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Turbulent Prandtl number effect on passive scalar advection

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Abstract

A generalization of Kraichnan's model of passive scalar advection is considered. Physically motivated regularizations of the model are considered which take into account both the effects of viscosity and molecular diffusion. The balance between these two effects on the inertial range behavior for the scalar is shown to be parameterized by a new turbulent Prandtl number. Three different regimes are identified in the parameter space depending on degrees of compressibility. In the strongly and weakly compressible regimes, the inertial range behavior of the scalar does not depend on the turbulent Prandtl number. In the regime of intermediate compressibility, the inertial range behavior does depend on the turbulent Prandtl number. © 2001 Published by Elsevier Science B.V.

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Kraichnan's model for the passive scalar advection [1] has become a popular benchmark in the studies of intermittency in hydrodynamic turbulence. In this model, one studies the behavior of a scalar field, $\theta(x,t)$, passively advected by a turbulent velocity, $u^\nu(x,t)$, and subject to molecular diffusion

$$\frac{\partial\theta}{\partial t}+(u^\nu(x,t)\cdot\nabla)\theta=\kappa\Delta\theta. \tag{1}$$

The velocity field is assumed to be a zero mean Gaussian random process, white-in-time, isotropic, and such that

$$\mathbf{E}(\delta u^\nu(x,y,t)\cdot\delta u^\nu(x,y,s))=D|x-y|^\xi\delta(t-s)\quad\text{for } \ell_\nu\ll|x-y|\ll\ell_0, \tag{2}$$

where $\delta u^\nu(x,y,t)=u^\nu(x,t)-u^\nu(y,t)$ and $0<\xi<2$. Here ℓ_0 is the integral scale (or correlation length) for u^ν, and ℓ_ν is the viscous length scale below which the velocity becomes smooth and dominated by viscous effects:

$$\mathbf{E}(\delta u^\nu(x,y,t)\cdot\delta u^\nu(x,y,s))=D\ell_\nu^{\xi-2}|x-y|^2\delta(t-s)\quad\text{for } |x-y|\ll\ell_\nu. \tag{3}$$

The Gaussian nature of u^ν and its white-in-time character are simplifying assumptions that make the Kraichnan model tractable. In contrast, the spatial dependence of u^ν is non-trivial and more realistic of real turbulent velocity fields. Indeed, interesting behavior for the scalar occurs when $0<\xi<2$ when u^ν is non-smooth and is only Hölder continuous on the inertial range of scales $\ell_\nu\ll|x-y|\ll\ell_0$. This is precisely the case of interest for fully developed turbulent velocity fields for which Kolmogorov's argument suggests $\xi=\frac{1}{3}$. In fact, the non-smoothness of u^ν is responsible for intermittency corrections in the behavior of the scalar θ, as we explain below.

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For some of the regimes discussed below, the transport equation (1) does not have a statistical steady state in the presence of forcing. Therefore, we will focus on the decaying situation. For simplicity, we will assume that the initial condition, $\theta(x,0)=\theta_0(x)$, is a zero-mean, isotropic Gaussian random process, independent of u^ν, and with covariance

$$\mathbf{E}_0(\theta_0(x)\theta_0(y)) = B(|x-y|), \tag{4}$$

where $B(r)$ is smooth and tends rapidly to 0 for $r \gg L$. The length L is the integral scale for the scalar field θ and typically one has $L \ll \ell_0$.

As usual, we are interested in the behavior of the structure functions of arbitrary order n, $S_n(r,t)$, defined as

$$S_n(r=|x-y|,t) = \mathbf{E}|\theta(x,t)-\theta(y,t)|^n \tag{5}$$

in the inertial range of scales for the passive scalar defined as [1]

$$\max(\ell_\nu, \ell_\kappa) \ll r \ll L \quad \text{(inertial range)}. \tag{6}$$

Here ℓ_κ is the diffusive length scale defined as $\ell_\kappa = (\kappa/D)^{1/\xi}$.

