Another note on forced burgers turbulence*

Weinan E[a] and Eric Vanden Eijnden[b]
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

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The power law range for the velocity gradient probability density function in forced Burgers turbulence has been an issue of intense discussion recently. It is shown in E and Vanden Eijnden, Phys. Rev. Lett. **83**, 2572 (1999) that the negative exponent in the assumed power law range has to be strictly larger than 3. Here we give another direct argument for that result, working with finite viscosity. At the same time we discuss viscous corrections to the power law range. This should answer the questions raised in Kraichnan, Phys. Fluids **11**, 3738 (1999) regarding the results of E and Vanden Eijnden, Phys. Rev. Lett. **83** 2572 (1999).
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The main purpose of this note is to clarify and extend to finite viscosity the results of our earlier paper[1] concerning the asymptotic behavior of the velocity gradient probability density function (PDF) for Burgers turbulence with homogeneous, smooth, Gaussian, and white-in-time forcing. In particular, the present note should answer the questions raised in Ref. 2 regarding Ref. 1.

Let $Q^\nu(\xi,t)$ denote the PDF of $\xi=u_x$, where u satisfies

$$u_t+uu_x=\nu u_{xx}+f, \tag{1}$$

and define

$$Q(\xi,t)=\lim_{\nu\to 0} Q^\nu(\xi,t). \tag{2}$$

Then the question of interest is the value of α, such that

$$Q\sim C_-|\xi|^{-\alpha} \quad \text{as} \quad \xi\to-\infty. \tag{3}$$

We emphasize that the existence of such a range (the so-called $-\alpha$ range) is not the issue here. The issue is the value of α. Notice that (3) is a statement about the inviscid limit. For fixed $\nu>0$, the left tail of Q^ν decays much faster due to the presence of the viscous range. In addition Q^ν satisfies

$$Q_t^\nu=\xi Q^\nu+(\xi^2 Q^\nu)_\xi+B_1 Q^\nu_{\xi\xi}+F^\nu, \tag{4}$$

where $F^\nu=-\nu\langle \xi_{xx}\delta(\xi-\xi(x,t))\rangle_\xi$ accounts for the effect of the viscosity, and $B_1=\int_0^\infty dt\langle f_x(x,t)f_x(x,0)\rangle$. Q satisfies an equation similar to (4) with F^ν replaced by $F=\lim_{\nu\to 0}F^\nu$:

$$Q_t=\xi Q+(\xi^2 Q)_\xi+B_1 Q_{\xi\xi}+F. \tag{5}$$

The expression for F is given explicitly in (13).

One main result of Ref. 1 is a statement to the effect that $\alpha>3$, expressed as

$$\lim_{\xi\to-\infty} \xi^3 Q(\xi,t)=0. \tag{6}$$

We emphasize that there is an important distinction between the strict inequality $\alpha>3$ and the bound $\alpha\geqslant 3$ advanced in Ref. 2: (6) rules out all the predictions in the literature (including those of Refs. 3–5) except that of Ref. 6 with $\alpha=7/2$. As is discussed in Ref. 2, $\alpha=3$ and $\alpha>3$ imply qualitatively different pictures concerning the contribution of the F^ν term in (4) in the viscous range.

In Ref. 1, (6) was derived from (5). Here we will work directly with (4). Consider statistical steady state ($Q_t^\nu=0$). For $\xi\ll-\xi_0(=-B_1^{1/3})$, the B_1 term in (4) can be neglected giving

$$\xi Q^\nu+(\xi^2 Q^\nu)_\xi+F^\nu\approx 0, \tag{7}$$

or, equivalently,

$$(\xi^3 Q^\nu)_\xi\approx-\xi F^\nu. \tag{8}$$

Because of the exponential decay in the viscous range, we get from (8)

$$Q^\nu\sim|\xi|^{-3}\int_{-\infty}^{\xi} d\xi'\,\xi' F^\nu(\xi') \quad \text{for} \quad \xi\ll-\xi_0. \tag{9}$$

