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A note on generalized flows

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Abstract

The problem of constructing flows associated with first order ordinary differential equations (ODEs) with spatially non-Lipschitz right-hand side is considered. Due to the lack of uniqueness of the solutions, standard flows cannot be defined in this case. For these situations, it is natural to introduce *generalized flows* which are random fields constructed by assigning a probability measure on the set of maps associated with the solutions of the ODEs. Some properties of the generalized flows are discussed here, in particular in terms of transport, via simple one-dimensional examples. These simple examples display a wide variety of behaviors and indicate that a general theory of generalized flows is likely to be inaccessible because they typically lack desirable properties such as stability with respect to perturbations or Markovianity in time.

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1. Introduction

Consider the transport equation for the scalar field $\theta(x, t)$ in $(x, t) \in \mathbb{R}^d \times [0, \infty)$:

$$\frac{\partial \theta}{\partial t} + (u(x, t) \cdot \nabla) \theta = 0, \quad \theta|_{t=0} = \theta_0. \tag{1}$$

From classical results it is known that if the velocity field $u \in \mathbb{R}^d$ is bounded and continuous in (x, t) and Lipschitz continuous in x , then (1) can be solved by the method of characteristics. Denote by $\varphi_{s,t}(x)$ the solution of

$$\frac{dx}{dt} = u(x, t), \tag{2}$$

starting at x at time s , i.e. with the initial condition $x(s) = x$. The solution of (1) is given by

$$\theta(x, t) = \theta_0(\varphi_{0,t}^{-1}(x)) = \theta_0(\varphi_{t,0}(x)). \tag{3}$$

It is also well known that the map $\varphi_{s,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfies the following properties:

- (a) $\varphi_{s,s}(x) = x$ for all s ;

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- (b) $\varphi_{s,t}(x)$ is continuous in s, t, x ;
- (c) $\varphi_{\tau,t}(\varphi_{s,\tau}(x)) = \varphi_{s,t}(x)$ for all s, t, τ and all x ;
- (d) $\varphi_{s,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a homeomorphism for all s, t .

The map $\varphi_{s,t}$ is called a flow of homeomorphism or, in short, a flow.

These classical results are inapplicable when u fails to be Lipschitz continuous in x . Then no standard flow can be associated with the ODE in (2) since the solution of this equation may fail to be unique. Equivalently, (1) has no unique solution. This situation is unfortunate since transport in non-Lipschitz velocity fields is relevant, e.g. for the problem of turbulence. The classical theory of Kolmogorov [7] predicts indeed that the solution of the Navier–Stokes equation in three dimension is only Hölder with exponent 1/3 in the limit of zero viscosity.

It is not possible to associate a forward-in-time map with the solutions of ODE in (2) because these solutions typically branch, i.e. (2) may have more than one solution for the same initial condition. (Similarly, it is impossible to associate a backward-in-time map with the solutions of (2) because they may coalesce on each other in finite time.) One way to circumvent this difficulty is as follows: make a selection of one outgoing solution of (2) at each branching point to define unambiguously a forward-in-time map in the future of this branching point. (Similarly, a backward-in-time map can be defined by selection at the points of coalescence. Of course by selection it not possible to define maps in both time directions simultaneously.) More generally, one can define the probability to pick one amongst all the outgoing trajectories at the branching point. In effect this amounts to defining a random field by randomizing the set of maps which can be associated with the solutions in (2) by selection at the branching points. This is the viewpoint we shall adopt in this paper to define the *generalized flows* associated with (2). Thus:

A generalized flow is a random field with parameter (s, t, x) constructed by assigning a probability measure on the set of all maps associated with the solutions of (2).

(We elaborate on this definition in Appendix A.) Notice that generalized flows are typically non-degenerate random fields (the measure is not concentrated on a single point) due to branching.

To choose this probability measure it is natural to try to make contact with the motivating physical problem and recall that (1) should be understood as the limiting equation for

$$\frac{\partial \theta}{\partial t} + (u^\varepsilon(x, t) \cdot \nabla) \theta = \kappa \Delta \theta, \quad \theta|_{t=0} = \theta_0. \quad (4)$$

Here κ is the molecular diffusivity and u^ε is a mollified version of u on the scales $|x| \ll \varepsilon$ (e.g. if u solves Navier–Stokes equation, ε is the characteristic length scale associated with the kinematic viscosity). (4) has a unique solution if either κ or ε are positive. Equivalently, a unique (stochastic) flow can be associated with the solutions of

$$dx = u^\varepsilon(x, t) dt + \sqrt{2\kappa} d\beta(t), \quad (5)$$

where $\beta(\cdot)$ is a d -dimensional Wiener process [13,14]. It is natural to define generalized flows by taking the limit as $\kappa, \varepsilon \rightarrow 0$ of the stochastic flow associated (5), provided this limit can be defined in a suitable way.

An important motivation of the present work is to illustrate via simple examples how the above construction procedure can be made in practice, and discuss the main characteristics of resulting generalized flows; these are more difficult to apprehend in more complicated models, like the problem for turbulent transport introduced originally by Kraichnan [8] (there the velocity field is a white-in-time random process; for studies of Kraichnan model with emphasis on—various versions of—generalized flows, see [1–6,10–12]). In particular, the simple examples below illustrate clearly the origin of anomalous dissipation via branching or coalescence, etc.

Another question we will address via these simple examples is whether a program aimed at a general theory of generalized flows is likely to produce mathematical objects with desirable properties as far as modeling of the

underlying physical problems is concerned. For instance, will the limiting generalized flows have a certain degree of universality (that is, be the same for a wide class of mollifiers for u , or do not depend on the particular sequence $\kappa, \varepsilon \rightarrow 0$), or define a random field with nice properties (like being Markov in time for instance). One of the main conclusion of this paper is that this is generally not the case. The limiting generalized flows depend sensitively on the regularization procedure, and they are non-Markov for generic regularizations.

2. A collection of simple illustrative examples

Example 2.1. Consider the ODE

$$\frac{dx}{dt} = [x]^\alpha, \quad x \in \mathbb{R}, \quad \alpha \in (0, 1), \tag{6}$$

where $[x]^\alpha = \text{sgn}(x)|x|^\alpha$. This ODE is a textbook example for non-uniqueness in velocity fields which are only Hölder continuous. Indeed,

$$x(t) = \pm((1 - \alpha)(t - \tau)_+)^{1/(1-\alpha)}, \tag{7}$$

where $f_+ = \max(f, 0)$, are solutions for an arbitrary τ . It follows that no standard flow can be build associated with (6) (Fig. 1).