The parameters are ξ, which is dimensionless, D, with dimension [length]$^{2-\xi}$[time]$^{-1}$, $B_0=B(0)$, with dimension [temperature]^2, and the various lengths ℓ_ν, ℓ_κ, L. Normal scaling means that the behavior of the structure functions in the inertial range is independent of ℓ_ν, ℓ_κ, and $1/L$ to leading order in these parameters. Dimensional analysis then gives

$$S_n(r,t) = B_0^{n/2} f_n\left(\frac{r^{2-\xi}}{Dt}\right) \quad \text{(normal scaling)}, \tag{7}$$

where f_n's are dimensionless functions. However, it was shown by several groups [2–5] that the normal scaling in (7) does not hold for the Kraichnan model. More precisely, to leading order in ℓ_ν, ℓ_κ, and $1/L$,

the structure functions depend on L in the inertial range. (8)

This makes the Kraichnan model a relevant example for the study of intermittency. Notice that due to (8), the scaling of structure functions cannot be obtained by dimensional analysis.

We will consider a generalization of the Kraichnan model due to Gawędzki and Vergassola [6] (see also [7]) where the velocity is allowed to be compressible. As in the original Kraichnan model, u^ν is assumed to be a zero mean Gaussian random process with covariance

$$\mathbf{E}u^\nu_\alpha(x,t)u^\nu_\beta(y,s) = (C_0\delta_{\alpha\beta} - c_{\alpha\beta}(x-y))\delta(t-s). \tag{9}$$

Compressibility is now incorporated into the model by taking $c_{\alpha\beta}$ as

$$c_{\alpha\beta}(x) = Ac^P_{\alpha\beta}(x) + Bc^S_{\alpha\beta}(x), \tag{10}$$

where to leading order

$$c^P_{\alpha\beta}(x) = D\left(\delta_{\alpha\beta} + \xi\frac{x_\alpha x_\beta}{|x|^2}\right)|x|^\xi, \qquad c^S_{\alpha\beta}(x) = D\left((d+\xi-1)\delta_{\alpha\beta} - \xi\frac{x_\alpha x_\beta}{|x|^2}\right)|x|^\xi \tag{11}$$

[1] In the so-called Batchelor regime, one typically assumes that $\ell_\kappa \ll \ell_\nu$ and studies the behavior of the structure functions in the range $\ell_\kappa \ll r \ll \ell_\nu$ where the velocity is smooth. We will not consider this case here.

for $\ell_\nu \ll |x| \ll \ell_0$. Here d is the spatial dimension. The dimensionless parameters A and B measure the divergence and rotation of the field u^ν. $A = 0$ corresponds to incompressible fields with $\nabla \cdot u^\nu = 0$. $B = 0$ corresponds to irrotational fields with $\nabla \times u^\nu = 0$. Following Ref. [6], we characterize compressibility by introducing

$$\mathcal{P} = \frac{\mathcal{C}^2}{\mathcal{S}^2}, \quad \mathcal{S}^2 = A + (d-1)B,\ \mathcal{C}^2 = A. \tag{12}$$

$\mathcal{P} = 0$ when the velocity is incompressible and $\mathcal{P} = 1$ when it is irrotational.

Gawędzki and Vergassola [6] studied the situation when $\ell_\nu = 0$ and identified two different regimes which can already be seen at the level of S_2 (see also [7]):

1. When $\mathcal{P} \geq d/\xi^2$, corresponding to a regime of weak compressibility, one has, to leading order in $\ell_\kappa,\ 1/L$,

$$S_2(r,t) = C\frac{B_0 r^{2-\xi}}{Dt} \quad \text{for } \ell_\nu = 0,\ \ell_\kappa \ll r \ll L. \tag{13}$$

   Here and in the formulas below, C is a generic numerical constant. (21) is normal scaling.

2. In contrast, when $\mathcal{P} < d/\xi^2$, corresponding to a regime of stronger compressibility, one has to leading order in $\ell_\kappa,\ 1/L$,

$$S_2(r,t) = C\frac{B_0 L^{2-\xi}}{Dt}\left(\frac{r}{L}\right)^\zeta \quad \text{for } \ell_\nu = 0,\ \ell_\kappa \ll r \ll L, \tag{14}$$

   where

$$\zeta = \frac{2 - d - \xi + 2\xi\mathcal{P}}{1 + \xi\mathcal{P}}. \tag{15}$$

   The scaling in (21) is anomalous.

