Chapter 4

Wiener Process

Because of its central role in what follows, it is worthwhile to devote one entire chapter to the study of Wiener process, also known as Brownian motion.

4.1 The Invariance Principle

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that

$$\mathbf{E}\xi_n = 0, \qquad \mathbf{E}\xi_n^2 = 1,$$

and define

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} \xi_i.$$

Viewed as a function of the discrete time n, S_n gives the instantaneous position of a random walker on $\mathbb{Z}$, see figure 4.1. We wish to rescale both time and space so as to define a random function define on $t \in [0,1]$ and taking value in $\mathbb{R}$.

Recall that the Central Limit Theorem asserts that

$$\frac{S_N}{\sqrt{N}} \to N(0,1) \tag{4.1}$$

in distribution as $N \to \infty$. This suggest to rescale S_n and define a piecewise constant random function $W^N(t)$ on $t \in [0,1]$ by letting

$$W^N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}, \tag{4.2}$$

This function is shown in figure 4.2 for different N. We have:

Theorem 4.1.1 (Donsker). *As $N \to \infty$,, $W^N(\cdot)$ converges to a limit $W(\cdot)$ in the sense of distributions*

$$W^N \xrightarrow{d} W \tag{4.3}$$

$W(\cdot)$ is the Wiener process.

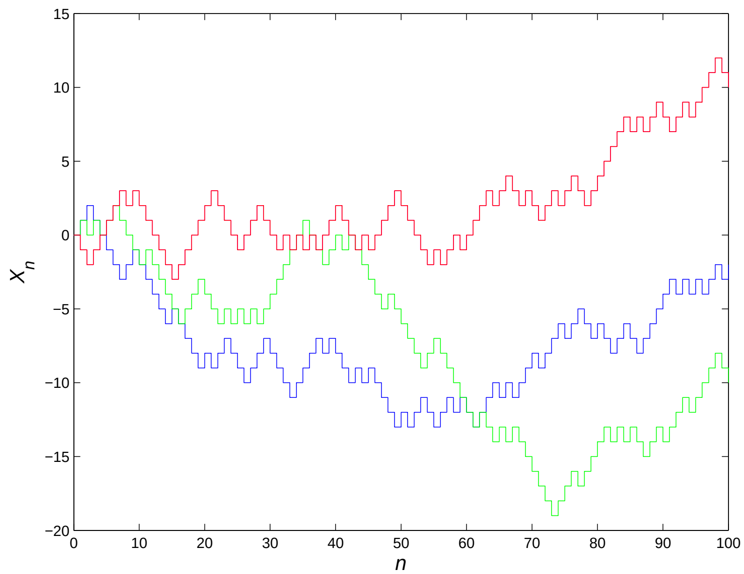

Figure 4.1: Three realizations of the (unrescaled) random walk S_n for $n \in [0, 100]$.

We shall not prove Theorem 4.1.1. Rather we will take for granted that the limiting process $W(\cdot)$ exists and we shall study its properties. In accordance with standard notations for stochastic processes, from now on we indicate the time-dependency of a random function by writing t as a subscript, e.g.

$$W_t^N = W^N(t), \qquad W_t = W(t), \qquad \text{etc.}$$

4.2 Elementary Properties of W_t

Since for $t > 0$

$$W_t^N = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}} = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{\lfloor Nt \rfloor}} \frac{\sqrt{\lfloor Nt \rfloor}}{\sqrt{n}} \xrightarrow{d} N(0,1)\sqrt{t} \stackrel{d}{=} N(0,t), \tag{4.4}$$

we deduce that W_t at fixed t is distributed as $N(0,t)$.

