Non-Gaussian Invariant Measures for the Majda Model of Decaying Turbulent Transport

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Abstract

The problem of turbulent transport of a scalar field by a random velocity field is considered. The scalar field amplitude exhibits rare but very large fluctuations whose typical signature is fatter than Gaussian tails for the probability distribution of the scalar. The existence of such large fluctuations is related to clustering phenomena of the Lagrangian paths within the flow. This suggests an approach to turn the large-deviation problem for the scalar field into a small-deviation, or small-ball, problem for some appropriately defined process measuring the spreading with time of the Lagrangian paths. Here such a methodology is applied to a model proposed by Majda consisting of a white-in-time linear shear flow and some generalizations of it where the velocity field has finite, or even infinite, correlation time. The non-Gaussian invariant measure for the (reduced) scalar field is derived, and, in particular, it is shown that the one-point distribution of the scalar has stretched exponential tails, with a stretching exponent depending on the parameters in the model. Different universality classes for the scalar behavior are identified which, all other parameters being kept fixed, display a one-to-one correspondence with an exponent measuring time persistence effects in the velocity field. © 2001 John Wiley & Sons, Inc.

1 Introduction

Consider the evolution of a scalar $T(x,t)$ passively advected by a turbulent velocity field $u(x,t)$,

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \Delta T\,, \quad T|_{t=0} = \phi\,, \quad \nabla \cdot u = 0\,. \tag{1.1}$$

The solutions of this equation generally exhibit the following remarkable property: Even for roughly Gaussian velocity fields as observed in turbulent flows, the scalar can experience rare but very large fluctuations in amplitude, and its statistics can depart significantly from Gaussianity. Other quantities, such as pressure or derivatives of velocity, also exhibit such intermittent behavior in turbulent flows, and the complete description of the phenomena is one of the main challenges for an eventual statistical theory of hydrodynamic turbulence. In this context, many recent efforts have been devoted toward establishing some universal properties of the solutions of (1.1) in the large time limit in terms of the velocity field u and the initial

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data for the scalar ϕ [1, 4, 5, 6, 10, 12, 19, 20, 21, 22, 23, 24, 25]. We shall focus on this problem.

A disclaimer is appropriate here since the evolution in (1.1) describes a nonstationary process [24]. Indeed, the mean energy of the scalar field,

$$\mathcal{E}(t) = \left\langle T^2 \right\rangle, \tag{1.2}$$

where $\langle \cdot \rangle$ denotes expectation over an appropriate ensemble that we assume to be homogeneous, satisfies

$$\frac{\partial \mathcal{E}}{\partial t} = -2\kappa \left\langle |\nabla T|^2 \right\rangle, \tag{1.3}$$

and thus $\mathcal{E} \to 0$ as $t \to \infty$ provided $\kappa > 0$. On the other hand, it is natural to assume that the reduced scalar field

$$X = \frac{T}{\mathcal{E}^{1/2}} \tag{1.4}$$

has an invariant measure and to ask about the generic properties of the latter. (The scalar field itself may reach a statistical steady state if a source term is added to (1.1) [2, 3, 8, 10, 11, 12, 23]. This is a different problem, though, and we shall focus on the decaying case here. Both the forced and the decaying situations can be relevant to physical situations; for a recent review on turbulent diffusion, see, e.g., the survey paper by Majda and Kramer [21], particularly chapter 5.)

In [20], Majda introduced a simple model for the equation in (1.1) to address in a rigorous fashion the question of the non-Gaussian statistical properties of the reduced scalar field $X(\cdot, t)$ in the limit as $t \to \infty$. The model is simple enough to be amenable to exact solution, yet it captures some essential properties of the solutions of the general equation in (1.1). Majda assumes that the velocity field in (1.1) is a rapidly fluctuating two-dimensional linear shear profile,

$$u = (u_x, u_y) = (0, \gamma \xi(t) x)\,, \tag{1.5}$$

where γ is a constant with dimension of $[\text{time}]^{-1/2}$, and $\xi(\cdot)$ is a white-noise process. Majda also assumes that the initial data for the scalar field, $T|_{t=0} = \phi$, is a Gaussian random process, statistically independent of the velocity, depending only on the variable y, and whose covariance satisfies (the precise definition for ϕ is given in (3.2) below)

$$\int_0^y \langle \phi(z)\phi(0)\rangle dz = O(y^{-\alpha}) \quad \text{as } y \to \infty\,. \tag{1.6}$$

The spectral parameter α in (3.3) controls the amount of energy on the large scales of the initial scalar field, and the correlation length for the initial scalar field is finite for $\alpha = 0$ (short-range correlated), infinite for $\alpha \in (-1, 0)$ (long-range correlated), and zero for $\alpha > 0$ (anticorrelated). Thus, upon varying α one may access situations where the spatial correlation effects in the initial data are weaker, comparable, or stronger than those in the velocity field (recall that the correlation length for a linear shear layer is formally infinite).

In the original papers [19, 20] (see also [4, 22]), Majda computed all the moments of X using the fact that the equation for the n-point correlator of the scalar field in his model can be converted to a Schrödinger equation with parabolic potential through partial Fourier representation in the y-variable. These moments were shown to be typically non-Gaussian. Using the information from the moments, Bronski and McLaughlin [5, 6] later obtained a rigorous estimate for the tails of the one-point distribution function of X. Their estimate is encompassed by the following result, which we state as a corollary of Theorem 3.1:

COROLLARY 1.1 *In the limit as $t \to \infty$, one has*

$$\left\langle \left(\int_{\mathbb{R}^2} \varphi(x,y)\big(X(x,y,t) - \bar{X}(t)\big)dx\,dy \right)^2 \right\rangle \to 0 \tag{1.7}$$

for all test functions $\varphi(x,y)$, where the probability distribution of the limiting $\bar{X}(t)$ satisfies

$$\lim_{t\to\infty} \mathsf{P}\{|\bar{X}| > \delta\} = C_1 \delta^{-2/(3+\alpha)} e^{-C_2\delta^{4/(3+\alpha)}} + O\big(\delta^{-6/(3+\alpha)} e^{-C_2\delta^{4/(3+\alpha)}}\big) \tag{1.8}$$

as $\delta \to \infty$, with the constants C_1 and C_2 given by (3.39) *below.*

Corollary 1.1 specifies the whole process $X(\,\cdot\,,t)$ in the limit as $t \to \infty$ with the following properties:

PROPERTY 1: From (1.7) it follows that $X(\,\cdot\,,t)$ becomes flat in the limit as $t \to \infty$. Thus, the one-point statistics of the reduced scalar contains all the relevant information. Notice that this implies that for large times large fluctuations of the reduced scalar field are observed *everywhere* (i.e., for all (x,y)) in *specific* realizations. This is to be contrasted with a situation where there are large fluctuations at specific positions in each realization, and it shows the nonergodicity with respect to spatial averaging of the process $X(\cdot,t)$ in the large time limit.

PROPERTY 2: The asymptotic result (1.8) demonstrates the non-Gaussian nature of the distribution of X, which displays stretched exponential tails. (1.8) also highlights the influence of the initial condition for the scalar on the long-time statistics of X. In particular, the less energy in the large scales of ϕ, i.e., the bigger $\alpha > -1$, the more important is the departure from Gaussianity, with an increase of weight in the tail of the distribution of X.