The main purpose of the present paper is to study the effect of the simultaneous presence of viscosity and molecular diffusion. We account for the effect of viscosity by assuming that the tensors $c^P_{\alpha\beta}(x)$ and $c^S_{\alpha\beta}(x)$ entering the covariance of u^ν behave for $|x| \ll \ell_\nu$ as

$$c^P_{\alpha\beta}(x) = D\ell_\nu^{\xi-2}\left(\delta_{\alpha\beta} + 2\frac{x_\alpha x_\beta}{|x|^2}\right)|x|^2, \qquad c^S_{\alpha\beta}(x) = D\ell_\nu^{\xi-2}\left((d+1)\delta_{\alpha\beta} - 2\frac{x_\alpha x_\beta}{|x|^2}\right)|x|^2. \tag{16}$$

Thus, $D\ell_\nu^\xi$ can be identified as the dynamic viscosity ν, and taking the limit as $\ell_\nu \to 0$ amounts to letting $\nu \to 0$.

The standard way to measure the relative strength of viscous and diffusive effects is through the Prandtl number. In the present model, the Prandtl number is given by

$$Pr = \frac{\nu}{\kappa} = \frac{D\ell_\nu^\xi}{\kappa} = \left(\frac{\ell_\nu}{\ell_\kappa}\right)^\xi. \tag{17}$$

Pr is the only non-dimensional parameter one can construct based on D, ℓ_ν and κ. However, it turns out for the present model that the relative strength of viscous and diffusive effects must be characterized by a different Prandtl number which we shall refer to as the *turbulent Prandtl number*:

$$Pr^{\mathrm{T}} = \frac{D\ell_\nu^{2-\xi+2\alpha}}{\kappa L^{2-2\xi+2\alpha}} = Pr\left(\frac{\ell_\nu}{L}\right)^{2-2\xi+2\alpha}, \tag{18}$$

where

$$\alpha = \frac{d - 1 + \xi - \xi\mathcal{P}}{1 + \xi\mathcal{P}}. \tag{19}$$

As shown below, in the range of parameters where the behavior of S_2 depends on Pr^{T}, we have $\xi - 1 < \alpha < 1$, i.e. $Pr^{\mathrm{T}} \ll Pr$ since $\ell_\nu \ll L$ (Pr^{T} tends to Pr as $\alpha \to \xi - 1$).

The turbulent Prandtl number Pr^{T} has the following interpretation. Let $\tau^{\nu,\kappa}$ be the average time it takes for two particles to be separated by distance L if their initial distance is zero, and decompose $\tau^{\nu,\kappa}$ as $\tau^{\nu,\kappa} = \tau_1^{\nu,\kappa} + \tau_2^{\nu,\kappa}$, where $\tau_1^{\nu,\kappa}$ (resp. $\tau_2^{\nu,\kappa}$) is the amount of time during which the distance between the two particles is less (resp. more) than the viscous length scale ℓ_ν during the separation process. The relative strength of viscous and diffusive effects can be then characterized by the ratio $\tau_1^{\nu,\kappa}/\tau_2^{\nu,\kappa}$: the latter turns out to be proportional to the square root of Pr^{T}.

We identify three different regimes according to their degree of compressibility:

1. In the weakly compressible regime when

$$\mathcal{P} \le \frac{d + \xi - 2}{2\xi}, \tag{20}$$

the scaling of S_2 is, to leading order in ℓ_ν, ℓ_κ, $1/L$ given by

$$S_2(r,t) = C\frac{B_0 r^{2-\xi}}{Dt} \quad \text{for} \max(\ell_\nu, \ell_\kappa) \ll r \ll L. \tag{21}$$

2. In the strongly compressible regime when

$$\mathcal{P} \ge \frac{d}{\xi^2}, \tag{22}$$

the scaling of S_2 is, to leading order in ℓ_ν, ℓ_κ, $1/L$ given by

$$S_2(r,t) = C\frac{B_0 L^{2-\xi}}{Dt}\left(\frac{r}{L}\right)^{\zeta} \quad \text{for} \max(\ell_\nu, \ell_\kappa) \ll r \ll L, \tag{23}$$

where ζ is given by (15).

3. In the intermediate regime when

$$\frac{d + \xi - 2}{2\xi} < \mathcal{P} < \frac{d}{\xi^2}, \tag{24}$$

the scaling of S_2 depends on the turbulent Prandtl number in (18). More precisely, if

$$Pr^{\mathrm{T}} \ll 1, \tag{25}$$

S_2 scales as in (23), whereas if

$$Pr^{\mathrm{T}} \text{ is of order one, or } Pr^{\mathrm{T}} \gg 1, \tag{26}$$

S_2 scales as in (21), with C depending on the precise value of Pr^{T}.