What about generalized flows? Let Ω be the set of all functions f^ω defined on $[0, T]$, $0 < T < \infty$, and taking values $-1, 0$, or 1 . For $s \leq t \in [0, T]$, we define the family of forward map $\{\varphi_{s,t}^\omega : \mathbb{R} \mapsto \mathbb{R}\}$, with $\omega \in \Omega$, as

$$\varphi_{s,t}^\omega(x) = \begin{cases} \text{sgn}(x)(|x|^{1-\alpha} + (1 - \alpha)(t - s))^{1/(1-\alpha)} & \text{if } x \neq 0, \\ f^\omega(\tau)((1 - \alpha)(t - \tau)_+)^{1/(1-\alpha)} & \text{if } x = 0, \end{cases} \tag{8}$$

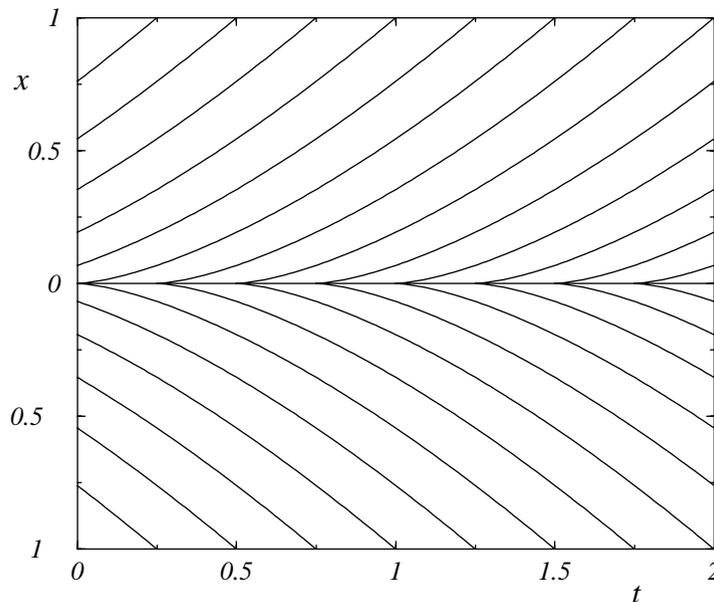


Fig. 1. The set of all solutions of (6) for $\alpha = 1/3$.

where τ is the first time after s where f^ω takes a non-zero value, $\tau = \inf\{t \geq s : f^\omega(s) \neq 0\}$. The ambiguity of where to map the point $x = 0$ is resolved with each of these maps. In fact, each of the maps $\varphi_{s,t}^\omega$ is a weak form of a flow, a *quasi-flow*, in the sense that it satisfies (compare (a)–(d)):

- (a') $\varphi_{s,s}^\omega(x) = x$ for all $s \in [0, T]$;
- (b') $\varphi_{s,t}^\omega(x)$ is continuous in s, t ;
- (c') $\varphi_{\tau,t}^\omega(\varphi_{s,\tau}^\omega(x)) = \varphi_{s,t}^\omega(x)$ for all x and all $s, \tau, t \in [0, T]$ with $s \leq \tau \leq t$.

Various random fields can now be defined by superposing quasi-flows. More precisely, we assign a probability measure, $\mu(\cdot)$, on Ω , and define n -dimensional transition probability distributions as

$$P_n(\underline{x}, \underline{s}, \underline{t}, \cdot) = \int_{\Omega \times \dots \times \Omega} \delta_{\varphi_{s_1,t_1}^{\omega_1}(x_1)}(\cdot) \cdots \delta_{\varphi_{s_n,t_n}^{\omega_n}(x_n)}(\cdot) \mu(d\omega_1) \cdots \mu(d\omega_n), \tag{9}$$

where $\underline{x} = (x_1, \dots, x_n)$, $\underline{s} = (s_1, \dots, s_n)$, $\underline{t} = (t_1, \dots, t_n)$, and $s_j \leq t_j$. Even though the stochastic process f^ω is rather pathological, by Kolmogorov construction the distributions P_n (together with arbitrary n -dimensional probability distributions at initial times, $Q_n(\underline{s}, \cdot)$) characterize uniquely a random field with parameters (s, t, x) which by (b') is continuous in (s, t) . This random field is by definition what we mean by a generalized flow associated with (6); it has the following properties:

1. The generalized flow defined in (9) is *factorizable* in the sense that

$$P_n(\underline{x}, \underline{s}, \underline{t}, \cdot) = P_1(x_1, s_1, t_1, \cdot) \cdots P_1(x_n, s_n, t_n, \cdot). \tag{10}$$

Non-factorizable generalized flows for which (10) is not satisfied are obtained if one introduces a family of measures, $\mu_n(\cdot)$, on $\Omega \times \dots \times \Omega$, consistent in the sense that

$$\mu_{n-1}(\cdot) = \int_{\Omega} \mu_n(\cdot, d\omega_j, \cdot), \tag{11}$$

but such that $\mu_n(\cdot) \neq \mu_1(\cdot) \cdots \mu_1(\cdot)$, and use $\mu_n(\cdot)$ instead of the product of $\mu(\cdot) \equiv \mu_1(\cdot)$ in (9).

2. The generalized flow in (9) is a *non-degenerate* random process—i.e. the measure P_n does not have atomic mass on a single point for all $(\underline{x}, \underline{s}, \underline{t})$ —as soon as f^ω has a non-zero probability to be either 1 or -1 at some values of t .
3. The generalized flow defined in (9) is Markov in time, though Markovianity is not a generic property of generalized flows. A *Markov* generalized flow is uniquely characterized by the n -point transition probabilities (using separability)

$$G_n(\underline{x}, s|\cdot, t) = G_1(x_1, s|\cdot, t) \cdots G_1(x_n, s|\cdot, t), \tag{12}$$

where

$$G_1(x, s|\cdot, t) = \int_{\Omega} \delta_{\varphi_{s,t}^\omega(x)}(\cdot) \mu(d\omega). \tag{13}$$

The Markov character requires that Chapman–Kolmogorov equation,

$$\int_{\mathbb{R}^d} \eta(y) G_1(x, s|dy, t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(y) G_1(x, s|dz, \tau) G_1(z, \tau|dy, t), \tag{14}$$

is satisfied for all test functions η and for all $s < \tau < t \in [0, T]$. By definition, (14) is given by

$$\int_{\Omega} \eta(\varphi_{s,t}^\omega(x)) \mu(d\omega) = \int_{\Omega \times \Omega} \eta(\varphi_{\tau,t}^\omega(\varphi_{s,\tau}^{\omega'}(x))) \mu(d\omega) \mu(d\omega'), \tag{15}$$

or, using (c'),

$$\int_{\Omega} \eta(\varphi_{\tau,t}^{\omega}(\varphi_{s,\tau}^{\omega}(x))) \mu(d\omega) = \int_{\Omega \times \Omega} \eta(\varphi_{\tau,t}^{\omega}(\varphi_{s,\tau}^{\omega'}(x))) \mu(d\omega) \mu(d\omega'). \tag{16}$$

It can be checked that this condition is satisfied for the generalized flow in (9), though clearly this equation need not be satisfied in general (non-Markov generalized flows are considered below).