Stretched exponential distributions such as (1.8) are well-known to fit reasonably well the experimental observations [7, 9, 13, 26]. In fact, we shall argue that properties 1 and 2 capture some essential features of the reduced scalar field associated with the solution of the general equation in (1.1). To support this claim, we shall base our analysis of the statistical properties of $X(\,\cdot\,,t)$ as $t \to \infty$ on the observation that the large fluctuations in the scalar field amplitude are associated with clustering phenomena of the Lagrangian paths within the flow. In other words, the large-deviation problem for the (reduced) scalar field can be turned into a small-deviation, or small-ball, problem for some appropriately defined process measuring the spreading of the Lagrangian paths. This methodology gives simple explanations for properties 1 and 2 and supports their generic nature. It can also be made rigorous for Majda's model and for some generalizations of it where the velocity field has finite, or even infinite, correlation time.

The analysis shows in particular that different universality classes for the behavior of the scalar can be identified that, all other parameters being kept fixed, display a one-to-one correspondence with an exponent measuring time persistence effects in the velocity field.

Let us note at this point that the observation that clustering of the Lagrangian paths leads to intermittency is not new (it appeared in [23] and is also used , e.g., in [1, 25]), and, in particular, it underlies the slow-mode analysis proposed in [3] for Kraichnan's model [14, 15] of turbulent transport. (Kraichnan's model is another popular model that we shall not consider here.) However, it is worth pointing out that the approach in [3] relies on the analysis of the Fokker-Planck operators governing the evolution of the n-point correlator of the scalar field and whose very existence depends on the white-in-time character of the velocity field in Kraichnan's model. In contrast, our approach does not rely on the possibility of writing Fokker-Planck operators, which is why we can consider non-white-in-time flows.

The remainder of this paper is organized as follows: In Section 2 we give a heuristic discussion about intermittency for decaying turbulent transport and introduce the methodology to turn the large deviation for the (reduced) scalar field into a small-ball problem. This methodology is applied to the original Majda model in Section 3, then to the generalizations where the velocity field can have both finite and infinite correlation times in Section 4. Finally, Section 5 contains the proofs of our main theorems.

2 Heuristics on Intermittency in Decaying Turbulent Transport

The mechanism underlying intermittency in passive scalar advection is easy to understand, at least qualitatively. For the time being, let us focus on the general situation and consider (1.1) on $(x, t) \in \mathbb{R}^d \times [0, \infty)$, assuming that the initial condition $T|_{t=0} = \phi$ has mean zero with respect to spatial average. (Since the transformation $\phi \to \phi + C$ leads simply to $T \to T + C$, there is no loss in generality in this assumption.) The solution of (1.1) can be expressed as

$$T(x, t) = \langle \phi(X_{-t}) \rangle_\beta \,. \tag{2.1}$$

Here X_t solves the characteristic equation associated with (1.1),

$$dX_t = u(X_t, t)dt + \sqrt{2\kappa}\, d\beta(t) \,, \quad X_0 = x \,, \tag{2.2}$$

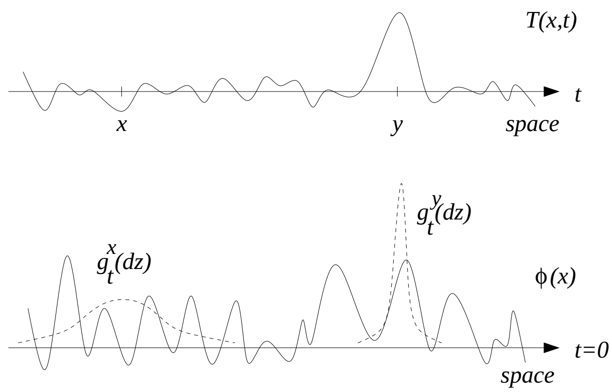

FIGURE 2.1. The field $T(\cdot, t)$ undergoes a large fluctuation at position y due to the abnormally small width of $g_t^y(dz)$.

where $\beta(\cdot)$ is a Wiener process, and the expectation in (2.1) is taken over β. An equivalent expression for (2.1) is

$$T(x,t) = \int_{\mathbb{R}^d} \phi(y) g_t^x(dy)\,, \tag{2.3}$$

where $g_t^x(dy)$ is the (random) measure giving the probability distribution of X_{-t} in each realization of the velocity field u. A key property of $g_t^x(dy)$ is that it is a nondegenerate distribution, i.e., broad in y, with a typical width that grows in time on the average. This is not surprising at finite diffusivity, but the nondegeneracy of $g_t^x(dy)$ may even survive in the limit as $\kappa \to 0$ if the velocity field is spatially non-Lipschitz, as is typical for a turbulent flow (this point is discussed in [10, 12, 16]).

Now, since $g_t^x(dy)$ broadens and ϕ has mean zero with respect to spatial average, it is clear from (2.3) that the dynamics will smooth out any spatial fluctuations in the initial data for the scalar field, with an average rate depending on the average growth rate of the width of $g_t^x(dy)$. This mechanism is responsible for energy decay in the model. On the other hand, by reversing the argument we conclude that (see Figure 2.1)

> In any realization where $g_t^x(dy)$ broadens abnormally slowly, one will observe a large fluctuation in the scalar-field amplitude at point x even if the initial data sampled by X_{-t} is very typical.

These large fluctuations are responsible for the intermittent corrections in the statistics of the scalar field.

These considerations suggest that the large-deviation problem for $T(x,t)$ is equivalent to a small-deviation, or small-ball, problem for some appropriately defined random process associated with the width of $g_t^x(dy)$. Let us now establish that such a connecting framework indeed exists and can be turned into a methodology for obtaining the statistical properties of the process $T(\cdot, t)$ or, more precisely,

$X(\cdot, t)$ in the large time limit. In the next few sections, we shall apply this methodology to the Majda model [19, 20] and some generalizations of it for which all calculations can be made rigorously. However, it is interesting to outline the main ideas beyond these calculations in the general case first and to give some general properties of the process $X(\cdot, t)$ in the large time limit that can be derived from them. For simplicity, we assume that both the velocity field and the initial data for the scalar field have mean zero and are statistically homogeneous and isotropic with respect to averaging over appropriate ensembles. For the velocity field, we assume stationarity as well.

The width of $g_t^x(dy)$ can be estimated by the trace of its covariance,

$$M_t(x) = \big\langle |X_{-t} - \langle X_{-t}\rangle_\beta|^2 \big\rangle_\beta \,, \tag{2.4}$$

or, equivalently,

$$M_t(x) = \int_{\mathbb{R}^d} |y|^2 g_t^x(dy) - \int_{\mathbb{R}^d\times\mathbb{R}^d} y\cdot z g_t^x(dy) g_t^x(dz)\,, \tag{2.5}$$

assuming these integrals exist. $M_t(x)$ is a positive random process that grows with time on the average. Using $M_t(x)$, we introduce the rescaled measure

$$p_t^x(d\eta) = g_t^x\big(\sqrt{M_t(x)}\, d\eta\big)\,, \tag{2.6}$$

which, by definition, has typical width of the order of unity. In terms of $p_t^x(d\eta)$, we can rewrite the expression in (2.3) for the scalar field as

$$T(x,t) = \int_{\mathbb{R}^d} \phi\big(\sqrt{M_t(x)}\eta\big)\, p_t^x(d\eta)\,. \tag{2.7}$$

We consider the large-time asymptotics of this expression. We proceed in three steps, each of which appeals to properties that we can expect to be observed for a large class of systems:

Step 1. If the velocity field is ergodic with respect to time average, one may appeal to homogenization theory and conjecture that $p_t^x(d\eta)$ must self-average for large times and have a nonrandom limit,

$$p_t^x(d\eta) \to P(d\eta) \quad \text{as } t\to\infty\,. \tag{2.8}$$

The limiting $P(d\eta)$ is the probability distribution of $X_{-t}/\sqrt{M_t(x)}$ in the limit as $t\to\infty$ with respect to expectation over the statistics of both β and the velocity u. It is independent of x by homogeneity of the velocity field.