Using the same analysis as in Ref. 1, for small ν we can obtain an explicit expression for Q^ν from (9). The calculation is performed in Appendix A and gives

$$Q^\nu\sim 2\rho|\xi|^{-3}\int_\xi^{+\infty} d\xi'\int_{-\infty}^{s_*} ds\,\xi' A(s,\xi-\xi')V(s,\xi')$$
$$-2\nu\rho|\xi|^{-2}\int_\xi^{+\infty} d\xi'\int_{-\infty}^{s_*} ds\,\frac{(\xi-\xi')}{A(s,\xi-\xi')}V(s,\xi')$$
$$\text{for} \quad \xi\ll-\xi_0, \tag{10}$$

where

$$A(s,\xi)=(\tfrac{1}{4}s^2+2\nu\xi)^{1/2}, \tag{11}$$

ρ is the number density of shocks, $s_*=-2(2\nu(\xi'-\xi))^{1/2}$, $V(s,\xi)=(V_-(s,\xi)+V_+(s,\xi))/2$, $V_\pm(s,\xi_\pm)$ are the PDFs of $s(y,t)$ (shock amplitude, $s\leqslant 0$), and $\xi_\pm(y,t)$ (gradients at the left and right of the shock in the inviscid limit), conditional on the property that there is a shock at $x=y$.

Without further information about $V(s,\xi)$ it is difficult to carry out asymptotics on (10), and we shall not dwell on this problem [see however (A19)–(A22)]. What is easier and

*This is a companion paper to R. H. Kraichnan, Phys. Fluids **11**, 3738 (1999).
[a]Electronic mail: weinan@cims.nyu.edu
[b]Electronic mail: eve2@cims.nyu.edu

more instructive is to actually take the limit as $\nu\to 0$ for fixed ξ in (10). Then, only the first term at the right hand side of (10) survives, and Q^ν converges to

$$Q(\xi)\sim\xi^{-3}\int_{\xi}^{+\infty}d\xi'\,\xi' F(\xi') \quad \text{for } \xi\ll-\xi_0, \tag{12}$$

with

$$F(\xi)=\lim_{\nu\to 0}F^\nu(\xi)=\rho\int_{-\infty}^{0}ds\, sV(s,\xi). \tag{13}$$

At this stage, we use the fact that

$$A=\int_{-\infty}^{+\infty}d\xi\,\xi F=\frac{\rho}{2}(\langle s\xi_-\rangle+\langle s\xi_+\rangle)=0. \tag{14}$$

This is a steady state consequence of

$$\frac{d}{dt}(\rho\langle s\rangle)=-\frac{\rho}{2}(\langle s\xi_-\rangle+\langle s\xi_+\rangle), \tag{15}$$

which is proven in Appendix B. As a result of (14), (12) may be rewritten as

$$Q(\xi)\sim|\xi|^{-3}\int_{-\infty}^{\xi}d\xi'\,\xi' F(\xi') \quad \text{for } \xi\ll-\xi_0. \tag{16}$$

(6) then follows because $\int_{-\infty}^{\xi}d\xi'\,\xi' F(\xi')\to 0$ as $\xi\to-\infty$. We stress that this argument, and in particular the use of (15) at steady state, does not imply that $Q\sim A|\xi|^{-3}$ at transient states because the whole analysis here rests on (9), which is only valid at statistical steady state. In fact, as we will now show, (6) also holds for transient states.[7] Again in Ref. 1 this argument was presented for Q. Here we reinterpret this using Q^ν for small ν.

Multiplying (4) by ξ and integrating from ξ_* to $+\infty$ (for fixed $\xi_*\ll-\xi_0$), we get

$$\int_{\xi_*}^{+\infty}d\xi\,\xi Q_t^\nu=-\xi_*^3Q^\nu(\xi_*,t)+\int_{\xi_*}^{+\infty}d\xi\,\xi F^\nu. \tag{17}$$

Here we neglected contribution from the B_1 term since it is small compared with the remaining terms. Taking the limit as $\nu\to 0$, we get:

$$\int_{\xi_*}^{+\infty}d\xi\,\xi Q_t=-\xi_*^3Q(\xi_*,t)+\int_{\xi_*}^{+\infty}d\xi\,\xi F. \tag{18}$$

Therefore

$$\lim_{\xi_*\to-\infty}\xi_*^3Q(\xi_*,t)=\int_{-\infty}^{+\infty}d\xi\,\xi F-\frac{d}{dt}\int_{-\infty}^{+\infty}d\xi\,\xi Q. \tag{19}$$