The measure $\mu(\cdot)$ involved in (9) is arbitrary. As explained in the introduction, to pick a $\mu(\cdot)$, it is natural to attempt defining generalized flows as limits of the standard (stochastic) flows associated with the regularized problems where the velocity in (6) is smoothed and/or some Brownian motion is added to the dynamics in (6). Consider the regularization by Brownian motion first, which we shall refer to as the κ -limit. The transition probability $G_1^{\kappa}(x, s | \cdot, t)$ associated with the process defined by the SDE

$$dx = [x]^{\alpha} dt + \sqrt{2\kappa} d\beta(t), \tag{17}$$

satisfies the backward equation

$$-\frac{\partial G_1^{\kappa}}{\partial s} = LG_n^{\kappa} := [x]^{\alpha} \frac{\partial G_1^{\kappa}}{\partial x} + \kappa \frac{\partial G_1^{\kappa}}{\partial x^2}. \tag{18}$$

The ellipticity of the operator L explains why the term $\sqrt{2\kappa} d\beta(t)$ regularizes (17). This can also be understood more intuitively because, with probability 1, the Brownian motion kicks instantaneously out of $x = 0$ any path that happens to go there, thereby resolving the ambiguity associated with this position. From this it is also easy to convince oneself that for all test functions η , one has

$$\left(\int_{\mathbb{R}} \eta(y) (G_1^{\kappa}(x, s | dy, t) - G_1(x, s | dy, t)) \right)^2 \rightarrow 0, \tag{19}$$

in the limit as $\kappa \rightarrow 0$, where

$$G_1(0, s | \cdot, t) = \frac{1}{2} (\delta_{a_{s,t}}(\cdot) + \delta_{-a_{s,t}}(\cdot)) \tag{20}$$

with $a_{s,t} = ((1 - \alpha)(t - s))^{1/(1-\alpha)}$, and, of course,

$$G_1(x, s | \cdot, t) = \delta_{b_{s,t}(x)}(\cdot) \tag{21}$$

with $b_{s,t}(x) = \text{sgn}(x)(|x|^{1-\alpha} + (1 - \alpha)(t - s))^{1/(1-\alpha)}$ if $x \neq 0$. A similar calculation with the n -points transition probabilities G_n^{κ} shows that the generalized flow obtained in the κ -limit is factorizable. Obviously, this generalized flow can be obtained by superposing the quasi-flows defined in (8) (take f^{ω} and $\mu(\cdot)$ such that for all $t_1 < t_2 < \dots < t_k \in [0, T]$, $f^{\omega}(t_1), \dots, f^{\omega}(t_k)$ are i.i.d. random variables taking values -1 or $+1$ with probability $1/2$).

The regularization by smoothing of the velocity around $x = 0$ (which we shall refer to as the ε -limit) is somewhat void in this first example. To see this, notice first that regularizing the set of solution of (6) rather the velocity itself is an equivalent (since there must be a corresponding regularized velocity field for any set of regularized solutions) but simpler way to look at the ε -limit. A regularized set of solutions of (6) is such that there exists an interval of size ℓ_{ε} centered around $x = 0$ with $\ell_{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, in which all solutions enter if going backward-in-time (more precisely, the solution satisfying $x \neq 0$ at time t was in the layer at time $s = t - |x|^{1-\alpha}/(1 - \alpha) + O(\ell_{\varepsilon})$), but in which the solutions never actually collapse on each other (backward-in-time again). Now, in order to distinguish where to map $x = 0$ as $\varepsilon \rightarrow 0$, one must be able to distinguish precisely the behavior of the solutions within the ℓ_{ε} -layer. Typically (that is, for generic smoothing), this is impossible due to oscillations as $\varepsilon \rightarrow 0$, and the ε -limit

is generally a weaker limit than the κ -limit. More precisely, we can only hope to find a G_1 such that for all test functions η , one has (compare (19))

$$\left(\int_{\mathbb{R} \times \mathbb{R}} \eta(x, y) (G_1^\varepsilon(x, s|dy, t) - G_1(x, s|dy, t)) dx \right)^2 \rightarrow 0, \tag{22}$$

in the limit as $\varepsilon \rightarrow 0$. The relation in (22) is satisfied with any G_1 , which makes indeed this first example void in the ε -limit. Here G_1^ε , which is the transition probabilities in the smoothed velocity field, is the degenerate measure

$$G_1^\varepsilon(x, s|\cdot, t) = \delta_{\varphi_{s,t}^\varepsilon(x)}(\cdot), \tag{23}$$

where $\varphi_{s,t}^\varepsilon$ is the standard flow associated with the smoothed velocity field. (Of course, we can consider special smoothing such as to avoid this problem and have strong convergence). It is easy to see, however, that in this case one must have

$$\left(\int_{\mathbb{R}} \eta(y) (G_1^\varepsilon(x, s|\cdot, t) - \bar{G}_1(x, s|\cdot, t)) dy \right)^2 \rightarrow 0, \tag{24}$$

in the limit as $\varepsilon \rightarrow 0$, with

$$\bar{G}_1(0, s|\cdot, t) = \delta_0(\cdot), \tag{25}$$

and $\bar{G}_1(x, s|\cdot, t) = \delta_{b_{s,t}(x)}(\cdot)$ (see (21)) for $x \neq 0$.

Mixed limits where both smoothing of the velocity field and Brownian motion are used can be considered as well.

Example 2.2. An interesting generalization of the ODE in (6) is

$$\frac{dx}{dt} = [x]^\alpha g(t), \quad x \in \mathbb{R}, \quad \alpha \in (0, 1), \tag{26}$$

where g is a bounded function. Some solutions of (26) branch at $x = 0$ on the time intervals where $g(t) > 0$, and other collapse at $x = 0$ where $g(t) < 0$ (see Fig. 2). Clearly, quasi-flows can be defined similarly as in (8) by resolving the ambiguity of where to map the point x if the solution of (26) for this initial condition is non-unique (this essentially amounts to specifying where to map $x = 0$ on the intervals where $g > 0$). By superposing these quasi-flows one defines generalized flows.

It is interesting to note that, in contrast with what happens in Example 2.1, the generalized flows associated with (26) are not necessarily Markovian in time. The reason is that, here, two solutions of (26) with non-zero initial condition can collapse on $x = 0$ (when $g < 0$), then escape again (when $g > 0$). From this, one can convince oneself that most generalized flows associated with (26) are non-Markov (for instance, suppose that we superpose only solutions with either $x(t) > 0$, or $x(t) < 0$ for all times, i.e. such that they always escape $x = 0$ by the side they came in: the generalized flows constructed this way are obviously non-Markov since the escaping side is history dependent). On the other hand, there obviously are Markov generalized flows associated with (26), as for instance the one obtained in the κ -limit considered below. For those ones, unlike what happens in Example 2.1, $G_1(x, s|\cdot, t)$ can be non-trivial for $x \neq 0$ since the dynamics in (26) maps intervals around $x = 0$ on $x = 0$ when $g < 0$, then possibly out again when $g > 0$.