Step 2. Since the initial data for the scalar field has mean zero with respect to spatial average, we have for any test function $\varphi(\cdot)$,

$$0 = \lim_{\lambda\to\infty} \frac{1}{\lambda^d}\int_{\mathbb{R}^d} \phi(x)\varphi(x/\lambda)dx = \lim_{\lambda\to\infty}\int_{\mathbb{R}^d} \phi(\lambda\eta)\varphi(\eta)d\eta\,.$$

Thus, we can assume that, for some specific value of the exponent $\gamma > 0$, the quantity

$$\lambda^{\gamma} \int_{\mathbb{R}^d} \phi(\lambda\eta) P(d\eta) \tag{2.9}$$

will converge in some suitable sense to a limit $\bar{\phi}$ as $\lambda \to \infty$. Then the limiting $\bar{\phi}$ is a random variable whose statistics depend on the statistical properties of $\phi(\cdot)$. For instance, if $\phi(\cdot)$ is a (mean zero) Gaussian random process, $\bar{\phi}$ is a Gaussian random variable with mean zero and covariance

$$\langle \bar{\phi}^2 \rangle = \lim_{\lambda\to\infty} \lambda^{2\gamma} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \phi(\lambda\eta)\phi(\lambda\eta') \rangle P(d\eta) P(d\eta') \,,$$

and the exponent γ simply depends on the spatial decay rate of the covariance of $\phi(\cdot)$. It is easy to see that the value $\gamma = d/2$ corresponds to the generic situation of initial data with short-range correlation for which

$$\int_{\mathbb{R}^d} \langle \phi(x+z)\phi(z) \rangle dz < \infty \,,$$

whereas $\gamma \in (0, d/2)$ corresponds to initial data with long-range correlation (infinite correlation length) for which the above integral diverges. Notice also that for initial data with short-range correlation, $\bar{\phi}$ must be Gaussian no matter what the statistics of $\phi(\cdot)$ are by the central limit theorem, since the limiting operation in (2.9) amounts to space-averaging the process $\phi(\cdot)$ over many correlation lengths.

Step 3. Under steps 1 and 2, it follows from (2.7) that

$$T(\,\cdot\,, t) \sim (M_t(\cdot))^{-\gamma/2}\bar{\phi} \tag{2.10}$$

for large times. One cannot take the limit as $t \to \infty$ on (2.10) since the scalar field itself cannot reach a statistical steady state. On the other hand, from (2.10) the reduced scalar field

$$X = \frac{T}{\mathcal{E}^{1/2}}$$

has an invariant measure if $M_t(\cdot)(\mathcal{E}(t))^{1/\gamma}$ has a limit μ as $t \to \infty$; the limiting μ must be space independent by homogeneity of the velocity field. For instance, if $\mathcal{E}(t) = C_{\mathcal{E}} t^{-\nu} + o(t^{-\nu})$ for some $\nu > 0$ as $t \to \infty$, it requires that

$$C_{\mathcal{E}}^{1/\gamma} t^{\nu/\gamma} M_t(\cdot) \to \mu \quad \text{as } t \to \infty \,. \tag{2.11}$$

In this case, (2.12) is equivalent to the following statement for the reduced scalar field X:

$$X(\,\cdot\,, t) \to \mu^{-\gamma/2}\bar{\phi} \quad \text{as } t \to \infty \,. \tag{2.12}$$

The expression in (2.12) is the final result of our considerations. Since $X(\,\cdot\,, t)$ is inversely proportional to μ in (2.12), the large fluctuations of the scalar fields are observed in those realizations where μ is abnormally small, i.e., where $M_t(\cdot)$

grows abnormally slowly. Thus we have reduced the large-deviation problem for the reduced scalar field to a small-deviation problem for μ, as announced. The limiting result in (2.12) is worth several comments.

Comment 1. The equation in (2.12) specifies completely the statistical properties of the process $X(\,\cdot\,, t)$ in the limit as $t \to \infty$ (i.e., the invariant measure of this process) in terms of μ and $\bar{\phi}$. Since μ and $\bar{\phi}$ are space independent, (2.12) predicts that the process $X(\,\cdot\,, t)$ becomes flat in the large-time limit, which is consistent with property 1. Of course, the simplicity of this statement is deceptive. First, establishing the statistical properties of μ for general turbulent velocity fields remains a formidable challenge (in contrast, the value of γ and the properties of $\bar{\phi}$ are essentially given data since they depend merely on $\phi(\cdot)$). Second, similar considerations applied to other quantities like the scalar gradient, ∇T, or the scalar difference, $\delta T = T(x_1, t) - T(x_2, t)$, lead to the similar conclusion that the reduced quantities $\nabla T / \langle |\nabla T|^2 \rangle^{1/2}$ and $\delta T / \langle \delta T^2 \rangle^{1/2}$ have limits proportional to some $\mu_{\nabla T}$ or $\mu_{\delta T}(x_1 - x_2)$, respectively. This simply means that under our assumptions the corresponding energies, $\langle |\nabla T|^2 \rangle$ and $\langle \delta T^2 \rangle$, decay faster than $\mathcal{E}(t)$ as $t \to \infty$.

Comment 2. Since $X(\,\cdot\,, t)$ is asymptotically equivalent to the product involving the two independent random variables μ and $\bar{\phi}$, the probability distribution of the reduced scalar field must be broader than the one of $\bar{\phi}$. From step 2, this automatically implies a broader-than-Gaussian probability distribution of $X(\,\cdot\,, t)$ if $\phi(\cdot)$ is a Gaussian random process or an arbitrary process with short-range correlation.

Comment 3. More specifically, it is well-known that the probability for small deviations of a random process are usually superexponential [18]. Thus, assuming that

$$-\ln(\mathsf{P}\{\mu < \varepsilon\}) = O\left(\frac{1}{\varepsilon^{\beta}}\right) \quad \text{as } \varepsilon \to 0 \tag{2.13}$$

for some exponent $\beta > 0$, one obtains from (2.12) with a Gaussian $\bar{\phi}$ after a calculation similar to the one presented in the proof of Corollary 1.1 in Section 3,

$$-\lim_{t\to\infty} \ln(\mathsf{P}\{|X| > \delta\}) = O(\delta^{2\gamma/(\beta+\gamma)}) \quad \text{as } \delta \to \infty\,. \tag{2.14}$$

This result is consistent with property 2; that is, the limiting distribution of X has stretched exponential tails.