A more precise formulation of this result is given in Proposition 1. The corresponding phase diagrams are shown in Fig. 1. The different scalings in (21) or in (23) for S_2 are related to different types of *generalized flows* that can be associated with the transport equation (1). Generalized flows for passive scalar were introduced in Ref. [8] and will be discussed in more details elsewhere. Essentially, a generalized flow is a family of probability distribution functions for the trajectories of n test particles advected by the velocity field u^ν and subject to molecular diffusion, in the limit where the regularization parameters, ℓ_ν and ℓ_κ, are both taken to zero. In this limit, the family of probability distribution functions exhibits properties of branching or coalescence between the test particle trajectories, related to the non-Lipschitz character of the velocity u^ν in the limit as $\ell_\nu \to 0$ and not observed for standard flows; these

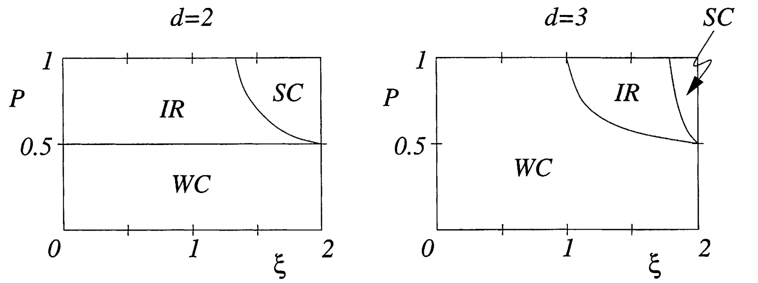

Fig. 1. Phase diagrams for the three regimes (WC: weakly compressible regime; IR: intermediate regime; SC: strongly compressible regime).

properties are formulated in a more precise way in terms of the pair distance probability density function in (29) and (30). Branching is associated with the scaling in (21), whereas coalescence yields the scaling in (23). The above classification is essentially equivalent to the property that the limiting generalized flow depends on the way the regularization parameters are removed, i.e. the way the limit as $\ell_\nu, \ell_\kappa \to 0$ is taken. Our proof of the above classification for S_2 essentially amounts to studying the behavior of the probability density function for the pair distance between two particles.

We now turn to more precise statement for the classification given earlier. It is easy to see that

$$S_2(r,t) = 2\int_0^\infty B(r')(P^{\nu,\kappa}(0|r',t) - P^{\nu,\kappa}(r|r',t))\,\mathrm{d}r'. \tag{27}$$

Here

$$\int_{r_1}^{r_2} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = \mathrm{Prob}\{|\varphi_t(x) - \varphi_t(y)| \in (r_1, r_2]\}, \tag{28}$$

where $|x-y| = \rho > 0$ and φ_t satisfies

$$\mathrm{d}\varphi_t(x) = u^\nu(\varphi_t(x),t)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\beta(t), \qquad \varphi_0(x) = x,$$

where β is a Wiener process. In other words, $P^{\nu,\kappa}(\rho|r,t)$ is the probability density function that the distance between two test particles is r at time t if it was ρ initially. Thus, to understand the behavior of S_2 in the inertial range, it is crucial to understand the behavior of $P^{\nu,\kappa}$ in the limit as $\ell_\nu, \ell_\kappa \to 0$ (or, equivalently, as $\nu, \kappa \to 0$). We will establish the following proposition.

Proposition 1. *Let $P^{\nu,\kappa}(\rho|r,t)$ be defined as in* (28). *We have*:

1. *In the weakly compressible regime when* (20) *is satisfied, for any fixed $\rho, t > 0$,*

$$\lim_{\nu,\kappa\to 0} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = P(\rho|r,t)\,\mathrm{d}r,$$

weakly as measures. The limiting measure is absolutely continuous with respect to the Lebesgue measure and P satisfies

$$\lim_{\rho\to 0+} P(\rho|r,t) > 0 \quad \textit{for}\, r > 0,\ t > 0. \tag{29}$$