Consider now the generalized flows obtained by regularization. It is rather easy to deduce that the κ -limit produces a non-degenerate, factorizable generalized flow where $G_1(0, s|\cdot, t)$ is concentrated with half-half weight on the two most external paths escaping $x = 0$ by $x > 0$ or $x < 0$ for $t \geq s$ (that is, as soon as g becomes positive, the paths escape; if ever the two paths collapse again on $x = 0$ —which happens if g is negative on a long enough interval—they re-escape as soon as g becomes positive again).

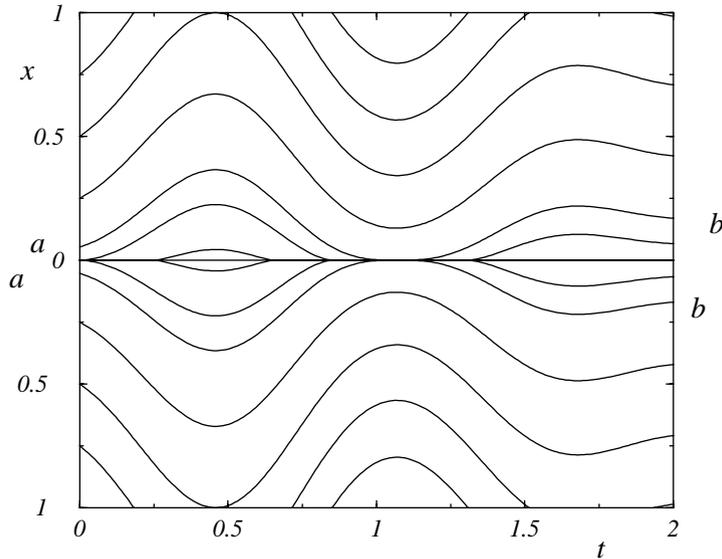


Fig. 2. The set of all solutions of (26) for $\alpha = 1/3$ and $g(t) = \sin(2\pi t) + \cos(t)$.

The regularization by smoothing is somewhat more subtle in this example. First, most generalized flows obtained in the ε -limit are non-Markovian. To see this, notice that Markov generalized flows are obtained if and only if any two paths that were simultaneously within the smoothed layer of size ℓ_ε at a given time and get separated by the forward or the backward dynamics by a distance much bigger than ℓ_ε , either stay separated forever, or, if they re-enter both the layer, can never separate again (since otherwise re-separation is history dependent). Since this property is not satisfied with most smoothing, it follows that most smoothings lead to ε -limit that are non-Markov. If one restricts to those smoothing such that the above property is satisfied—such a smoothing must exist—then it is easy to see that, in the limit as $\varepsilon \rightarrow 0$, one has for all test functions $\eta(x, y)$

$$\left(\int_{\mathbb{R} \times \mathbb{R}} \eta(x, y) (G_1^\varepsilon(x, s|dy, t) - \bar{G}_1(x, s|dy, t)) dx \right)^2 \rightarrow 0, \tag{27}$$

where

$$\bar{G}_1(x, s|dy, t) = \delta_{b_{s,t}(x)}(\cdot) \tag{28}$$

with

$$b_{s,t}(x) = \text{sgn}(x) (|x|^{1-\alpha} + (1-\alpha)(t-s))^{1/(1-\alpha)} \tag{29}$$

if $b_{s,\tau}(x) \neq 0$ for all $\tau \in [0, t]$, and

$$\bar{G}_1(x, s|\cdot, t) = \delta_{c_{\tau,t}}(\cdot), \tag{30}$$

where $c_{\tau,t}$ is one (that is, positive or negative) of the most external paths starting from $x = 0$ at time τ and τ is the first time after s such that $b_{s,\tau}(x) = 0$. A similar argument shows that the generalized flow obtained in the ε -limit is factorizable.

The generalized flow corresponding to (27) also has a property of *coalescence*:

$$\bar{G}_2(x, x', s|dy, dy', t) = \bar{G}_1(x, s|dy, t) \delta_y(dy') \tag{31}$$

for all times $t \geq \tau \geq s$, where τ is the first time after s , such that both $b_{s,\tau}(x) = 0$ and $b_{s',\tau'}(x') = 0$. (31) follows from the fact that:

$$\bar{G}_1(x, s|\cdot, t) = \bar{G}_1(x', s'|\cdot, t) \tag{32}$$

for all times $t \geq \tau \geq s$, where τ is defined as before. The κ -limit and ε -limit are thus essentially different in this example.

It is interesting to study the transport in the above flow, that is, to consider

$$\frac{\partial \theta}{\partial t} + [x]^\alpha g(t) \frac{\partial \theta}{\partial x} = 0, \quad \theta|_{t=0} = \theta_0. \tag{33}$$

The solution of this equation is naturally interpreted as

$$\theta(x, t) = \int_{\mathbb{R}} \theta_0(y) G_1(x, t|dy, 0). \tag{34}$$

Here $G_1(x, t|dy, s)$ with $s < t$ is the transition probability for the *backward* generalized flow, constructed in a way similar as the forward generalized flow but with the opposite ordering in time. Just as the κ -limit and the ε -limit give rise to different forward generalized flow, they lead to different backward generalized flow and, thereby, different solution for the transport equation in (33). Because of the non-degeneracy of the associated backward generalized flow, it is easy to see that the scalar field obtained in the κ -limit is such that the value of $\theta(x, t)$ for x in some interval around $x = 0$ is equal to the average value of θ_0 at the two positions on the most external paths. For instance, for the velocity field represented in Fig. 2, we have

$$\theta(x, t = 2) = \frac{1}{2}(\theta_0(a) + \theta_0(-a)) \quad \text{for } x \in [-b, b], \tag{35}$$

whereas the value of $\theta(x, t = 2)$ for $|x| > b$ can be read simply by following the appropriate (unique) characteristic. In contrast, the scalar field obtained in the ε -limit for those generalized flows that are Markov is such that the value of $\theta(x, t)$ for x in some interval around $x = 0$ is equal to the value of θ_0 at one of the two positions on the most external paths; that is due to the coalescence property of the associated backward generalized flow. For instance, for the velocity field represented in Fig. 2, we have

$$\theta(x, t = 2) = \theta_0(a) \quad \text{for } x \in [-b, b] \tag{36}$$

for some smoothing, and $\theta(x, t = 2) = \theta_0(-a)$ for others.