In the following sections we shall make rigorous the statements above and obtain the statistical properties of the reduced scalar field in the large time limit for Majda's model and some generalizations of it with non-white-in-time velocity fields. The methodology outlined above applies to these models with minor modifications due to the anisotropy and inhomogeneity of the velocity fields. Thus, we shall study the equivalent of (2.7), compute explicitly what M_t and μ are, and characterize the invariant measure for the reduced scalar field for these models.

3 Majda's Original Shear Model

In [20], Majda introduced the following model where the scalar field is advected by a rapidly fluctuating linear shear profile:

$$\frac{\partial T}{\partial t} + \gamma \xi(t) x \frac{\partial T}{\partial y} = \kappa \Delta T \,, \quad T|_{t=0} = \phi \,, \tag{3.1}$$

where γ is a constant with dimension [time]$^{-1/2}$, and $\xi(\cdot)$ is a white noise, defined as the derivative (in the sense of distributions) of a Wiener process $\xi(t) = \dot{B}(t)$.

Majda also assumed that the initial data for the scalar field, $T|_{t=0} = \phi$, is a Gaussian random process, statistically independent of the velocity and depending only on the variable y,

$$\phi(y) = \int\limits_{\mathbb{R}} e^{2i\pi k y} E^{1/2}(k) dW(k) \,, \tag{3.2}$$

with the energy spectrum $E(k)$ given by

$$E(k) = C_E |k|^\alpha \psi(k) \,, \quad \alpha > -1 \,. \tag{3.3}$$

Here C_E is a constant with dimension [scalar]^2[length]$^{1+\alpha}$, $\psi(k)$ is a cutoff function rapidly decaying for $L|k| > 1$ and satisfying $\psi(k) = \psi(-k)$, $\psi(k) = 1 - L^2k^2 + O(L^4k^4)$ (L is an ultraviolet cutoff length), and dW denotes the complex white-noise process with

$$\langle dW(k) d\bar{W}(q) \rangle = \delta(k-q) dk\, dq \,. \tag{3.4}$$

As explained in the introduction, the spectral parameter α in (3.3) controls the amount of energy on the large scales of the initial scalar field, and the model in (3.2) satisfies the property in (1.6).

For Majda's model, we have the following:

THEOREM 3.1 *Let* $X = T/\mathcal{E}^{1/2}$ *where* T *solves* (3.1) *and* $\mathcal{E} = \langle T^2 \rangle$. *In the limit as* $t \to \infty$, *one has for all test functions* $\varphi(x, y)$

$$\left\langle \left(\int\limits_{\mathbb{R}^2} \varphi(x,y) \big(X(x,y,t) - \bar{X}(t) \big) dx\, dy \right)^2 \right\rangle \to 0 \,, \tag{3.5}$$

where

$$\bar{X}(t) = \frac{1}{\sqrt{\sigma}} \left(\frac{1}{8\pi^2} \int_0^1 B_t^2(s) ds \right)^{-(1+\alpha)/4} \int\limits_{\mathbb{R}} |z|^{\alpha/2} e^{-2\pi^2 z^2} dW_t(z) \,. \tag{3.6}$$

Here σ *is given by*

$$\sigma = \int_0^\infty \frac{\lambda^{(\alpha-1)/2} d\lambda}{\sqrt{\cosh \sqrt{\lambda}}} \,, \tag{3.7}$$

$B_t(\cdot)$ is the rescaled process

$$B_t(s) = \frac{B(ts)}{\sqrt{t}}\,, \tag{3.8}$$

and $dW_t(\cdot)$ is the rescaled white noise

$$dW_t(z) = A_t^{1/4}\, dW\left(\frac{z}{\sqrt{A_t}}\right), \quad A_t = 2\kappa\gamma^2 \int_0^t B^2(s)ds\,. \tag{3.9}$$

Furthermore, the law of $\bar{X}(t)$ satisfies

$$\bar{X}(t) \overset{D}{=} \frac{a}{\sqrt{\bar{\sigma}}} \left(\int_0^1 B^2(s)ds\right)^{-(1+\alpha)/4}, \tag{3.10}$$

where $\bar{\sigma} = \sigma/\,\Gamma(\frac{1}{2}(1+\alpha))$, $B(\cdot)$ is a standard Wiener process, and a is a Gaussian random variable with zero mean and unit covariance.

Theorem 3.1 follows as a special case of Theorem 4.1, which is proven in Section 5. A sketch of the proof of Theorem 3.1 is given below. At the end of this section, we also use the theorem to prove Corollary 1.1.

We now sketch the idea of the proof of Theorem 3.1. We essentially follow the methodology proposed in Section 2. This will highlight the origin of intermittency in Majda's model and also allow us to estimate the rate of convergence of X to $\bar{X}$. In fact, we will connect (3.5) (or (3.6)) with the general expression in (2.12) by identifying

$$\begin{aligned} \gamma = \frac{1+\alpha}{2}\,, \quad \mu &\overset{D}{=} \frac{1}{8\pi^2}(\sigma C_E)^{2/(1+\alpha)} \int_0^1 B^2(s)ds\,, \\ \bar{\phi} &\overset{D}{=} \sqrt{C_E} \int_{\mathbb{R}} e^{-2\pi^2 z^2} |z|^{\alpha/2}\, dW(z)\,. \end{aligned} \tag{3.11}$$

We start from the following representation formula for the scalar field

$$T = \int_{\mathbb{R}} e^{2i\pi k(y-\gamma B(t)x) - 4\pi^2\kappa k^2 t - 2\pi^2 k^2 A_t} E^{1/2}(k) dW(k)\,, \tag{3.12}$$

with A_t given by (3.9). The formula in (3.12) is the explicit representation for the equivalent of (2.1) for Majda's model

$$T(x, y, t) = \langle \phi(Y_{-t}) \rangle_\beta\,, \tag{3.13}$$

where Y_{-t} must be obtained from the characteristic equations associated with (3.1) (using $\xi(t)dt = dB(t)$)

$$\begin{aligned} 2dX_t &= \sqrt{2\kappa}\, d\beta_x(t)\,, && X_0 = x, \\ dY_t &= \gamma X_t dB(t) + \sqrt{2\kappa}\, d\beta_y(t)\,, && Y_0 = y\,, \end{aligned} \tag{3.14}$$

where β_x and β_y are independent Wiener processes. Thus,

$$\begin{aligned} X_{-t} &= x - \sqrt{2\kappa}\beta_x(t)\,, \\ Y_{-t} &= y - \gamma x B(t) - \sqrt{2\kappa}\beta_y(t) + \sqrt{2\kappa}\gamma \int_0^t B(s)d\beta_x(s)\,, \end{aligned} \tag{3.15}$$

and the expression in (3.13) reduces to (3.12) after using the explicit formula in (3.2) for ϕ.