2. *In the strongly compressible regime when* (22) *is satisfied*,

$$\lim_{\nu,\kappa\to 0} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = P(\rho|r,t)\,\mathrm{d}r,$$

weakly as measures. Moreover,

$$P(\rho|r,t)\,\mathrm{d}r = A(\rho,t)\delta(r) + \bar{P}(\rho|r,s)\,\mathrm{d}r, \tag{30}$$

where $A(\rho,s) = 1 - \int_0^\infty \bar{P}(\rho|r,s)\,\mathrm{d}r > 0$. $\bar{P}$ *is integrable and satisfies*

$$\lim_{\rho\to 0+} \bar{P}(\rho|r,t) = 0 \quad \textit{for}\, r > 0,\ t > 0. \tag{31}$$

3. *In the intermediate regime when* (24) *is satisfied, we must distinguish three situations. For any fixed constant* $C > 0$,

$$\lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to 0}} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = P_1(\rho|r,t)\,\mathrm{d}r, \tag{32}$$

$$\lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to C}} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = P_2(\rho|r,t)\,\mathrm{d}r, \tag{33}$$

$$\lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to \infty}} P^{\nu,\kappa}(\rho|r,t)\,\mathrm{d}r = P_3(\rho|r,t)\,\mathrm{d}r, \tag{34}$$

weakly as measures. P_1 *satisfies* (29), *and the measure* $P_1\,\mathrm{d}r$ *is absolutely continuous with respect to the Lebesgue measure*. P_2 *depends on* C, *and satisfies both* (29) *and* (30). P_3 *satisfies both* (30) *and* (31).

The properties in (29) and (30), respectively, reflect the branching or the coalescence behaviors of the generalized flow. In contrast, the property in (31) reflects the absence of branching, and the absolute continuity of the distribution associated with P with respect to the Lebesgue measure reflects the absence of coalescence. Notice that, in the intermediate regime, if there is coalescence in the sense of (30), then it dominates the scaling of S_2. This explains that the scaling is the same if $\lim_{\nu,\kappa\to 0} Pr^{\mathrm{T}} = C \in (0,\infty)$ or $\lim_{\nu,\kappa\to 0} Pr^{\mathrm{T}} = \infty$.

Proof. Because the velocity field in Kraichnan model is an isotropic white-noise, the density $P^{\nu,\kappa}$ satisfies a Fokker–Planck equation given explicitly by

$$\frac{\partial P^{\nu,\kappa}}{\partial t} = -\frac{\partial}{\partial r}(b^{\nu,\kappa}(r)P^{\nu,\kappa}) + \frac{\partial^2}{\partial r^2}(a^{\nu,\kappa}(r)P^{\nu,\kappa}), \tag{35}$$

where

$$a^{\nu,\kappa}(r) = \kappa + a^{\nu}(r), \qquad b^{\nu,\kappa}(r) = (d-1)\kappa r^{-1} + b^{\nu}(r), \tag{36}$$

and a^ν, b^ν behave for $\ell_\nu \ll r \ll \ell_0$ outside the viscous layer as

$$a^{\nu}(r) = a(r)\left(1 + \mathrm{O}\left(\frac{r}{\ell_0}\right) + \mathrm{O}\left(\frac{\ell_\nu}{r}\right)\right), \qquad b^{\nu}(r) = b(r)\left(1 + \mathrm{O}\left(\frac{r}{\ell_0}\right) + \mathrm{O}\left(\frac{\ell_\nu}{r}\right)\right) \tag{37}$$

with

$$a(r) = D(\mathcal{S}^2 + \xi\mathcal{C}^2)r^{\xi}, \qquad b(r) = D((d-1+\xi)\mathcal{S}^2 - \xi\mathcal{C}^2)r^{\xi-1}, \tag{38}$$

and for $r \ll \ell_\nu$ inside the viscous layer as

$$a^{\nu}(r) = D\ell_\nu^{\xi-2}(\mathcal{S}^2 + 2\mathcal{C}^2)r^2\left(1 + \mathrm{O}\left(\frac{r}{\ell_\nu}\right)\right), \qquad b^{\nu}(r) = D\ell_\nu^{\xi-2}((d+1)\mathcal{S}^2 - 2\mathcal{C}^2)r\left(1 + \mathrm{O}\left(\frac{r}{\ell_\nu}\right)\right). \tag{39}$$