Example 2.3. Consider the ODE

$$\frac{dx}{dt} = u(x, t), \quad (x, u) \in \mathbb{R} \times \mathbb{R}, \tag{37}$$

where $u(x, t)$ is given by

$$u(x, t) = \begin{cases} \frac{2x}{t} & \text{if } |x| \leq t^2, \\ 2 \operatorname{sgn}(x)t & \text{if } |x| > t^2. \end{cases} \tag{38}$$

This velocity field is continuous in (x, t) but non-Lipschitz (in fact, not even Hölder) at $(x, t) = (0, 0)$. The set of solutions of (37) (see Fig. 3) can be parameterized as

$$x(t) = \operatorname{sgn}(a)t^2 + a, \quad x(t) = bt^2 \tag{39}$$

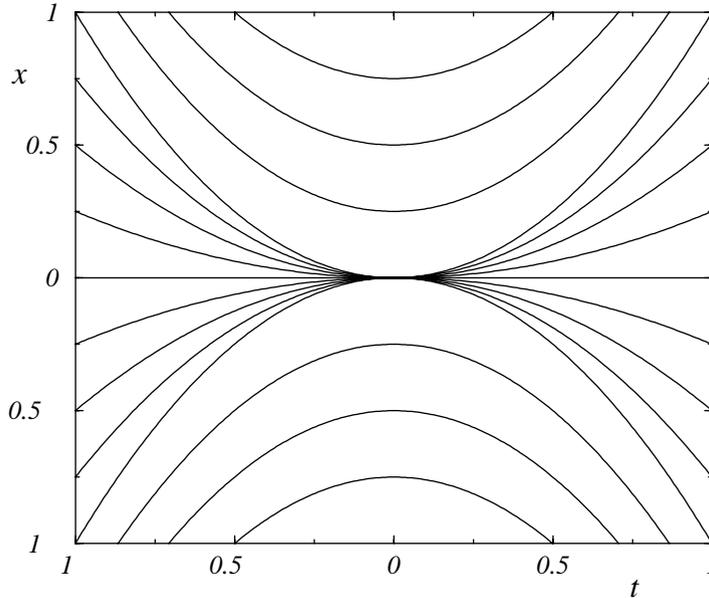


Fig. 3. The set of all solutions of (37).

with $a > 0, b \in [-1, 1]$. Accordingly, for $s \leq t$ we define the family of forward maps $\{\varphi_{s,t}^\omega : \mathbb{R} \mapsto \mathbb{R}\}$ with parameter $\omega \in [-1, 1]$ as

$$\varphi_{s,t}^\omega(x) = \begin{cases} x + t^2 - s^2 & \text{if } |x| > s^2, \\ \frac{xt^2}{s^2} & \text{if } |x| \leq s^2 \text{ and } \text{sgn}(st) = 1, \\ \omega t^2 & \text{if } |x| \leq s^2 \text{ and } \text{sgn}(st) = -1. \end{cases} \tag{40}$$

(Each of these maps picks one single exiting path for all the trajectories which collapse on the singular node $(0, 0)$, thereby solving the ambiguity created by that point.) Each of these map is a quasi-flow satisfying (a')–(c'). Generalized flows can be constructed by superposing these quasi-flows, that is, by assigning a probability measure, $\mu(\cdot)$, on $\Omega := [-1, 1]$ (or, more generally, consistent measures, $\mu_n(\cdot)$, on $[-1, 1]^n$ in order to get non-factorizable generalized flows). As in Example 2.2, these generalized flows are usually non-Markov; the somewhat surprising fact here is that even the one obtained in the κ -limit is non-Markov, as shown next.

To characterize the generalized flow obtained from the κ -limit we consider the SDE

$$dx = u(x, t) dt + \sqrt{2\kappa} d\beta(t). \tag{41}$$

For this regularization it can be shown that

$$\left(\int_{\mathbb{R}} \eta(y) (G_1^\kappa(x, s|dy, t) - G_1(x, s|dy, t)) \right)^2 \rightarrow 0, \tag{42}$$

in the limit as $\kappa \rightarrow 0$, where $G_1(x, s|\cdot, t) = \delta_{x+t^2-s^2}(\cdot)$ if $|x| > s^2$, $G_1(x, s|\cdot, t) = \delta_{xt^2/s^2}(\cdot)$ if $|x| \leq s^2$ and $\text{sgn}(st) = 1$, and

$$G_1(x, s|\cdot, t) = A(x, s, t) \mathbf{1}_{|y| \leq t^2}(\cdot) \tag{43}$$

$(\mathbf{1}_S(\cdot)$ is the uniform measure on the set S) for $|x| \leq s^2$ and $\text{sgn}(st) = -1$. The function $A(x, s, t)$ is difficult to obtain analytically, but it can be shown that this function depend on (x, s, t) , and this automatically implies that the corresponding generalized flow is non-Markov since

$$G_1(x, s|\cdot, t) \neq \int_{\mathbb{R}} G_1(x, s|dz, 0)G_1(z, 0|\cdot, t) \tag{44}$$

for $s < 0, t > 0$. To deduce (43), notice that in terms of the variable z , with $z = x/t^2$ for $|x| < t^2$ and $z = x \mp t^2$ for $|x| \geq t^2$, (41) becomes

$$dz = \sqrt{2\kappa}\sigma(z, t) d\beta(t), \tag{45}$$

where

$$\sigma(z, t) = \begin{cases} \frac{1}{t^2} & \text{if } |z| < 1, \\ 1 & \text{if } |z| \geq 1. \end{cases} \tag{46}$$

In terms of z , it is easy to see that all the mass is expelled out of the range $z \in [-1, 1]$ as the process approaches $t = 0$ by the left (which is equivalent to say that the probability for x to reach the singular node $(0, 0)$ is zero for the regularized SDE in (41)), yet the probability for z to end up at $t = 0^-$ in the ranges $z > 1$ or $z < -1$ is asymmetric and depends on the initial condition. Furthermore, there is a non-zero probability of re-entrance in $z \in [-1, 1]$ as the process leaves $t = 0$ by the right, no matter how small κ is, due to the singular nature of the diffusion in (45), and this probability is again asymmetric and depends on how much mass was in $z > 1$ or $z < -1$ at $t = 0^+$. This leads to (43); the shape of the function A in this expression can be inferred numerically (see Fig. 4).