Notice that even though by homogeneity

$$\langle\xi\rangle_\nu=\int_{-\infty}^{+\infty}d\xi\,\xi Q^\nu=0, \tag{20}$$

in the limit as $\nu\to 0$,

$$\langle\xi\rangle=\int_{-\infty}^{+\infty}d\xi\,\xi Q=-\rho\langle s\rangle\neq 0. \tag{21}$$

In other words, a finite amount of $\xi=u_x$ has gone to the shocks in the limit as $\nu\to 0$. Hence, using (21) at the right-hand side of (19), this equation becomes

$$\begin{aligned}\lim_{\xi_*\to-\infty}\xi_*^2Q(\xi_*,t)&=\frac{\rho}{2}(\langle s\xi_-\rangle+\langle s\xi_+\rangle)-\frac{d}{dt}\langle\xi\rangle\\ &=\frac{\rho}{2}(\langle s\xi_-\rangle+\langle s\xi_+\rangle)+\frac{d}{dt}(\rho\langle s\rangle)=0.\end{aligned} \tag{22}$$

The last equality follows from (15).

One main message in Ref. 2 is the claim that the argument in Ref. 1 which led to the strict bound (6) is insufficient. Here we paraphrase the argument which led to this claim. One may always write (9) as

$$|\xi|^3Q^\nu(\xi)=\int_{-\infty}^{\xi_M}d\xi'\;\xi' F^\nu(\xi')+\int_{\xi_M}^{\xi}d\xi'\;\xi' F^\nu(\xi'), \tag{23}$$

where ξ_M is defined as the value at which the $-\alpha$ range is masked by the viscous range. Assuming the latter behaves as $\nu C|\xi|^{-1}$ [see Refs. 2 and (A20)] then ξ_M is determined from solving

$$C_-|\xi|^{-\alpha}=\nu C|\xi|^{-1}, \tag{24}$$

which gives $\xi_M=-C_0\nu^{-1/(\alpha-1)}$ with $C_0=(C/C_-)^{-1/(\alpha-1)}$. In the limit as $\nu\to 0$, the second term at the right-hand side of (23) gives

$$\lim_{\nu\to 0}\int_{\xi_M}^{\xi}d\xi'\;\xi' F^\nu(\xi')=\int_{-\infty}^{\xi}d\xi'\;\xi' F(\xi'), \tag{25}$$

which goes to 0 as $\xi\to-\infty$. Therefore, whether $\alpha>3$ depends on whether

$$\lim_{\nu\to 0}\int_{-\infty}^{\xi_M}d\xi'\;\xi' F^\nu(\xi')=0 \tag{26}$$

holds. Since

$$\xi_M^3Q^\nu(\xi_M)=\int_{-\infty}^{\xi_M}d\xi'\;\xi' F^\nu(\xi'), \tag{27}$$

an equivalent form of (26) is

$$\lim_{\nu\to 0}\xi_M^3Q^\nu(\xi_M)=0. \tag{28}$$

This argument gives another way of appreciating the difference between $\alpha=3$ and $\alpha>3$. However, it does not allows one to address the issue of whether $\alpha=3$, for the simple reason that the validity of (26) and (28) depends sensitively on the value of ξ_M, which cannot be known prior to knowing α. Unlike what is claimed in Ref. 2, ξ_M cannot be an arbitrary choice that satisfies $\xi_0/\xi_M\to 0$, $\nu\xi_M\to 0$ as $\nu\to 0$. For example if we choose $\xi_M=\xi_N=-C_1\nu^{-1/2}$, from (A20)

$$\lim_{\nu\to 0}\xi_N^3Q^\nu(\xi_N)=CC_1^2\neq 0, \tag{29}$$

regardless the value of α. On the other hand, it is easy to see that if $\xi_M=o(\nu^{-1/2})$, then (26) and (28) hold. The important technical point in Ref. 1 is to find ways to circumvent this path. That was done by studying directly the inviscid limit of Q^ν. Here we have presented a direct argument based on the integral expression for Q^ν. As a by-product, we have

$$\xi_M = o(\nu^{-1/2}). \tag{30}$$

It is also worth stressing that even though (9) and (16) look similar, they are not equivalent. (9) is meant for the case of finite ν and is derived using straightforward integration from (7). (16) is valid for the limiting PDF Q and its derivation is much more nontrivial. In Appendix C, we show that (16) can be derived directly from (5) using the realizability constraint over the *whole* ξ line, as well as the additional information provided by (15). It shows that the only realizable steady state solutions of (5) has the form (16). Other solutions violate non-negativity either for $\xi \to -\infty$ or for $\xi \to +\infty$. The argument in Ref. 1 was a global argument, not localized at very large negative values of ξ.