Now consider the random variable $W_t - W_s$ for $0 \le s < t$. Since $S_n - S_m$ for $0 \le m < n$ has the same distribution as S_{n-m}, it follows that

$$W_t - W_s \stackrel{d}{=} W_{t-s}, \qquad 0 \le s < t.$$

Similarly, $S_n - S_m$ and $S_p - S_n$ are independent random variables when $0 \le m < n < p$. This implies that

$$W_t - W_s \text{ and } W_u - W_t \text{ are independent when } 0 \le s < t < u.$$

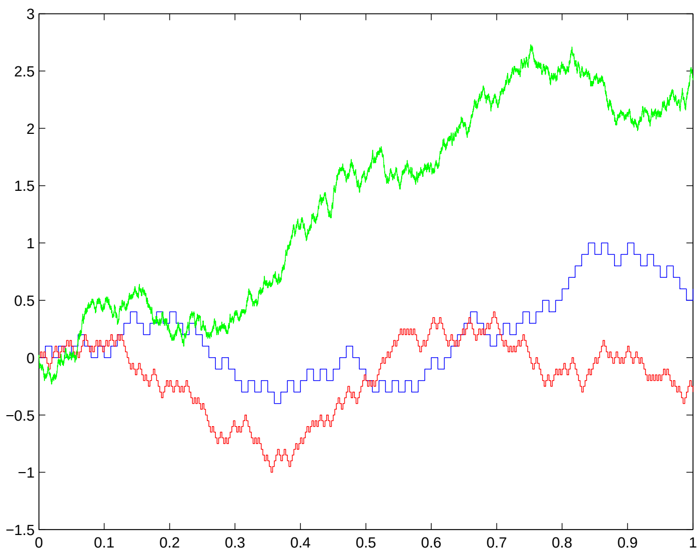

Figure 4.2: Realizations of W_t^N for $N = 100$ (blue), $N = 400$ (red), and $N = 10000$ (green).

This is enough information to deduce the joint density of $(W_{t_1}, W_{t_2}, \ldots, W_{t_n})$ for $0 \le t_1 < t_2 \ldots < t_n \le 1$, $n \in \mathbb{N}$ arbitrary. The density is given by

$$\rho_{t_1,t_2,\ldots,t_n}(x_1, \ldots, x_n) = \rho_{t_n - t_{n-1}}(x_n|x_{n-1}) \ldots \rho_{t_2 - t_1}(x_2|x_1)\rho_{t_1}(x_1|0), \qquad (4.5)$$

where

$$\rho_t(x|y) = \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} \qquad (t > 0).$$

Exercise 4.2.1. Show that

$$\mathbf{E}(W_t - W_s)^2 = |t - s|, \qquad \text{annd} \qquad \mathbf{E}W_t W_s = \min(t, s).$$

Exercise 4.2.2. Show that the Wiener process is self-similar, in the sense that for $\lambda > 0$

$$W_t \stackrel{d}{=} \lambda^{-1/2} W_{\lambda t}.$$

This indicates that it suffices to study the Wiener process on $t \in [0, 1]$ to deduce its statistical properties on any time interval.

$\rho_t(x|y)$ also is the conditional probability density of W_{t+s} given that $W_s = y$. It can be shown by direct calculation that

$$\rho_{t+s}(x|y) = \int_{\mathbb{R}} \rho_t(x|z)\rho_s(z|y)dz, \qquad (0 < s < t)$$

This equation is called the *Chapman-Kolmogorov equation*, and it implies that W_t is Markov. Indeed, by integrating by sides of this equation over x in the interval I, it can also be written as ($u \geq 0$)

$$\mathbf{P}(W_{t+s+u} \in I | W_u = y) = \int_{\mathbb{R}} \mathbf{P}(W_{t+s+u} \in I | W_{s+u} = z)\mathbf{P}(W_{s+u} \in dz | W_u = y).$$

This states that the conditional probability that $W_{t+s+u} \in I$ given that $W_u = y$ is the conditional probability density that $W_{s+u} = z$ for some z given that $W_u = y$, times the conditional probability that $W_{t+s+u} \in I$ given that $W_{s+u} = z$.