We notice now that if the limiting P is well defined it must satisfies the equation obtained by setting $\nu, \kappa \to 0$ in (35). Thus,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial r}(b(r)P) + \frac{\partial}{\partial r^2}(a(r)P). \tag{40}$$

This equation makes sense for $r \in (0, \infty)$ but is singular at $r = 0$. Identifying P as a suitable limit of $P^{\nu,\kappa}$ amounts to classifying the boundary $r = 0$ for Eq. (40). This is a well-known problem in probability theory [9] and a complete classification of the boundary $r = 0$ can be given as follows. Let

$$A(r) = \exp\left(-\int_\beta^r \frac{b(\rho)}{a(\rho)}\,\mathrm{d}\rho\right), \tag{41}$$

where $\beta > 0$ is arbitrary. We have

1. If A and $(Aa)^{-1}$ are integrable at $r = 0$, $r = 0$ is a *regular* boundary.
2. If A is not integrable at $r = 0$, but $(Aa)^{-1}\int_\beta^r A\,\mathrm{d}\rho$ is integrable, $r = 0$ is an *entrance* boundary.
3. If $(Aa)^{-1}$ is not integrable at $r = 0$, but $A\int_\beta^r (Aa)^{-1}\,\mathrm{d}\rho$ is integrable, $r = 0$ is an *exit* boundary.
4. In all the other cases, $r = 0$ is a *natural* boundary.

A boundary condition at $r = 0$ is required and only allowed if $r = 0$ is a regular boundary. The condition may be an absorbing condition for which

$$\lim_{r\to 0+} a(r)A(r)P = 0, \tag{42}$$

a reflecting condition for which

$$\lim_{r\to 0+}\left(-b(r)P + \frac{\partial}{\partial r}(a(r)P)\right) = 0, \tag{43}$$

or a linear combination of the two (mixed condition).

For Eq. (35), we obtain

$$A(r) = Cr^{-\alpha}, \qquad (a(r)A(r))^{-1} = \bar{C}r^{\alpha-\xi}, \tag{44}$$

where α is given by (19). By analyzing the integrability of the functions in (44) according to the classification given above, we have

1. In the weakly compressible regime, $r = 0$ is an entrance boundary where no boundary condition is allowed. It follows then that $\lim_{\nu,\kappa\to 0} P^{\nu,\kappa} = P$ is well defined on every subsequences $\nu, \kappa \to 0$ (i.e. P is independent of Pr^{T}). It also follows from Feller's theory [9] that the distribution associated with P is absolutely continuous with respect to the Lebesgue measure and P satisfies (29).
2. In the strongly compressible regime, $r = 0$ is an exit boundary where no boundary condition is allowed. Again $\lim_{\nu,\kappa\to 0} P^{\nu,\kappa} = P$ is well defined on every subsequences. It also follows from Feller's theory [9] that P satisfies (30) and (31).
3. In the intermediate regime, $r = 0$ is a regular boundary where a boundary condition is required. In this case, P is well defined only if the limit as $\nu, \kappa \to 0$ is taken on subsequences $\nu, \kappa \to 0$ where Pr^{T} is appropriately constrained, which specifies the effective boundary condition at $r = 0$ for Eq. (40), as we show now.

In the intermediate regime, we shall obtain the type of boundary condition at $r = 0$ for the limiting equation for P by studying the behavior as $\nu, \kappa \to 0$ of the average time, $\tau^{\nu,\kappa}$, it takes for two particles to be separated by a *finite* (i.e. independent of ℓ_ν, ℓ_κ) distance d_1 if their initial distance is zero; since the integral scale L is the only length scale besides ℓ_ν, ℓ_κ in the model, it is natural to take $d_1 = L$. We shall compare the limit of $\tau^{\nu,\kappa}$ as $\nu, \kappa \to 0$ to the average time τ_R it takes for two particles to be separated by L if their initial distance is zero for the process associated with the limit equation (40) when the latter is solved with a reflecting boundary condition at $r = 0$. By definition of the various types of boundary conditions at $r = 0$ that are allowed it follows indeed that

1. If the ratio $\tau^{\nu,\kappa}/\tau_R$ tends to 1 as $\nu, \kappa \to 0$, we obtain a reflecting boundary condition at $r = 0$.
2. If this ratio tends to a finite constant $C \in (1, \infty)$, we obtain a mixed boundary condition.
3. If this ratio tends to infinity, we obtain an absorbing boundary condition.