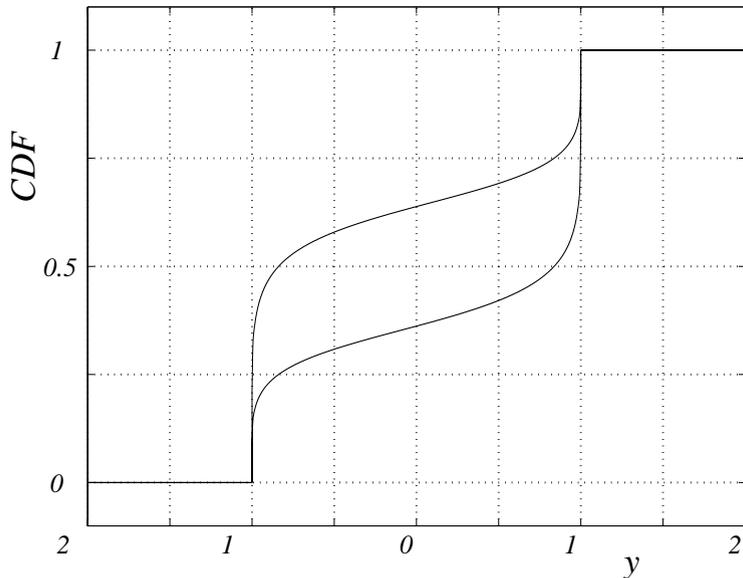


Fig. 4. The cumulative distribution, $\int_{-\infty}^y G_1(x, s|dz, t)$, inferred from numerical integration of (41) at small values of κ . The upper curve corresponds to $x = -0.8$, the lower one to $x = 0.8$; in both cases, $s = -1$, and the distribution is show at $t = 1$. This result shows that the limiting measure is concentrated in $|y| \leq t^2$, but with a non-trivial shape that depends on the initial condition.

The ε -limit in this example lead to similar difficulties as in [Example 2.2](#). Most smoothings do not yield a random field that is Markov in time and the only ones that do produces a (Markov) generalized flow are those such that in the limit as $\varepsilon \rightarrow 0$ all the paths that collapse on the singular node $(0, 0)$ exit on a single trajectory (because maps are monotonous in one dimension the only other possibilities are such that different groups of path that collapse on $(0, 0)$ exit on different single trajectories but those maps cannot be Markov because of the very presence of the singular node). Thus, the only Markov generalized flows obtained in the ε -limit form a one parameter family such that

$$\left(\int_{\mathbb{R} \times \mathbb{R}} \eta(x, y) (G_1^\varepsilon(x, s|dy, t) - \bar{G}_1^a(x, s|dy, t)) dx \right)^2 \rightarrow 0, \tag{47}$$

where

$$\bar{G}_1^a(x, s|\cdot, t) = \delta_{at^2}(\cdot) \tag{48}$$

with $a \in [-1, 1]$ for $s < 0, t > 0$ and $|x| \leq s^2$ (and G_1 given by the obvious formula otherwise). Notice that these generalized flows have the property of coalescence, [\(31\)](#), and they are factorizable.

It is also interesting to note that a much broader class of Markov generalized flows can be obtained by the ε -limit if one allows for more exotic type of convergence. Consider in particular a smoothing of [\(37\)](#) such that all the solutions for the initial condition $x \in [-s^2 + O(\ell_\varepsilon), s^2 - O(\ell_\varepsilon)]$ at $s < 0$ exit the singular node into a tube of width $O(\ell_\varepsilon)$ (and, of course, the solutions for the initial conditions in the tubes of width $O(\ell_\varepsilon)$ around s^2 and $-s^2$ fill the two complementary intervals in between t^2 and the outgoing tube, and the outgoing tube and $-t^2$, respectively). Suppose in addition that the location of the center of the outgoing tube is specified by $a^\varepsilon(t) = t^2 f(1/\varepsilon) + O(\ell_\varepsilon)$, where f is some oscillatory function taking values in $[-1, 1]$. Then,

$$G_1^\varepsilon(x, s|\cdot, t) = \delta_{a^\varepsilon(t)}(\cdot) \tag{49}$$

for $s < 0, t > 0, x \in [-s^2 + O(\ell_\varepsilon), s^2 - O(\ell_\varepsilon)]$, and due to the oscillations in $a^\varepsilon(t)$, G_1^ε has no (standard) limit as $\varepsilon \rightarrow 0$. It is possible, however, to choose f such that $f(1/\varepsilon)$ converges to the random variable r in the sense of Young measure, that is, such that there exist a measure $\nu(\cdot)$ supported on the range of f satisfying

$$\int_{\mathbb{R}} \eta(z) \psi(f(z/\varepsilon)) dz \rightarrow \int_{-1}^1 \psi(r) \nu(dr), \tag{50}$$

as $\varepsilon \rightarrow 0$, for all test functions η satisfying $\int_{\mathbb{R}} \eta(z) dz = 1$, and all test functions ψ . For instance, if $f(\cdot) = \sin(\cdot)$, one has

$$\nu(dr) = \frac{dr}{\pi \sqrt{1 - r^2}}.$$

Then,

$$G_1^\varepsilon(x, s|\cdot, t) \rightharpoonup G_1(x, s|\cdot, t) := \int_{-1}^1 \delta_{t^2 r}(\cdot) \nu(dr) \tag{51}$$

as $\varepsilon \rightarrow 0$, where the convergence is understood in the same sense as in [\(50\)](#). The generalized flows defined in this way are Markov and generally non-degenerate. They are also non-factorizable since

$$G_n^\varepsilon(\underline{x}, s|d\underline{y}, t) \rightharpoonup G_n(\underline{x}, s|d\underline{y}, t) := G_1(x_1, s|dy_1, s) \delta_{y_1}(dy_2) \cdots \delta_{y_1}(dy_n) \tag{52}$$

as $\varepsilon \rightarrow 0$. From [\(52\)](#), one also sees that these generalized flows have a property of coalescence (implying in particular that a generalized flow can be both non-degenerate and with coalescence at the same time).

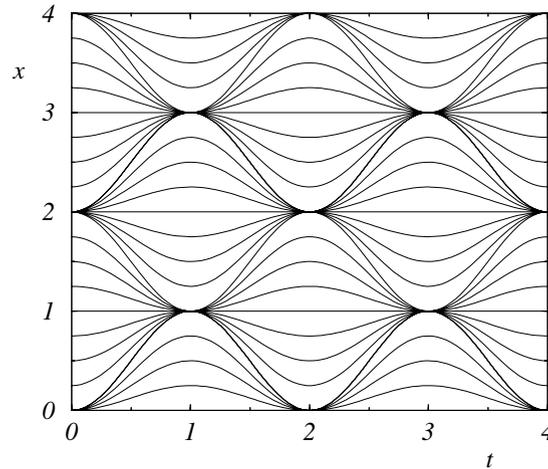


Fig. 5. The set of all solutions of a generalization of (37).

The transport by the above flows can be studied as in [Example 2.2](#).

[Fig. 5](#) shows the set of solutions of an obvious generalization of the present example which can be studied along the same lines. While Markov generalized flows can be constructed in this example—for instance they may lead to a random walk on the separatrix joining the singular nodes, where the path exit the node the left or the right separatrix with the same probability $1/2$ —the analysis of the generalized flows obtained by regularization will already be fairly complicated due to the difficulties mentioned above—non-Markovianity, etc.

