)$$

This property is readily proved from the relation

$$\partial_t n(v; t)|_{t=0} = \partial_v D_0(v) \partial_v \delta(v - v_0),$$

which one uses in the definition (9). However, Eqs. (10) do not imply a linear relation between $\langle \Delta v^2 \rangle$ and t for finite times:

$$\langle \Delta v^2(t) \rangle \neq 2D_0(v_0)t, \quad \text{for } t > 0. \quad (11)$$

Again, this is obvious since, using Eq. (9) in Eq. (8), $\langle \Delta v^2(t) \rangle$ is shown to obey

$$\frac{d}{dt} \langle \Delta v^2(t) \rangle = 2 \int_{-\infty}^{\infty} dv n(v; t) \frac{d}{dv} [(v - v_0) D_0(v)]. \quad (12)$$

For a velocity-dependent $D_0(v)$, the right hand side of this expression cannot, in general, be evaluated without the explicit knowledge of the density n and, clearly, Eq. (12) may yield a non-linear dependence of $\langle \Delta v^2 \rangle$ on t even for a time-independent diffusion coefficient $D_0(v)$. The same is of course true for any $\langle \Delta v^n \rangle$, which also shows that Eq. (8) cannot be translated into a finite set of equations for the first few moments of n , as already mentioned.

Summarizing, the existence of a diffusion coefficient $D_0(v)$ entering the Fokker–Planck Eq. (8) *does not* imply that the process is diffusive. Of course, the Fokker–Planck equation (8) and, hence, Eqs. (10), only provide a relevant approximation for times larger than τ_c : the exact dynamics requires $\Delta v \propto t$ if $t \ll \tau_c$ and, hence, the time-derivative of $\langle \Delta v^2 \rangle$ is zero at the initial time. Consequently, the exact mean square displacement of a test particle obtained from experiments or simulations needs neither obey Eq. (10) for $t < \tau_c$ nor be diffusive for $t > \tau_c$. The identification at time $t > 0$ between $D_0(v)$ and half the time derivative of $\langle \Delta v^2 \rangle$ may thus be illicit in any case and lead to the wrong conclusion that the quasilinear scheme breaks down even within its range of applicability. Such a pitfall is well known but seems to be responsible for some confusion in the literature.^{3–5}

For the purpose of illustration, we now consider an explicit form for the correlation (2):

$$C(x, t) = \alpha^2 e^{-x^2/4\lambda^2} \cos(kx - \omega t).$$

This correlation has been used to describe a weak-beam plasma instability^{3,5} and corresponds to a monochromatic wave-packet with frequency ω , wave number k and spectrum width λ ; $\alpha = q \langle E^2 \rangle^{1/2} / m$ measures the intensity of the fluctuation. Rescaling $\omega t \rightarrow t$, $kx \rightarrow x$ and $k v / \omega \rightarrow v$, it gives

$$C(x, t) = \sigma^2 e^{-x^2/4\epsilon^2} \cos(x - t), \quad (13)$$

where $\sigma = \alpha k / \omega^2$ and $\epsilon = \lambda k$. For identifying the correlation time τ_c , the calculations should be performed on the new variable $z = x - t$. From Eq. (13), one can then estimate $\tau_c \simeq \epsilon$. The explicit form of the quasilinear diffusion coefficient is obtained by using Eq. (13) in Eq. (7). In the original variables $\{x, v\}$, one obtains

$$D_0(v) = \sqrt{\pi} \sigma^2 \frac{\epsilon}{|v|} e^{-\epsilon^2(1-1/v)^2}. \quad (14)$$

The validity of the quasilinear approximation requires that the profile n does not vary significantly on the time scale $\tau_c = \epsilon$. An order of magnitude estimation based on Eqs. (8) and (14) yields, after some algebra,

$$(\sigma \epsilon)^2 \ll 1. \quad (15)$$

Equation (14) implies that $v = 0$ is a natural boundary for the problem and that no transition between the ranges $v > 0$ and $v < 0$ is possible. For D_0 given by Eq. (14), the evaluation of the solution of Eq. (8) requires numerical integration; it is