We shall show now that the ratio $\tau^{\nu,\kappa}/\tau_R$ behaves as $1 + \mathrm{O}((Pr^{\mathrm{T}})^{1/2})$ as $\nu, \kappa \to 0$ consistent with our above classification and the results in (32)–(34).

It is a standard result in probability theory that $\tau^{\nu,\kappa} \equiv T^{\nu,\kappa}(0)$, where $T^{\nu,\kappa}(r)$ satisfies

$$b^{\nu,\kappa}(r)\frac{\mathrm{d}T^{\nu,\kappa}}{\mathrm{d}r} + a^{\nu,\kappa}(r)\frac{\mathrm{d}^2T^{\nu,\kappa}}{\mathrm{d}r^2} = -1 \tag{45}$$

for the boundary conditions

$$\left.\frac{\mathrm{d}T^{\nu,\kappa}}{\mathrm{d}r}\right|_{r=0} = 0, \quad T^{\nu,\kappa}(L) = 0. \tag{46}$$

This gives

$$\tau^{\nu,\kappa} = \int_0^L (a^{\nu,\kappa}(r)A^{\nu,\kappa}(r))^{-1}\int_r^L A^{\nu,\kappa}(r')\,\mathrm{d}r'\,\mathrm{d}r, \tag{47}$$

where $A^{\nu,\kappa}$ is defined as in (41) (for convenience we take $\beta = L$)

$$A^{\nu,\kappa}(r) = \exp\left(\int_r^L \frac{b^{\nu,\kappa}(\rho)}{a^{\nu,\kappa}(\rho)}\,\mathrm{d}\rho\right). \tag{48}$$

The integrals involved in (47) exist because for $r \ll \ell_\kappa$, we have

$$A^{\nu,\kappa}(r) = C^{\nu,\kappa}r^{-d+1} + \mathrm{o}(r^{-d+1}), \qquad (a^{\nu,\kappa}(r)A^{\nu,\kappa}(r))^{-1} = \bar{C}^{\nu,\kappa}r^{d-1} + \mathrm{o}(r^{d-1}). \tag{49}$$

(This implies that $r = 0$ is an entrance boundary for the regularized equation (35) for $d > 1$, and a regular boundary for $d = 1$.) Let us decompose

$$\tau^{\nu,\kappa} = \tau_1^{\nu,\kappa} + \tau_2^{\nu,\kappa}, \tag{50}$$

where

$$\tau_1^{\nu,\kappa} = \int_0^{\ell_\nu} (a^{\nu,\kappa}(r)A^{\nu,\kappa}(r))^{-1}\int_r^L A^{\nu,\kappa}(r')\,\mathrm{d}r'\,\mathrm{d}r, \qquad \tau_2^{\nu,\kappa} = \int_{\ell_\nu}^L (a^{\nu,\kappa}(r)A^{\nu,\kappa}(r))^{-1}\int_r^L A^{\nu,\kappa}(r')\,\mathrm{d}r'\,\mathrm{d}r. \tag{51}$$

The time $\tau_1^{\nu,\kappa}$ (resp. $\tau_2^{\nu,\kappa}$) is the amount of time during which the distance between the two particles is less (resp. more) than the viscous length scale ℓ_ν during the separation process.

We now estimate the behaviors of $\tau_1^{\nu,\kappa}$ and $\tau_2^{\nu,\kappa}$ as $\nu, \kappa \to 0$. We denote by $f^{\nu,\kappa} \sim g^{\nu,\kappa}$ if $f^{\nu,\kappa}/g^{\nu,\kappa} \to 1$ as $\nu, \kappa \to 0$. From (51), we have

$$\tau_1^{\nu,\kappa} \sim \frac{L^{1-\alpha}\ell_\nu^\alpha}{1-\alpha}\int_0^{\ell_\nu} (\bar{a}^{\nu,\kappa}(r))^{-1}\exp\left(-\int_r^{\ell_\nu}\frac{\bar{b}^{\nu,\kappa}(r')}{\bar{a}^{\nu,\kappa}(r')}\,\mathrm{d}r'\right)\,\mathrm{d}r, \tag{52}$$

where

$$\bar{a}^{\nu,\kappa}(r) = \kappa + D\ell_\nu^{\xi-2}(\mathcal{S}^2 + 2\mathcal{C}^2)r^2, \qquad \bar{b}^{\nu,\kappa}(r) = \kappa(d-1)r^{-1} + D\ell_\nu^{\xi-2}((d+1)\mathcal{S}^2 - 2\mathcal{C}^2)r. \tag{53}$$