Consistent with the rescaling by $M_t(x)$ proposed in (2.6), we now change the integration variable in (3.12) as

$$k = \frac{z}{\sqrt{M_t}}\,, \quad t > 0\,, \tag{3.16}$$

where

$$M_t = \left\langle (Y_{-t} - \langle Y_{-t}\rangle_\beta)^2 \right\rangle_\beta = 2\kappa t + A_t\,, \tag{3.17}$$

with A_t given by (3.9). (The independence in x of $M_t(x) \equiv M_t$ in Majda's model is related to the absence of a positive Lyapunov exponent for a linear shear flow.) This gives

$$T = \int_{\mathbb{R}} \pi_t^{x,y}(z)(M_t)^{-(1+\alpha)/4}\sqrt{C_E}|z|^{\alpha/2}\psi\left(\frac{z}{\sqrt{M_t}}\right)d\hat{W}_t(z)\,, \tag{3.18}$$

where we used the explicit expression in (3.3) for $E(k)$, and we defined

$$\begin{aligned} \pi_t^{x,y}(z) &= \exp\left(2i\pi z\frac{y-\gamma B(t)x}{\sqrt{M_t}} - 2\pi^2 z^2\right), \\ d\hat{W}_t(z) &= (M_t)^{1/4}\,dW\left(\frac{z}{\sqrt{M_t}}\right). \end{aligned} \tag{3.19}$$

The expression in (3.18) is the equivalent of (2.7) in Fourier representation. The function $\pi_t^{x,y}(z)$ is the Fourier representation of the measure $p_t^{x,y}(d\eta)$ for the rescaled process $Y_{-t}/\sqrt{M_t}$ (i.e., the anisotropic equivalent of $p_t^x(d\eta)$ as defined in (2.8)),

$$\int_{\mathbb{R}} e^{2i\pi z\eta}p_t^{x,y}(d\eta) = \pi_t^{x,y}(z)\,. \tag{3.20}$$

The factor

$$(M_t)^{-(1+\alpha)/4}\sqrt{C_E}|z|^{\alpha/2}\psi\left(\frac{z}{\sqrt{M_t}}\right)d\hat{W}_t(z) \tag{3.21}$$

accounts for the initial data for the scalar field after appropriate rescaling as in (2.7). We now show that the properties in steps 1 through 3 hold for (3.18), which yields the equivalent of (2.10) and (2.12) for large times.

First, since

$$M_t \overset{D}{=} 2\kappa t + 2\kappa\gamma^2 t^2 \int_0^1 B^2(s)ds\,, \tag{3.22}$$

we have

$$\frac{y-\gamma B(t)x}{\sqrt{M_t}} \overset{D}{=} \frac{y-\gamma\sqrt{t}B(1)x}{\sqrt{2\kappa t + 2\kappa\gamma^2 t^2 \int_0^1 B^2(s)ds}} = O\big(t^{-1/2}\big) \tag{3.23}$$

as $t \to \infty$. It follows that

$$\pi_t^{x,y}(z) \to e^{-2\pi^2 z^2} \quad \text{as } t \to \infty\,. \tag{3.24}$$

(Here and below, the convergence is understood in the sense of (3.5).) Thus, consistent with the assumption in step 1, $p_t^{x,y}(d\eta)$ self-averages for large times and tends to a limiting measure $P(d\eta)$ whose Fourier representation is simply $e^{-2\pi^2 z^2}$.

Next, from (3.24) and the property that $M_t = O(t^2)$ as $t \to \infty$, it follows that

$$\bar\phi - \sqrt{C_E}\int_{\mathbb{R}} \pi_t^{x,y}(z)|z|^{\alpha/2}\psi\left(\frac{z}{\sqrt{M_t}}\right)d\hat W_t(z) \to 0 \tag{3.25}$$

in the limit as $t \to \infty$, with

$$\bar\phi = \sqrt{C_E}\int_{\mathbb{R}} e^{-2\pi^2 z^2}|z|^{\alpha/2}\,dW_t(z)\,, \tag{3.26}$$

where $dW_t(z)$ is the rescaled white-noise defined in (3.9). Since $dW_t(z) \overset{D}{=} dW(z)$, where $dW(z)$ is a complex white noise as defined in (3.4), $\bar\phi$ is a Gaussian random variable with mean zero and covariance

$$\langle\bar\phi^2\rangle = (2\pi)^{-(1+\alpha)}\Gamma\big(\tfrac{1}{2}(1+\alpha)\big)C_E\,. \tag{3.27}$$

The existence of the limiting $\bar\phi$ in (3.26) confirms the assumption in step 2 for Majda's model.

Finally, using (3.25), we obtain that (3.18) reduces for large times to

$$T(\,\cdot\,,t) \sim (M_t)^{-(1+\alpha)/4}\bar\phi\,, \tag{3.28}$$

which is the equivalent of (2.10). After rescaling by the square root of the energy $\mathcal{E}(t)$, whose explicit expression is easily obtained from the formula for the scalar field given in (3.12),

$$\mathcal{E}(t) = \sigma C_E\big(16\pi^2\kappa\gamma^2 t^2\big)^{-(1+\alpha)/2}\,, \tag{3.29}$$

the expression in (3.28) gives for the rescaled scalar field

$$X(\,\cdot\,,t) - \bar X(t) \to 0 \quad \text{as } t \to \infty \tag{3.30}$$

with

$$\bar X(t) = \mu^{-(1+\alpha)/4}\bar\phi\,. \tag{3.31}$$

Here μ is such that

$$\mu - \frac{M_t}{(\mathcal{E}(t))^{2/(1+\alpha)}} \to 0 \quad \text{as } t \to \infty \tag{3.32}$$

and is explicitly given by

$$\mu = \frac{1}{8\pi^2}(\sigma C_E)^{2/(1+\alpha)} \int_0^1 B_t^2(s)ds\,, \tag{3.33}$$

with $B_t(s)$ defined in (3.8) (hence the law of μ satisfies the equality in (3.11)). (3.30) is the equivalent of (2.12), and it confirms step 3; after some elementary reorganization using (3.26) and (3.33), (3.31) gives (3.6).

Summarizing, the essence of (3.6), (3.10), or (3.31) can be understood as follows: The probability of observing very large, non-Gaussian fluctuations in X is directly related to the probability of having $\int_0^1 B^2(s)ds$ very small, which in turn is equivalent to the probability of observing abnormally slow broadening of the probability distribution of Y_{-t}. As proposed in Section 2 as a general principle, this property is at the heart of the origin of intermittency in Majda's model. Notice that it also explains the stretching exponent in (1.8). Indeed, the more long-range-correlated the initial data for the scalar (i.e., the smaller $\alpha > -1$), the less efficient is the smoothing through the dynamics and hence the less important are the fluctuations in the rescaled scalar. This implies less weight in the tail for smaller α, as observed.