The transition probability $\rho_t(x|y)$ is nothing but the heat kernel, i.e. the fundamental solution of

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho_t}{\partial x^2}$$

So far we have considered Wiener process that starting at 0, $W_0 = 0$. Given $y \in \mathbb{R}^1$, it is trivial to define Wiener process that starts at y, by letting

$$\tilde{W}_t = W_t + y.$$

The properties of $\{\tilde{W}_t\}$ are the same as those of $\{W_t\}$. Denote by $\mathbf{E}_y$ expectation with respect to Wiener process that starts at y, let f be a smooth function, and let

$$\begin{aligned} u(y,t) &= \mathbf{E}_y f(\tilde{W}_t) \\ &= \int_{\mathbb{R}} f(x)\rho_t(x|y)dx \\ &= \int_{\mathbb{R}} f(x)\frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi}}dx \end{aligned}$$

Hence u satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial y^2} \qquad u(y,0) = f(y).$$

4.3 Alternative Expression for W_t.

A different way of constructing the Wiener process is the following. Recall that a real function $g(t)$ defined on $[0,1]$ is in $L^2[0,1]$ if

$$\int_0^1 g^2(t)dt < \infty.$$

Let $\{f_k\}_{k\in S}$ where S is a numerable set like e.g. $\mathbb{N}$ or $\mathbb{Z}$ be an orthonormal basis of $L^2[0,1]$, meaning that

$$\int_0^1 f_k(t)f_q(t)dt = \begin{cases} 1 & \text{if } k = q \\ 0 & \text{otherwise} \end{cases}$$

and any $g(t) \in L^2[0,1]$ can be represented as

$$g(t) = \sum_{k\in S} \alpha_k f_k(t) \qquad \text{where} \qquad \alpha_k = \int_0^1 g(t)f_k(t)dt$$

Let $\{\xi_k\}$ be a sequence of i.i.d. normal random variables. Then

$$W_t = \sum_{k \in S} \xi_k \int_0^t f_k(s) ds.$$

is the Wiener process. This can be verified as follows. $\{W_t\}$ is obviously Gaussian since it is the linear combination of Gaussian random variables. Furthermore, $\mathbf{E}W_t = 0$, and

$$\begin{aligned} \mathbf{E}W_t W_s &= \sum_{k,j \in S} \mathbf{E}(\xi_k \xi_q) \int_0^t f_k(\tau) d\tau \int_0^s f_q(\tau') d\tau' \\ &= \sum_{k \in S} \int_0^t f_k(\tau) d\tau \int_0^s f_k(\tau') d\tau'. \end{aligned}$$

Denote by χ_t the indicator function of the interval $[0, t]$, i.e.

$$\chi_t(\tau) = \begin{cases} 1 & \text{if } \tau \in [0, t] \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\chi_t(\tau) = \sum_{k \in S} \Big(\int_0^t f_k(\tau') d\tau' \Big) f_k(\tau).$$

Using Parseval equality[1] we have

$$\sum_k \Big(\int_0^t f_k(\tau) d\tau \Big) \Big(\int_0^s f_k(\tau') d\tau' \Big) = \int_0^1 \chi_t(\tau) \chi_s(\tau) d\tau = s \wedge t.$$

Hence

$$\mathbf{E}W_t W_s = t \wedge s.$$

This proves that W_t is a Wiener process.

There are many ways to construct $\{f_k\}$. One possibility is the Haar basis where

$$f_k^j(t) = 2^{j/2} f(2^j t - k), \qquad j, k \in \mathbb{Z}$$

with

$$f(t) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2}] \\ -1 & \text{if } x \in (\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases}$$

[1] If $g(t) = \sum_k \alpha_k f_k(t)$ and $h(t) = \sum_k \beta_k f_k(t)$, Parseval equality states that

$$\int_0^1 g(t) h(t) dt = \sum_{k \in S} \alpha_k \beta_k$$

This can be shown by inserting the representations of $g(t)$ and $h(t)$ in terms of $f_k(t)$ in the integral at the left hand-side, and using the orthonormality of the $f_k(t)$.