3. Concluding remarks

Summarizing the highlights of this paper, we have illustrated via simple examples how generalized flow can be constructed in non-Lipschitz velocity fields, and we have discussed how properties specific to these flows—like non-degeneracy due to branching—lead to interesting features in terms of transport by such velocity fields. Unfortunately, while a theory of such generalized flows might possibly be developed at an abstract level along the lines discussed in [Appendix A](#), the really interesting question on how to construct such generalized flows as limits of regularized standard flows—stochastic or not—seems inaccessible owing to the wide variety of limits that one obtains even in very simple examples. It would be interesting to see if other selection criteria (like, e.g., the one introduced in [\[9,14\]](#) in the different context of non-unique martingales) may be conceived which would single out a single—or at least, only a few—physically relevant generalized flows. Another very interesting question along this line is to understand which—if any—additional properties of the velocity field may lead to more universality in behavior of the generalized flow. For instance, it is rigorously known from the work in [\[10\]](#) (see also [\[2\]](#)) that Kraichnan’s model leads to generalized flows which, at least in the incompressible case, are very robust against the regularization procedure chosen to construct them. It remains a challenge to understand why this is the case.

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Appendix A. Tentative general definition

Here we give a tentative general definition for generalized flows and study some of their properties. We focus on Markov generalized flows for simplicity: non-Markov generalized flows can be defined similarly. We also focus on an abstract definition of generalized flow; in particular, we make no attempt to relate them to limits of regularized flows since the latter procedure is likely to be extremely complicated in the general case, as illustrated by the examples above.

Suppose that u is bounded, continuous in (x, t) , but non-necessarily Lipschitz in x . Then the solution of (2) exists by Peano’s theorem but it may fail to be unique and leads to no standard flow. In this case, we introduce:

Definition A.1. A Markov generalized flow associated with the velocity field $u \in \mathbb{R}^d$ is a random field with parameter (s, t, x) , continuous and Markov in time, characterized by a family of n -points transition probabilities, $\{G_n(\underline{x}, s | d\underline{y}, t)\}$, satisfying:

1. The forward equation:

$$\int_s^t \int_{\mathbb{R}^{nd}} \left(\frac{\partial \eta}{\partial \tau}(\underline{y}, \tau) + \sum_j u(y_j, \tau) \cdot \nabla_{y_j} \eta(\underline{y}, \tau) \right) G_n(\underline{x}, s | d\underline{y}, \tau) d\underline{y} d\tau - \int \eta(\underline{y}, t) G_n(\underline{x}, s | d\underline{y}, t) + \eta(\underline{x}, s) = 0 \tag{A.1}$$

for all test functions $\eta \in C^\infty([0, T] \times \mathbb{R}^{nd})$.

2. The Chapman–Kolmogorov equation:

$$G_n(\underline{x}, s | \cdot, t) = \int_{\mathbb{R}^{nd}} G_n(\underline{x}, s | d\underline{z}, \tau) G_n(\underline{z}, \tau | \cdot, t) \tag{A.2}$$

for all $s \leq \tau \leq t \in [0, T]$.

3. The consistency constraint:

$$G_{n-1}(x_1, \dots, x_n, s | \cdot, t) = \int_{\mathbb{R}^d} G_n(\underline{x}, s | \cdot, dy_j, \cdot, \tau). \tag{A.3}$$

Non-Markov generalized flows can be defined using equations similar to (A.1) and (A.3) for the transition probabilities $P_n(\underline{x}, \underline{s}, \underline{t}, \cdot)$, and relaxing (A.2). We shall show below that this definition guarantees existence of at least one generalized flow associated with u . However, generalized flows are typically not unique. Notice that (A.1) implies

$$\lim_{\tau \rightarrow 0^+} G_n(\underline{x}, s | \cdot, s + \tau) = \delta_{\underline{x}}(\cdot). \tag{A.4}$$

It also implies the random field is continuous in time since Lindeberg condition,

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_{|\underline{x} - \underline{y}| > \delta} G_n(\underline{x}, s | d\underline{y}, s + \tau) = 0, \tag{A.5}$$

is satisfied for all $\delta > 0$. (A.5) can be checked from (A.1) by taking η independent of τ and such that $\eta(\underline{y}) = 0$ if $|\underline{x} - \underline{y}| < \delta$. This gives, using (A.4),

$$\begin{aligned} \frac{1}{\tau} \int_{|\underline{x}-\underline{y}|>\delta} \eta(\underline{y}) G_n(\underline{x}, s | d\underline{y}, s + \tau) &= \frac{1}{\tau} \int_s^{s+\tau} \int_{|\underline{x}-\underline{y}|>\delta} \left(\sum_j u(y_j, \tau) \cdot \nabla_{y_j} \eta(\underline{y}) \right) G_n(\underline{x}, s | d\underline{y}, \tau) d\underline{y} d\tau \\ &= \int_{|\underline{x}-\underline{y}|>\delta} \left(\sum_j u(y_j, \tau) \cdot \nabla_{y_j} \eta(\underline{y}) \right) \delta(\underline{x} - \underline{y}) d\underline{y} + o(1) = o(1), \end{aligned}$$

and (A.5) follows.

Alternatively, generalized flows can be obtained by superposing members of the set of all solutions of (2). Consider the set of all solutions of (2) for the initial condition $x(s) = x, s \in [0, T], x \in \mathbb{R}^d$. Recall that these solutions typically branch (two solutions exist for a single initial condition), which is the very reason why no standard flow exist. Yet, if we order time and relax continuity in x , it is possible to construct maps associated with these solutions through selection at each branching point. Of course, typically there are many different maps and they are not homeomorphisms. Let us parameterize as $\{\varphi_{s,t}^\omega\}$ with $\omega \in \Omega$ the set of all maps from \mathbb{R}^d to \mathbb{R}^d associated with the solutions of (2) on $s, t \in [0, T], T < \infty$, satisfying the following properties for each ω :

- (a') $\varphi_{s,s}^\omega(x) = x$ for all s ;
- (b') $\varphi_{s,t}^\omega(x)$ is continuous in s, t ;
- (c') $\varphi_{\tau,t}^\omega(\varphi_{s,\tau}(x)) = \varphi_{s,t}^\omega(x)$ for all x and all s, τ, t with $s \leq \tau \leq t$.

In other words, for each $\omega, \varphi_{s,t}^\omega$ is a weak form of a flow. We now use the set $\{\varphi_{s,t}^\omega\}$ to show the following theorem.