The characterization that "the analysis in (Ref. 1) is done in terms of a split of u and ξ into a part exterior to shocks and a part interior to shocks"[2] also needs more clarification. What was done in Ref. 1 was a derivation of an approximation to $\nu\xi_{xx}$ (or $\nu\xi_x^2$) using boundary layer analysis and matched asymptotics in order to evaluate the limit of F^ν as $\nu \to 0$. The same technique was used here to evaluate directly the limit of Q^ν (for large negative ξ) as $\nu \to 0$. This approximation is uniformly valid except at shock creation and collision whose contributions to F^ν is of lower order. It can also be systematically improved if additional information is required.

Going back to the statistical stationary state, what is the actual value of α? Reference 6 predicted that $\alpha = 7/2$ under the assumption that the main contribution to Q for large negative values of ξ comes from neighborhoods of shock creation points (preshocks). This geometric argument was expanded in Ref. 1 in the context of (5) and in particular the form of F: under the geometric argument $F(\xi) \sim C|\xi|^{-5/2}$ with $C<0$ for $\xi \ll -\xi_0$. Reference 2 further expanded the geometric argument and obtained values of α in (3, 7/2) by considering special singular data. Polyakov[8] gave an example that gives $\alpha = 3$. However these are rather pathological situations that lie outside the regime of interest here, i.e., the case of smooth forcing. By studying the master equation for the environment of shocks, Ref. 9 verifies that indeed the main contribution does come from shock creation points, and thereby confirms $\alpha = 7/2$.

In conclusion, we stress that there are many ways to exclude the possibility of having $|\xi|^{-3}$ behavior for the left tail of Q in the case of smooth Gaussian force. The discussion in Ref. 2 provides yet another way of understanding the different consequences of $\alpha = 3$ and $\alpha > 3$, but it is not the right way to address the issue of whether $\alpha = 3$.

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APPENDIX A: EVALUATION OF Q^ν

In this appendix we perform the computation of Q^ν to $O(\nu)$. We first put (9) in a form which is more convenient for the calculation. Notice that

$$F^\nu(\xi,t) = G^\nu_{\xi\xi}(\xi,t), \tag{A1}$$

where

$$G^\nu(\xi,t) = -\nu\langle \xi_x^2(x,t)\,\delta(\xi - \xi(x,t))\rangle. \tag{A2}$$

Thus, from (9),

$$Q^\nu \sim -|\xi|^{-3}G^\nu - |\xi|^{-2}G^\nu_\xi \quad \text{for } \xi \ll -\xi_0. \tag{A3}$$

We now evaluate G^ν. For statistically homogeneous situations, the averages at the right-hand side of (A2) can be evaluated upon resorting to spatial ergodicity to replace the ensemble average by the space average:

$$G^\nu(\xi,t) = -\nu \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dx\, \xi_x^2\,\delta(\xi - \xi(x,t)). \tag{A4}$$

In the limit of small ν only small intervals around the shocks will contribute to this integral. In these layers, we use boundary layer analysis to evaluate $\xi(x,t)$. This analysis was outlined in Ref. 1 (for details see Ref. 9): Let y be a shock position. Near y, u can be expressed as

$$u(x,t) = u^{\text{in}}(x,t) = v\left(\frac{x-y}{\nu}, t\right), \tag{A5}$$

and the expression for $v(z,t)$ can be obtained via a series expansion in ν. It yields $v = v_0 + \nu v_1 + O(\nu^2)$, with

$$v_0(z,t) = \bar{u} - \frac{s}{2}\tanh\left(\frac{sz}{4}\right). \tag{A6}$$

v_1 is a solution of

$$v_{0_t} + (v_0 - \bar{u})v_{1_z} + v_1 v_{0_z} = v_{1_{zz}} + f. \tag{A7}$$

The actual expression of v_1 is rather complicated, and the only information really needed about v_1 to evaluate (A4) is the values of v_{1z} as $z \to \pm\infty$. Let $\lim_{z\to\pm\infty} v_{1z} = \xi_\pm$. Then

$$\xi_\pm = \mp \frac{2\bar{u}_t}{s} - \frac{s_t}{s} \pm \frac{2f}{s}, \tag{A8}$$

or, equivalently,

$$s_t = -\frac{s}{2}(\xi_- + \xi_+), \quad \bar{u}_t = \frac{s}{4}(\xi_- - \xi_+) + f. \tag{A9}$$

In these expressions, the values of s and $\bar{u} = dy/dt$ must be obtained by matching $u^{\text{in}}(x,t)$ with the solution of the Burgers equation outside the shock layer, say, $u^{\text{out}}(x,t)$: this eventually produces an approximation for $u(x,t)$ uniformly valid except at shock creation and collision.