Finally, let us briefly comment on the statistics of the scalar field at *finite* time. From the expression in (3.28) for the large-time behavior of the scalar and the property in (3.22) for M_t, it follows that for very small values of $\int_0^1 B^2(s)ds$ the large fluctuations in T or X will eventually be cut off at $t < \infty$, since for

$$\gamma^2 \int_0^1 B^2(s)ds < \frac{1}{t}\,, \tag{3.34}$$

the term $2\kappa t$ in M_t dominates. In other words, for finite time, the distribution of X has a non-Gaussian core that agrees with the distribution of the variable $\bar{X}$ in (3.10), but the tails of the distribution of X eventually relax to a Gaussian shape for $|X| \gg X^\star$ with

$$X^\star = \frac{t^{(1+\alpha)/4}}{\sqrt{\bar{\sigma}}}\,. \tag{3.35}$$

PROOF OF COROLLARY 1.1: Let

$$P_\varepsilon = \mathsf{P}\left\{\int_0^1 B^2(s)ds \le \epsilon\right\}.$$

From the law in (3.10), it follows that

$$\mathsf{P}\{|\bar{X}| > \delta\} = \int_0^\infty \int_{|a|\frac{\varepsilon^{-(1+\alpha)/4}}{\sqrt{\bar{\sigma}}}\ge\delta} \frac{e^{-\frac{1}{2}a^2}}{\sqrt{2\pi}}\, da\, dP_\varepsilon\,.$$

After integration by parts in ε, we obtain

$$\mathsf{P}\{|\bar{X}| > \delta\} = \frac{(1+\alpha)\delta\sqrt{\bar{\sigma}}}{2\sqrt{2\pi}} \int_0^\infty \varepsilon^{\frac{\alpha-3}{4}} e^{-\frac{1}{2}\delta^2\bar{\sigma}\varepsilon^{\frac{1+\alpha}{2}}} P_\varepsilon \, d\varepsilon \,. \tag{3.36}$$

To proceed further, we now need an estimate for P_ε whose explicit value is not available but whose Laplace representation is given by

$$\mathcal{L}_\lambda(P_\varepsilon) = \left(\lambda\sqrt{\cosh\sqrt{2\lambda}}\right)^{-1} . \tag{3.37}$$

This result can be obtained from the general expression for Gaussian integrals,

$$\left\langle e^{-\lambda\int_0^1 B^2(s)ds}\right\rangle = \prod_{n=1}^{\infty}(1+2\lambda\lambda_n)^{-1/2} \,,$$

where λ_n are the eigenvalues for the Gaussian process $B(t)$, together with the explicit expression for the λ_n's for the Wiener process

$$\lambda_n = \frac{4}{\pi^2(2n+1)^2} \,.$$

For $\delta \gg 1$, because of the exponential factor in the integral in (3.36), we essentially need to know P_ε for small ε. The small ε-expansion of P_ε can be obtained from the large λ-expansion of (3.37):

$$P_\varepsilon = \sqrt{\frac{\varepsilon}{\pi}} e^{-2/\varepsilon} + O\left(\varepsilon^{3/2} e^{-2/\varepsilon}\right) . \tag{3.38}$$

With the help of this relation, the integral in (3.36) can be evaluated for large δ by the Laplace method. The result of this quite standard, though tedious, calculation is (1.8) with the constants C_1 and C_2 given by

$$\begin{aligned} C_1 &= \sqrt{\frac{2}{\pi}}(3+\alpha)^{-\frac{1}{2}}(1+\alpha)^{\frac{1+\alpha}{2(3+\alpha)}}\left(\frac{\bar{\sigma}}{8}\right)^{-\frac{1}{3+\alpha}} , \\ C_2 &= 2(3+\alpha)(1+\alpha)^{-\frac{1+\alpha}{3+\alpha}}\left(\frac{\bar{\sigma}}{8}\right)^{\frac{2}{3+\alpha}} . \end{aligned} \tag{3.39}$$

□

4 Generalizations

In this section we study generalizations of Majda's model where we relax the assumption that the velocity field be white-in-time, and instead consider situations with finite, or even infinite, correlation time for the velocity. We shall show that the results in Section 3 are in fact valid for all velocity fields with finite correlation time. On the other hand, for velocity fields with infinite correlation time the intermittent corrections in the statistics of the scalar become more important as the persistence effects in the velocity fields increase. We quantify these effects by relating explicitly the stretching exponent in the exponential for the probability distribution of the reduced scalar to a measure of persistence with time in the velocity fields. In the limit case of a static velocity field, the probability distribution of the reduced scalar is no longer a stretched exponential; instead, it becomes a power law.

To be more specific, we need to make some assumption about the statistics of the velocity field. We will assume that the function $\xi(\cdot)$ entering the velocity field $(0, \gamma\xi(t)x)$ in (3.1) is a Gaussian random process with mean zero and covariance

$$\langle \xi(t)\xi(s) \rangle = R(|t-s|)\,, \tag{4.1}$$

where R satisfies

$$\int_0^t R(s)ds = Ht^{2H-1} + o(t^{2H-1}) \quad \text{with } H \in \left[\tfrac{1}{2}, 1\right] \tag{4.2}$$

in the limit as $t \to \infty$. (We require essentially $\int_0^t R(s)ds = O(t^{2H-1})$: The proportionality constant is taken to be H in (4.2) for convenience only and is irrelevant since γ in the velocity $(0, \gamma\xi(t)x)$ is arbitrary.) The parameter H is a measure of the correlation with time, or persistence, of the velocity field and, upon varying H, the model in (4.1) and (4.2) allows us to consider a wide variety of Gaussian processes. The case $H = \frac{1}{2}$ corresponds to the generic situation where the covariance function is integrable,

$$\int_0^\infty R(s)ds = \frac{1}{2}\,,$$

i.e., the correlation time of the velocity is finite with no persistence. For instance, $H = \frac{1}{2}$ covers the situation where $\xi(t)$ is a white noise as in Majda's original model or an Ornstein-Uhlenbeck process with covariance

$$\langle \xi(t)\xi(s) \rangle = \frac{\nu}{2} e^{-\nu|t-s|}\,.$$

The case with $H \in (\frac{1}{2}, 1]$ corresponds to the situation where the velocity is long-range correlated, or (strongly) persistent, with an infinite correlation time (the limit case $H = 1$ corresponding to a static ξ).

We study the long-time statistical property of the reduced scalar field X advected by the velocity field $(0, \gamma\xi(t)x)$, with $\xi(\cdot)$ satisfying the properties in (4.1) and (4.2). In particular, we ask about the dependence in H of the distribution of X. These are specified by the following:

THEOREM 4.1 *Let* $X = T/\mathcal{E}^{1/2}$, *where* T *solves* (3.1) *with* ξ *satisfying* (4.1) *and* (4.2), *and* $\mathcal{E} = \langle T^2 \rangle$. *Define*

$$G(t) = \int_0^t \xi(s)ds\,. \tag{4.3}$$

In the limit as $t \to \infty$ one has for all test functions $\varphi(x,y)$

$$\left\langle \left(\int\limits_{\mathbb{R}^2} \varphi(x,y)\big(X(x,y,t) - \bar{X}_H(t)\big) dx\, dy \right)^2 \right\rangle \to 0\,, \tag{4.4}$$

where the process $\bar{X}_H(t)$ is defined as

$$\bar{X}_H(t) = \frac{C_\alpha}{\sqrt{\sigma_H}} \left(\int_0^1 (G_t^H(s))^2\, ds \right)^{-(1+\alpha)/4} \int\limits_{\mathbb{R}} |z|^{\alpha/2} e^{-2\pi^2 z^2}\, dW_t^\sharp(z)\,. \tag{4.5}$$

Here $C_\alpha = (2\pi)^{1+\alpha}/\,\Gamma(\frac{1}{2}(1+\alpha))$, $G_t^H(\cdot)$ is the rescaled process

$$G_t^H(s) = G\frac{ts}{t^H}\,, \tag{4.6}$$

and $dW_t^\sharp(\cdot)$ is the rescaled white noise

$$dW_t^\sharp(z) = A_t^{\sharp\,1/4}\, dW\left(\frac{z}{\sqrt{A_t^\sharp}}\right), \quad A_t^\sharp = 2\kappa\gamma^2 \int_0^t G^2(s) ds\,. \tag{4.7}$$

The parameter σ_H is given by

$$\sigma_H = \left\langle \left(\int_0^1 B_H^2(s) ds \right)^{-(1+\alpha)/2} \right\rangle, \tag{4.8}$$

where $B_H(\cdot)$ is a fractional Brownian motion of index H. Furthermore, the law of $\bar{X}_H(t)$ satisfies

$$\bar{X}_H(t) \overset{D}{=} \frac{a}{\sqrt{\sigma_H}} \left(\int_0^1 B_H^2(s) ds \right)^{-(1+\alpha)/4} + o(1) \tag{4.9}$$

in the limit as $t \to \infty$, where a is a Gaussian random variable with zero mean and unit covariance.