Theorem A.1. *The following family of $\{G_n\}$ defines a generalized flow associated with the velocity field $u \in \mathbb{R}^d$:*

$$G_n(\underline{x}, s | \cdot, t) = \int_{\Omega^n} \delta_{\varphi_{s,t}^{\omega_1}(x_1)}(\cdot) \cdots \delta_{\varphi_{s,t}^{\omega_n}(x_n)}(\cdot) \mu_n(d\underline{\omega}). \tag{A.6}$$

Here $\{\varphi_{s,t}^\omega\}$ is the set of all maps satisfying (a')–(c') parameterized by $\omega \in \Omega$, and $\{\mu_n(\cdot)\}$ with $\underline{\omega} = (\omega_1, \dots, \omega_n)$, is an arbitrary family of consistent measure on $\Omega^n = \Omega \times \dots \times \Omega$ satisfying

$$\int_{\Omega^n \times \Omega^n} \eta(\varphi_{\tau,t}^{\omega_1}(\varphi_{s,\tau}^{\omega'_1}(x_1)), \dots, \varphi_{\tau,t}^{\omega_n}(\varphi_{s,\tau}^{\omega'_n}(x_n))) \mu(d\underline{\omega}) \mu(d\underline{\omega}') = \int_{\Omega^n} \eta(\varphi_{s,t}^{\omega_1}(x_1), \dots, \varphi_{s,t}^{\omega_n}(x_n)) \mu(d\underline{\omega}) \tag{A.7}$$

for all test functions $\eta \in C^\infty(\mathbb{R}^{nd})$.

It would actually be interesting to determine if all the generalized flows associated with u can be constructed in this way.

(A.7) guarantees that the resulting generalized flow is Markov, and this condition is natural in the following sense. Denote by $\Omega_{[0,T]}$ the set of all maps associated with the solutions of (2) on $[0, T]$. Clearly, given $s, \tau, t \in [0, T], s \leq \tau \leq t$, one has $\Omega_{[s,t]} = \Omega_{[s,\tau]} \cup \Omega_{[\tau,t]} \subseteq \Omega_{[0,T]}, \Omega_{[s,\tau]} \cap \Omega_{[\tau,t]} = \emptyset$. Since, using (c'), (A.7) can be written as

$$\begin{aligned} &\int_{\Omega^n \times \Omega^n} \eta(\varphi_{\tau,t}^{\omega_1}(\varphi_{s,\tau}^{\omega'_1}(x_1)), \dots, \varphi_{\tau,t}^{\omega_n}(\varphi_{s,\tau}^{\omega'_n}(x_n))) \mu(d\underline{\omega}) \mu(d\underline{\omega}') \\ &= \int_{\Omega^n} \eta(\varphi_{s,t}^{\omega_1}(x_1), \dots, \varphi_{s,t}^{\omega_n}(x_n)) \mu(d\underline{\omega}), \end{aligned}$$

this condition means that, given any time $t \in [0, T]$, the measure μ on $\Omega_{[0,T]} = \Omega_{[0,t]} \cup \Omega_{[t,T]}$ is always a product of independent measures on the subsets $\Omega_{[0,t]}, \Omega_{[t,T]}$. Notice also that if μ is a degenerate measure on a single

element in Ω , (A.7) is automatically satisfied: thus, at least one generalized flow associated with u exists, as asserted before.

Proof. For each ω , we have by definition

$$\frac{d}{dt} \eta(\varphi_{s,t}^\omega(x_1), \dots, \varphi_{s,t}^\omega(x_n), t) = \frac{\partial \eta}{\partial t}(\underline{y}, t) + \sum_j u(y_j, \tau) \cdot \nabla_{y_j} \eta(\underline{y}, t) \Big|_{y_j = \varphi_{s,t}^\omega(x_j)}.$$

It follows that (A.1) is satisfied since, inserting (A.6) on the left-hand side of this equation, we obtain

$$\int_s^t \int_{\Omega^n} \frac{d}{d\tau} \eta(\varphi_{s,\tau}^\omega(x_1), \dots, \varphi_{s,\tau}^\omega(x_n), \tau) d\mu(\underline{\omega}) d\tau - \int_{\Omega^n} \eta(\eta_{s,\tau}^\omega(x_1), \dots, \eta_{s,\tau}^\omega(x_n), t) d\mu(\underline{\omega}) + \eta(\underline{x}, s) = 0,$$

where the second equality follows by integration by parts of the first term. (A.2) follows directly from (A.6) and (A.7), and (A.3) follows directly from (A.6). \square

A generalized flow is usually a non-degenerate. A necessary and sufficient condition for non-degeneracy is that

$$\int_{\mathbb{R}^d} y^2 G_1(x, s|dy, t) > \left(\int_{\mathbb{R}^d} y G_1(x, s|dy, t) \right)^2. \tag{A.8}$$

If we construct a generalized flow as in Theorem A.1, we see that we have non-degeneracy if the set Ω contains more than one element (i.e. the solutions of (2) branch), and the $\mu(d\underline{\omega})$ is not degenerate on a single $\underline{\omega} \in \Omega^n$. Non-degeneracy of the generalized is important since it yields a dissipation mechanism.

Another property is *factorizability*. A generalized flow is factorizable if

$$G_n(\underline{x}, s|\cdot, t) = G_1(x_1, s|\cdot, t) \cdots G_1(x_n, s|\cdot, t), \tag{A.9}$$

and *non-factorizable* otherwise. Non-factorizability is related to anomalous scaling of the scalar field.

All the definitions above generalize trivially to define backward generalized flows (with $t \leq s \in [0, T]$ instead of $s \leq t \in [0, T]$). Given any backward generalized flow associated with u , it is then natural to define the solution of (1) as

$$\theta(x, t) = \int_{\mathbb{R}^d} \theta_0(y) G_1(x, t|dy, 0). \tag{A.10}$$

More generally, we define the n -point solution of

$$\frac{\partial \Theta_n}{\partial t} + \sum_{j=1}^n u(x_j, t) \cdot \nabla_{x_j} \Theta_n = 0 \tag{A.11}$$

for the initial condition $\Theta_n(\underline{x}, 0) = \theta_0(x_1) \cdots \theta_0(x_n)$ as

$$\Theta_n(\underline{x}, t) = \int_{\mathbb{R}^{nd}} \theta_0(y_1) \cdots \theta_0(y_n) G_n(\underline{x}, t|d\underline{y}, 0), \tag{A.12}$$

$\Theta_n(\underline{x}, t)$ may be taken as the definition for the product field $\theta(x_1, t) \cdots \theta(x_n, t)$. Different generalized flows yield different Θ_n and, hence, different types of transport properties in the system. Notice also that non-degeneracy of the generalized flow in the sense of (A.8) yields a dissipation mechanism for the scalar field, whereas non-factorizability yields an anomalous scaling in the sense that, e.g.,

$$\Theta_n(\underline{x}, t) \neq \theta(x_1, t) \cdots \theta(x_n, t). \tag{A.13}$$

(Anomalous scaling of this kind is actually very strong. Even for factorizable generalized flows, one may observe anomalous scaling if, e.g.

$$\langle |\theta(x, t) - \theta(y, t)|^n \rangle \quad (\text{A.14})$$

scales in x, y differently than $\langle |\theta(x, t) - \theta(y, t)| \rangle^n$, where $\langle \cdot \rangle$ denotes averaging over a suitable ensemble (for instance spatial averaging.)

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