It follows that

$$\tau_1^{\nu,\kappa} \sim \frac{A_1^{\nu,\kappa} L^{1-\alpha}\ell_\nu^\alpha}{1-\alpha}\left(\frac{\ell_\nu^{2-\xi}}{D\kappa}\right)^{1/2} \tag{54}$$

with

$$A_1^{\nu,\kappa} = \int_0^{Pr^{1/2}} (1 + (\mathcal{S}^2 + 2\mathcal{C}^2)z^2)^{-1} \exp\left(-\int_z^{Pr^{1/2}} \frac{(d-1) + ((d+1)\mathcal{S}^2 - 2\mathcal{C}^2)z'^2}{1 + (\mathcal{S}^2 + 2\mathcal{C}^2)z'^2}\frac{\mathrm{d}z'}{z'}\right)\mathrm{d}z. \tag{55}$$

It is easy to check that

$$A_1^{\nu,\kappa} = \mathrm{O}(Pr^{1/2}) \quad \text{as } Pr \to 0, \qquad \lim_{Pr\to\infty} A_1^{\nu,\kappa} \in (0,\infty). \tag{56}$$

On the other hand, it can be verified by direct calculation that in the limit as $\nu, \kappa \to 0$,

$$\tau_2^{\nu,\kappa} \to \tau_R = \frac{L^{2-\xi}}{D(\mathcal{S}^2 + \xi\mathcal{C}^2)(1-\xi+\alpha)(2-\xi)}, \tag{57}$$

where α is given by (19).

Combining the above expressions, we obtain that

$$\frac{\tau^{\nu,\kappa}}{\tau_R} \sim 1 + \bar{C}^{\nu,\kappa}\left(\frac{D\ell_\nu^{2-\xi+2\alpha}}{\kappa L^{2-2\xi+2\alpha}}\right)^{1/2} = 1 + \bar{C}^{\nu,\kappa}(Pr^{\mathrm{T}})^{1/2} \tag{58}$$

with $\bar{C}^{\nu,\kappa}$ given by

$$\bar{C}^{\nu,\kappa} = \frac{A_1^{\nu,\kappa}(1-\xi+\alpha)(2-\xi)(\mathcal{S}^2 + \xi\mathcal{C}^2)}{1-\alpha}. \tag{59}$$

Since $Pr \to C \in [0,\infty)$ if $Pr^{\mathrm{T}} \to 0$ and $Pr \to \infty$ if $Pr^{\mathrm{T}} \to (0,\infty]$, it follows using the properties of $A_1^{\nu,\kappa}$ that

$$\lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to 0}} \frac{\tau^{\nu,\kappa}}{\tau_R} = 1, \qquad \lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to C}} \frac{\tau^{\nu,\kappa}}{\tau_R} \in (0,\infty), \qquad \lim_{\substack{\nu,\kappa\to 0\\ Pr^{\mathrm{T}}\to \infty}} \frac{\tau^{\nu,\kappa}}{\tau_R} = \infty, \tag{60}$$

where $C \in (0,\infty)$. This concludes the proof. □

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References

[1] R.H. Kraichnan, Phys. Fluids 11 (1968) 945–963.
[2] B. Shraiman, E. Siggia, Phys. Rev. E 49 (1994) 2912–2927.
[3] M. Chertkov, G. Falkovich, I. Kolokolov, V. Lebedev, Phys. Rev. E 51 (1995) 5609–5627.
[4] K. Gawędzki, A. Kupiainen, Phys. Rev. Lett. 75 (1995) 3834–3837.
[5] E. Balkovsky, V. Lebedev, Phys. Rev. E 58 (1998) 5776–5795.
[6] K. Gawędzki, M. Vergassola, Physica D 138 (2000) 63–90.
[7] Y. Le Jan, O. Raimond, 1999. math. PR/9909147.
[8] W. E, E. Vanden-Eijnden, Proc. Natl. Acad. Sci. USA 97 (2000) 8200–8205.
[9] W. Feller, Ann. Math. 55 (1952) 468–519.