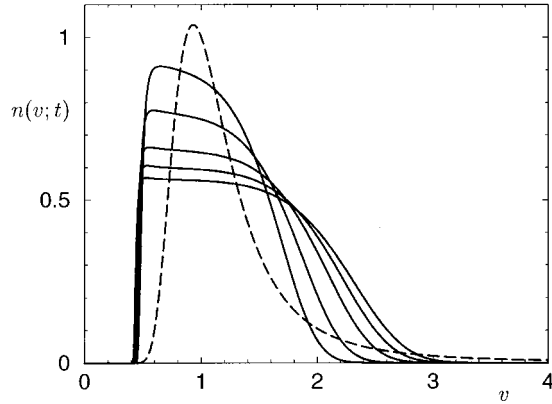


FIG. 1. The velocity profile $n(v; t)$ is plotted for successive times $t/2\pi = 5, 10, 20, 30, 40$ (solid line). The dashed line is $D_0(v)/d_0$.

represented in Figs. 1 and 2 for $v_0 = 0.931$, $\sigma = 0.05$ and $\epsilon = 2.5$ [$(\sigma\epsilon)^2 \approx 0.016$]. Clearly, the velocity profile is far from Gaussian, its spreading is slow and, consequently, the second moment $\langle \Delta v^2 \rangle$ grows slower than linearly with time. These features are observed in the direct numerical simulations of Eq. (1) (see Refs. 3 and 5). An analytical approximation of the solution of Eq. (8) is obtained by the WKB method. Defining

$$z(v) = \int_{v_0}^v du \sqrt{u} e^{\epsilon^2(1-1/u)^2/2}, \quad (16)$$

the zeroth order WKB approximation of n is

$$n(v; t) = c(d_0 t)^{-1/3} e^{-z^2(v)/[4d_0 t]}, \quad (17)$$

where $d_0 = \sqrt{\pi} \sigma^2 \epsilon$ and c is a normalization constant. The solution (17) may be used either for $v > 0$ if $v_0 > 0$ or for $v < 0$ if $v_0 < 0$; n is zero if v and v_0 have opposite signs. For $|v| \gg 1$, one has $D_0(v) \sim d_0 e^{-\epsilon^2/|v|}$ and Eq. (17) reduces to the asymptotic solution of Eq. (8),

$$n(v; t) \sim \frac{c}{(d_0 t)^{1/3}} \exp\left(-\frac{|v|^3}{9d_0 e^{-\epsilon^2} t}\right), \quad |v| \gg 1. \quad (18)$$

For very long times, i.e. for $d_0 e^{-\epsilon^2} t \gg 1$, the main contribution to the moments comes from the tail of n and, hence, Eq. (18) can be safely used for their evaluation:

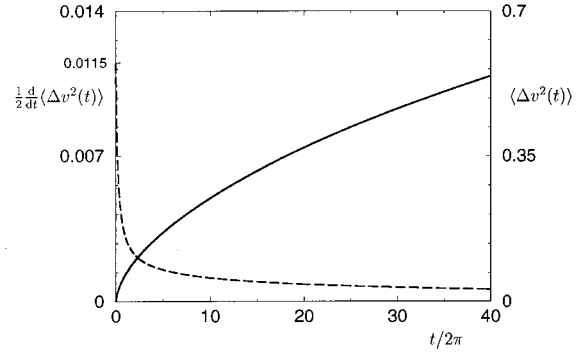


FIG. 2. The evolution with time of the second moment of n (solid line) and half its time derivative (dashed line). The $\langle \Delta v^2(t) \rangle$ evaluated from the WKB solution (17) is also represented but cannot be distinguished from the exact one. The initial value of $\frac{1}{2} \frac{d}{dt} \langle \Delta v^2(t) \rangle$ is $D_0(v_0)$.

$$\langle \Delta v^n(t) \rangle \propto [\text{sgn}(v_0)]^n (d_0 t)^{n/3}, \quad d_0 e^{-\epsilon^2} t \gg 1. \quad (19)$$

To summarize, the use of the quasilinear approximation is of course restricted to situations with a small turbulence level. However, when applicable, it does not imply the Gaussianity of the profile or the linear growth with time of $\langle \Delta v^2 \rangle$. Indeed, the quasilinear Fokker–Planck equation can reproduce non-usual “diffusion” features. In this kinetic picture, the mean square displacement of the particle or the profile itself must be compared with the data obtained from direct numerical simulations. On the contrary, the quasilinear diffusion coefficient is, in general, not related to simple observable quantities (except when it is constant).

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