4.4 The Karhunen-Loève expansion of W_t

Since W_t is a Gaussian process, it can be represented by the Karhunen-Loève expansion introduced in the Appendix. Since the covariance function of W_t is $K(t,s) = t \wedge s$ the eigenvalue problem reads

$$\int_0^1 t \wedge s\varphi(s)ds = \lambda\varphi(t)$$

or, equivalently,

$$\int_0^t s\varphi(s)ds + t\int_t^1 \varphi(s)ds = \lambda\varphi(t)$$

Note that this equation implies that $\varphi(0) = 0$. Taking the time-derivative of this equation gives

$$\int_t^1 \varphi(s)ds = \lambda\dot{\varphi}(t),$$

where $\dot{\varphi} = d\varphi/dt$. This equation implies that $\dot{v}arphi(1) = 0$. Taking the time-derivative one more time gives

$$-\varphi(t) = \lambda\ddot{\varphi}(t),$$

where $\ddot{\varphi} = d^2\varphi/dt^2$. The general solution of this equation for $\lambda > 0$ is

$$\varphi(t) = A\sin(t/\sqrt{\lambda}) + B\cos(t/\sqrt{\lambda}),$$

where A, B are constants. The boundary condition $\varphi(0) = 0$ implies that $B = 0$. On the other hand, the boundary condition
$dot\varphi(1) = 0$ can only be satisfied for specific values of λ:

$$\lambda_k = \frac{4}{(2k+1)^2\pi^2}, \qquad k = 0, 1, \ldots.$$

And A is then fixed by the orthonormality condition of $\varphi_k(t)$:

$$1 = \int_0^1 \varphi_k^2(t)dt = A^2\int_0^1 \sin^2\left((k+\tfrac{1}{2})\pi t\right)dt = \frac{A^2}{2},$$

i.e. $A = \sqrt{2}$. Therefore the W_t can be represented via Karhunen-Loève expansion as

$$W_t = \sqrt{2}\sum_{k\geq 0}\xi_k\frac{2}{(2k+1)\pi}\sin\left((k+\tfrac{1}{2})\pi t\right),$$

where $\{\xi_k\}$ are i.i.d. Gaussian random variables with mean zero and variance one.

As an application of the Karhunen-Loève expansion, we compute

$$A = \mathbf{E}\exp\Big(-\alpha\int_0^1 W_t^2 dt\Big),$$

where $\alpha \geq 0$ is a parameter. This expectation is the Laplace transform of the density of the random variable $Z = \int_0^1 W_t^2 dt$. Inserting the Karhunen-Loève expansion for W_t in this expectation gives

$$A = \mathbf{E} \exp\Big(-\alpha \sum_{k,q\geq 0} \xi_k \xi q \sqrt{\lambda_k \lambda_q} \int_0^1 \varphi_k(t)\varphi_q(t)dt\Big)$$

with λ_k, $\varphi_k(t)$ as above. Using the orthonormality of the $\{\varphi_k(t)\}$, this reduces to

$$A = \mathbf{E} \exp\Big(-\alpha \sum_{k\geq 0} \xi_k^2 \lambda_k\Big) = \mathbf{E} \prod_{k\geq 0} e^{-\alpha\lambda_k \xi_k^2}.$$

Since the ξ_k are independent and for $\mu \geq 0$

$$\mathbf{E}e^{-\mu\xi_k^2} = \int_{\mathbb{R}} e^{-muz^2} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz = \frac{1}{\sqrt{2\mu+1}},$$

we obtain

$$A = \prod_{k\geq 0} \frac{1}{\sqrt{2\alpha\lambda_k + 1}} = \frac{1}{\sqrt{g(\alpha)}},$$

where

$$g(\alpha) = \prod_{k\geq 0} (2\alpha\lambda_k + 1).$$

From the expression above the zeroes of $h(z) = g(z^2)$, viewed as a function of the complex variable z, are

$$z_k = \frac{i}{\sqrt{2\lambda_k}} = \tfrac{1}{4}\sqrt{2}(2k+1)\pi.$$

This implies that $g(z^2) = \sinh(\sqrt{2}z)$, i.e.