As a corollary, we also have the following:

COROLLARY 4.2 *In the limit as $t \to \infty$, the one-point distribution of X satisfies one of the following two properties depending on the value of H:*

(i) *For $H \in [\frac{1}{2}, 1)$, there exist positive constants c and C with $c \le C$ such that for $\delta \gg 1$,*

$$\exp\left(-c\delta^{\beta_H^\alpha}\right) \le \lim_{t\to\infty} \mathsf{P}\{|X| \ge \delta\} \le \exp\left(-C\delta^{\beta_H^\alpha}\right) \tag{4.10}$$

with

$$\beta_H^\alpha = \frac{2}{H(1+\alpha)+1}\,. \tag{4.11}$$

(ii) *For $H = 1$, we have*

$$\lim_{t\to\infty} \mathsf{P}\{|X| \geq \delta\} = C_3\delta^{-4/(1+\alpha)} + O\left(\delta^{-12/(1+\alpha)}\right) \tag{4.12}$$

in the limit as $\delta \to \infty$, where

$$C_3 = \frac{1}{\pi} 2^{(3+\alpha)/(2+2\alpha)} \sigma_H^{-1/(1+\alpha)} \Gamma\left(\frac{5+\alpha}{2+2\alpha}\right). \tag{4.13}$$

Remarks. (1) The fractional Brownian motion of order $H = 1$ is simply the random straight line, $B_1(t) = tB_1(1)$.

(2) One may consider velocity fields such that $H < \frac{1}{2}$ as well. These correspond to anticorrelated velocity fields since, from (4.2), one then has

$$\int_0^\infty R(s)ds = 0\,.$$

It is not difficult to generalize our results to these cases. In fact, it can be shown that Theorem 4.1 and (4.10) in Corollary 4.2 apply for $H \in (0, 1]$. For $H < 0$, on the other hand, X converges in law to a Gaussian random variable with zero mean and unit variance. (In the limit case $H = 0$, X converges to a random variable that is non-Gaussian in the core but with Gaussian tails.)

The results in Theorem 4.1 and Corollary 4.2 are self-explanatory, and we only make two comments. First, they show that, for a given α, the universality classes for the invariant measure for the reduced scalar field are completely specified by the exponent H, with different values of H yielding different universality classes. In particular, all situations where the velocity fields have finite correlation time, corresponding to $H = \frac{1}{2}$, belongs to the same universality class as the situations for white-in-time velocity fields.

Second, we note that the estimate in (4.10) confirms our understanding of the origin of intermittency in passive scalar advection. Indeed, it shows that the more persistent the velocity field, i.e., the higher H, the more weight there is in the tail of the distribution of the reduced scalar. In other words, persistence with time of the velocity field helps to build up large fluctuations in the reduced scalar field. This is consistent with the fact that high values of H facilitate slow broadening of the distribution of Y_{-t} (i.e., slow growth of $D_t^\sharp$ given by (4.15) below) since the velocity is long-range correlated in time. Here Y_{-t} and $D_t^\sharp$ are given by (cf. (3.15) and (3.17))

$$Y_{-t} = y - \gamma x G(t) - \sqrt{2\kappa}\beta_y(t) + \sqrt{2\kappa}\gamma \int_0^t G(s)d\beta_x(s) \tag{4.14}$$

and

$$D_t^\sharp = \left\langle (Y_{-t} - \langle Y_{-t}\rangle_\beta)^2 \right\rangle_\beta = 2\kappa t + A_t^\sharp\,, \tag{4.15}$$

where $A_t^\sharp$ is defined in (4.7).

5 Proof of Theorem 4.1 and Corollary 4.2

In our manipulations, we will need the following two lemmas:

LEMMA 5.1 *For $s \in (0,1]$, the rescaled process $G_t^H(s)$ defined in (4.6) satisfies*

$$G_t^H(s) \overset{D}{=} B_H(s) + o(1) \tag{5.1}$$

in the limit as $t \to \infty$.

PROOF: Since $\xi(t)$ is Gaussian by assumption,

$$G_t^H(s) = G\frac{ts}{t^H} = t^{-H}\int_0^t \xi(u)du$$

is Gaussian, and both processes have mean zero. Thus, to prove the lemma, we only have to verify that the covariances of $G_t^H(s)$ and $B_H(s)$ coincide as $t \to \infty$. For $0 < s' \le s \le 1$, we have

$$\begin{aligned}
\left\langle\left(G_t^H(s) - G_t^H(s')\right)^2\right\rangle &= t^{-2H}\left\langle\left(G(ts) - G(ts')\right)^2\right\rangle \\
&= t^{-2H}\left\langle\left(\int_{ts'}^{ts}\xi(u)du\right)^2\right\rangle \\
&= t^{-2H}\int_{ts'}^{ts}\int_{ts'}^{ts} R(|u-u'|)du'\,du \\
&= 2t^{-2H}\int_0^{t(s-s')}(t(s-s')-u)R(u)du \\
&= (s-s')^{2H} + o(1)\,,
\end{aligned}$$

where we used the assumption in (4.2) in the last step. Thus,

$$\lim_{t\to\infty}\left\langle\left(G_t^H(s) - G_t^H(s')\right)^2\right\rangle = \left\langle\left(B_H(s) - B_H(s')\right)^2\right\rangle = (s-s')^{2H}\,,$$

and we are done. □

LEMMA 5.2 *In the limit as $t \to \infty$, the energy $\mathcal{E}(t) = \langle T^2\rangle$ satisfies*

$$\mathcal{E}(t) = C_E C_\alpha^{-1}\sigma_H\left(2\kappa\gamma^2 t^{2H}\right)^{-(1+\alpha)/2} + o(t^{-H(1+\alpha)})\,, \tag{5.2}$$

where $C_\alpha = (2\pi)^{1+\alpha}/\,\Gamma(\frac{1}{2}(1+\alpha))$ and σ_H is given by (4.8).

PROOF: Using the expression in (4.14) for Y_{-t} in $T = \langle\phi(Y_{-t})\rangle_\beta$, together with the expression in (3.2) for ϕ, we have the following representation formula for the scalar field:

$$T = \int_{\mathbb{R}} e^{2i\pi k(y-\gamma G(t)x) - 4\pi^2\kappa k^2 t - 2\pi^2 k^2 A_t^\sharp} E^{1/2}(k)dW(k)\,. \tag{5.3}$$

It follows that

$$\mathcal{E}(t) = \int_{\mathbb{R}} \langle e^{-8\pi^2\kappa k^2 t - 4\pi^2 k^2 A_t^\sharp} \rangle E(k) dk$$
$$= C_E \left\langle (D_t^\sharp)^{-(1+\alpha)/2} \int_{\mathbb{R}} e^{-4\pi^2 z^2} |z|^\alpha \psi\left(\frac{z}{\sqrt{D_t^\sharp}}\right) dz \right\rangle .$$

Now, Lemma 5.1 implies that

$$D_t^\sharp = 2\kappa t + A_t^\sharp \stackrel{D}{=} 2\kappa t + 2\kappa\gamma^2 t^{2H} \int_0^1 B_H^2(s) ds + o(t^{2H}) = O(t^{2H}) \tag{5.4}$$

as $t \to \infty$. It follows that

$$\mathcal{E}(t) = C_E \left\langle \left(2\kappa\gamma^2 t^{2H} \int_0^1 B_H^2(s) ds \right)^{-(1+\alpha)/2} \right\rangle \int_{\mathbb{R}} e^{-4\pi^2 z^2} |z|^\alpha \, dz + o(t^{-H(1+\alpha)})$$

as $t \to \infty$, which is readily shown to be equivalent to (5.2). □

We are now ready to prove Theorem 4.1 and Corollary 4.2.

PROOF OF THEOREM 4.1: Notice first that the equality in law in (4.9) follows immediately from (5.4) and the property that $dW_t^\sharp(z) \stackrel{D}{=} dW(z)$. Consider now the average in (4.4). Assuming with no loss in generality that $\int_{\mathbb{R}^2} \varphi(x, y) dx dy = 1$, (4.4) can be written as

$$\left\langle \left(\int_{\mathbb{R}^2} \varphi(x, y) \big(X(x, y, t) - \bar{X}_H(t) \big) dx\, dy \right)^2 \right\rangle = A_1 + A_2 - 2A_3$$

with

$$A_1 = (\mathcal{E}(t))^{-1} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x, y) \varphi(\bar{x}, \bar{y}) \langle T(x, y, t) T(\bar{x}, \bar{y}, t) \rangle dx\, dy\, d\bar{x}\, d\bar{y}\, , \tag{5.5}$$

$$A_2 = \left\langle \bar{X}_H^2(t) \right\rangle , \tag{5.6}$$

$$A_3 = (\mathcal{E}(t))^{-1/2} \int_{\mathbb{R}^2} \varphi(x, y) \langle T(x, y, t) \bar{X}_H(t) \rangle dx\, dy\, . \tag{5.7}$$

By definition of $X_H(t)$ in (4.5) and the property in (5.4) for $M_t^\sharp$, we have $\langle \bar{X}_H^2(t) \rangle = 1 + o(1)$ and hence

$$A_2 = 1 + o(1)$$

as $t \to \infty$. To evaluate A_1 and A_3, notice that the expression in (5.3) for the scalar field can be written as

$$T = \int_{\mathbb{R}} \pi_t^{x,y}(z)(D_t^\sharp)^{-(1+\alpha)/4}\sqrt{C_E}|z|^{\alpha/2}\psi\left(\frac{z}{\sqrt{D_t^\sharp}}\right) d\hat{W}_t^\sharp(z)\,, \tag{5.8}$$

where we defined

$$\begin{aligned} \pi_t^{x,y}(z) &= \exp\left(2i\pi\frac{z(y-\gamma G(t)x)}{\sqrt{D_t^\sharp}} - 2\pi^2 z^2\right),\\ d\hat{W}_t^\sharp(z) &= (D_t^\sharp)^{1/4}\, dW\left(\frac{z}{\sqrt{D_t^\sharp}}\right). \end{aligned} \tag{5.9}$$

Since $D_t^\sharp = A_t^\sharp + O(t)$ and $A^\sharp(t) = O(t^{2H})$ as $t \to \infty$ by (5.4), it follows that

$$d\hat{W}_t^\sharp(z) = dW_t^\sharp(z) + o(1) \tag{5.10}$$

as $t \to \infty$, where $dW_t^\sharp(z)$ is the rescaled process defined in (4.7). Furthermore, since

$$\frac{y-\gamma G(t)x}{\sqrt{D_t^\sharp}} \overset{D}{=} \frac{y-\gamma t^H B_H(1)x}{\sqrt{2\kappa\gamma^2 t^{2H+1}\int_0^1 B_H^2(s)ds}} + o(t^{-1/2}) = O(t^{-1/2})\,,$$

as $t \to \infty$, it follows that

$$\left\langle \left(\int_{\mathbb{R}^2} \varphi(x,y)\big(\pi_t^{x,y}(z) - e^{-2\pi^2 z^2}\big)\right)^2 \right\rangle \to 0\,, \tag{5.11}$$

as $t \to \infty$. Using the expression in (5.8) for the scalar together with the estimates in (5.10) and (5.11), it is now straightforward to show that

$$A_1 = 1 + o(1)\,, \qquad A_3 = 1 + o(1)$$

as $t \to \infty$. Thus,

$$\lim_{t\to\infty}(A_1 + A_2 - 2A_3) = 0\,,$$

which concludes the proof. □

PROOF OF COROLLARY 4.2: We proceed as in the proof of Corollary 1.1. Let

$$P_\varepsilon^H = \mathsf{P}\left\{\int_0^1 B_H^2(s)ds \le \epsilon\right\}.$$

From the law in (4.9), it follows that

$$\mathsf{P}\{|\bar{X}_H| > \delta\} = \int_0^\infty \int_{|a|\varepsilon^{-(\alpha+1)/4}/\sqrt{\sigma_H} \ge \delta} \frac{e^{-\frac{1}{2}a^2}}{\sqrt{2\pi}}\, da\, dP_\varepsilon^H\,.$$

After integration by parts in ε, we obtain

$$\mathsf{P}\{|\bar{X}_H| > \delta\} = \frac{(1+\alpha)\delta\sqrt{\sigma_H}}{2\sqrt{2\pi}} \int_0^\infty \varepsilon^{(\alpha-3)/4} e^{-\frac{1}{2}\delta^2\bar{\sigma}\varepsilon^{(1+\alpha)/2}} P_\varepsilon^H \, d\varepsilon\,. \tag{5.12}$$

For $H \in [\frac{1}{2}, 1)$, we have the following estimate for P_ε^H [17]: There exists positive constants c' and C' with $c' \geq C'$ such that for $\varepsilon \ll 1$

$$\exp\left(-\frac{c'}{\varepsilon^{1/2H}}\right) \leq P_\varepsilon^H \leq \exp\left(-\frac{C'}{\varepsilon^{1/2H}}\right).$$

Using this estimate in (5.12), the bounds in (4.10) follow by standard application of the Laplace method. For $H = 1$, we have

$$P_\varepsilon^1 = \operatorname{erf}\left(\frac{\varepsilon}{\sqrt{2}}\right).$$

Using this expression in (5.12) gives (4.12) after standard application of the Laplace method. □

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