$$A = \frac{1}{\sqrt{\sinh(\sqrt{2\alpha})}}.$$

4.5 The Wiener Measure

The distribution on $C[0,1]$ of the Wiener process that we just constructed is the Wiener measure. Note that we can express

$$\rho_{t_1,\ldots,t_n}(y_1,\ldots,y_n) = Z_n^{-1} \exp\{-I_{t_1\ldots,t_n}(y_1,\ldots,y_n)\}$$

Here Z_n is a normalization factor,

$$Z_n = (2\pi)^{n/2}(t_1(t_2-t_1)\ldots(t_n-t_{n-1}))^{1/2},$$

and

$$I_{t_1,\ldots,t_n}(y_1,\ldots,y_n) = \tfrac{1}{2}\sum_{i=1}^{n} \frac{(y_i - y_{i-1})^2}{t_i - t_{i-1}}$$

where $t_0 = 0$, $y_0 = 0$. Note that $I_{t_1,\dots,t_n}(y_1,\dots,y_n)$

$$I_{t_1,\dots,t_n}(y_1,\dots,y_n) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\frac{W_{t_i} - W_{t_{i-1}}}{t_i - t_{i-1}}\right)^2 (t_i - t_{i-1})$$

Therefore if we think of the y_i as the values of some function $h(t)$ at times t_i, i.e. $y_i = h(t_i)$, $I_{t_1,\dots,t_n}(y_1,\dots,y_n)$ becomes an approximation to the functional

$$I[h(\cdot)] = \tfrac{1}{2}\int_0^1 \dot{h}^2(t)dt.$$

Since the Wiener measure should result in the limit as $n \to \infty$, except for the factor Z_n, we can express the Wiener measure formally as

$$d\mu_W = Z^{-1}\exp(-I[h(\cdot)])Dh(\cdot).$$

Of cousre, this expression is purely formal. The right hand side is in the form of an infinite dimensional Lebesgue measure $Dh(\cdot) = \prod_{0\le t\le 1} dh(t)$ (which does not exist) with a density $Z^{-1}e^{-I[W]}$, and Z is an normalization factor, which viewed as the limit of Z_n as $n \to \infty$ is zero. Nevertheless this expression is the basis for path integral techniques and is useful. The idea is that, given a functional $A[W.]$ of the Wiener process, its expectation can in principle be computed as

$$\mathbf{E}A[W.] = \frac{\int A[h(\cdot)]\exp(-I[h(\cdot)])Dh(\cdot)}{\int \exp(-I[h(\cdot)])Dh(\cdot)}.$$

The path integral at the numerator in this expression accounts for the normalization factor Z. Since the latter is zero, we already know that the path integral at the denominator must also vanish in order for the expectation to be finite. To see how this comes about, let us compute the expectation

$$A = \mathbf{E}\exp\Big(-\alpha\int_0^1 W_t^2 dt\Big),$$

which we have already determined in the last section using Karhunen-Loève expansion. Here we shall use the expression

$$A = \frac{B_1}{B_2},$$

where B_1, B_2 are the following path integrals

$$B_1 = \int \exp\Big(-\alpha\int_0^1 h^2(t)dt - \tfrac{1}{2}\int_0^1 \dot{h}^2(t)dt\Big)Dh(\cdot),$$

$$B_2 = \int \exp\Big(-\tfrac{1}{2}\int_0^1 \dot{h}^2(t)dt\Big)Dh(\cdot).$$

Both are Gaussian integrals, which mimicking what one does in the finite dimensional setting, we know how to evaluate by determining the eigenvalues of the symmetric Kernel of the quadratic forms in the exponential. Letting $h(0) = \dot{h}(1) = 0$, after integration by parts, these quadratic forms for B_1 and B_2 are respectively

$$\langle h, K_1 h\rangle = \int_0^1 h(t)\left(2\alpha - \frac{d}{dt^2}\right)h(t)dt,$$

and

$$\langle h, K_2 h\rangle = \int_0^1 h(t)\left(-\frac{d}{dt^2}\right)h(t)dt.$$

The eigenvalues of the symmetric operators K_1 and K_2 can be determined by solving the boundary value problems

$$2\alpha\varphi - \ddot{\varphi} = \mu\varphi, \quad \text{and} \quad -\ddot{\psi} = \eta\psi,$$

with boundary conditions $\varphi(0) = \psi(0) = \dot{\varphi}(1) = \dot{\psi}(1) = 0$. This is done as in the last section, and the eigenvalues one obtains are

$$\mu_k = 2\alpha + \tfrac{1}{4}(2k+1)^2\pi^2 \quad \text{and} \qquad \eta_k = \tfrac{1}{4}(2k+1)^2\pi^2, \quad k = 0, 1, \ldots$$

Therefore, A can be expressed as the ratio of the following two infinite products,

$$A = \frac{\sqrt{\prod_{k\geq 0} \frac{1}{4}(2k+1)^2\pi^2}}{\sqrt{\prod_{k\geq 0}\left(2\alpha + \frac{1}{4}(2k+1)^2\pi^2\right)}}$$

Both these products are infinite. However, pairing terms at the denominator and the numerator, their ratio can also be expressed as

$$A = \prod_{k\geq 0}\sqrt{\frac{\frac{1}{4}(2k+1)^2\pi^2}{\left(2\alpha + \frac{1}{4}(2k+1)^2\pi^2\right)}} = \prod_{k\geq 0}\frac{1}{\sqrt{(2\alpha\lambda_k + 1)}},$$

where $\lambda_k = 4/(2k+1)^2\pi^2$ are the eigenvalues off the Karhunen-Loève expansion we determined in the last section. Therefore we recover the correct result for A.

Here is another example. Suppose $V(x)$ is a

4.6 Properties of the Wiener Path

Lemma 4.6.1 (Independent increments). *Given* $0 \leq t_1 < t_2 < t_3 < t_4 \leq 1$, *then* $W_{t_2} - W_{t_1}, W_{t_4} - W_{t_3}$ *are independent.*

This is a direct consequence of (4.5).

Let

$$\Omega_\alpha = \left\{ f \in C[0,1] : \sup_{0\le s,t\le 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty \right\},$$
$$\Omega_{BV} = \{ f \in C[0,1] : f \text{ has bounded variation} \},$$
$$\Omega_{H^1} = \left\{ f \in C[0,1] : \int_0^1 \left(\frac{df}{d\tau} \right)^2 d\tau < \infty \right\}.$$

Ω_α is the set of functions that are Hölder continuous with exponent α. Then

Theorem 4.6.1 (Regularity of the path).

1. *If* $0 \le \alpha < \frac{1}{2}$, *then* $\mathbf{P}\{W \in \Omega_\alpha\} = 1$. *If* $\alpha \ge \frac{1}{2}$, *then* $\mathbf{P}\{W \in \Omega_\alpha\} = 0$.
2. $\mathbf{P}\{W \in \Omega_{BV}\} = 0$.
3. $\mathbf{P}\{W \in \Omega_{H^1}\} = 0$.

In fact, we can define the quadratic variations of f by

$$Q(f) = \overline{\lim_{\|\triangle\|\to 0}} \sum_j \left(f(t_j) - f(t_{j-1}) \right)^2,$$

where $\triangle$ denotes the partition of $[0,1]$, $\triangle = \{t_0 = 0, t_1, \ldots, t_{n-1}, t_n = 1\}$, and

$$\|\triangle\| = \max_{1\le j\le n} (t_j - t_{j-1}).$$

Then

Theorem 4.6.2. $\mathbf{P}\{Q(W) = 1\} = 1$ *i.e. almost all the paths have the same quadratic variation.*

Moreover, for any fixed $T_1, T_2 \in [0,1]$, with $T_1 < T_2$, we can define

$$Q_{T_1,T_2}(f) = \overline{\lim_{\|\triangle\|\to 0}} \sum_{t_{j-1},t_j \in (T_1,T_2]} \left(f(t_j) - f(t_{j-1}) \right)^2.$$

Then

Theorem 4.6.3. $\mathbf{P}\{Q_{T_1,T_2}(W) = T_2 - T_1\} = 1$.

This statement is sometimes written formally as

$$(dW_t)^2 = dt.$$