Using the results of the boundary layer analysis, to $O(\nu)$ (A4) can be estimated as ($\xi^{\text{in}} = u_x^{\text{in}}$)

$$G^{\nu}(\xi,t)=-\nu\lim_{L\to\infty}\frac{N}{2L}\frac{1}{N}\sum_j\int_{j\text{th } layer}dx\,(\xi_x^{\text{in}})^2\delta(\xi-\xi^{\text{in}}(x,t)),\tag{A10}$$

or, picking any particular shock layer, going to the stretched variable $z=(x-y)/\nu$, and taking the limit as $L\to\infty$,

$$G^{\nu}(\xi,t)=-\rho\int ds\,d\bar{u}\,d\xi_-d\xi_+T(\bar{u},s,\xi_-,\xi_+,s,t)\times\int_{-\infty}^{+\infty}dz\;\eta_z^2\delta(\xi-\eta(z,t)),\tag{A11}$$

where we defined $\eta=\nu^{-1}v_{0z}+v_{1z}\approx\nu^{-1}v_{0z}+\xi_\pm$. Here $T(\bar{u},s,\xi_-,\xi_+,t)$ is the PDF of $\bar{u},s,\xi_-,\xi_+$ conditional on the existence of a shock at $x=y$; it arises because $\eta(z,t)$ depends parametrically on these (random) variables. To perform the z integral in (A11), we change the integration variable to $\xi'=\eta-\xi_-$ for $z<0$ and to $\xi'=\eta-\xi_+$ for $z>0$. Then, using the equation for $v_0,(v_0-\bar{u})v_{0z}=v_{0zz}$, as well as (A6), we have for $z<0$,

$$\eta_z=((v_0-\bar{u})(\eta-\xi_-))=(\tfrac14 s^2+2\nu(\eta-\xi_-))^{1/2}(\eta-\xi_-),\tag{A12}$$

and for $z>0$,

$$\eta_z=((v_0-\bar{u})(\eta-\xi_+))=-(\tfrac14 s^2+2\nu(\eta-\xi_+))^{1/2}(\eta-\xi_+).\tag{A13}$$

Thus

$$dzn_z^2=\begin{cases}d\xi' & \xi'(\tfrac14 s^2+2\nu\xi')^{1/2}\quad \text{for } z<0\\ -d\xi' & \xi'(\tfrac14 s^2+2\nu\xi')^{1/2}\quad \text{for } z>0\end{cases}.\tag{A14}$$

Also, $\xi'\in[-s^2/8\nu,0]$ as $z\in[-\infty,0]$ or $z\in[0,+\infty]$. Combining these results gives

$$\begin{aligned}-\int_{-\infty}^{+\infty}&dz\;\eta_z^2\delta(\xi-\eta(z,t))\\ &=\int_{-s^2/8\nu}^{0}d\xi'\;\xi'(\tfrac14 s^2+2\nu\xi')^{1/2}\delta(\xi-\xi'-\xi_-)\\ &\quad+\int_{-s^2/8\nu}^{0}d\xi'\;\xi'(\tfrac14 s^2+2\nu\xi')^{1/2}\delta(\xi-\xi'-\xi_+)\\ &=(\xi-\xi_-)(\tfrac14 s^2+2\nu(\xi-\xi_-))^{1/2}\times(H(\xi_--\xi)\\ &\quad-H(\xi_--s^2/8\nu-\xi))+(\xi-\xi_+)(\tfrac14 s^2+2\nu(\xi-\xi_+))^{1/2}\\ &\quad\times(H(\xi_+-\xi)-H(\xi_+-s^2/8\nu-\xi)),\end{aligned}\tag{A15}$$

where $H(x)$ is the Heaviside function. Inserting this expression in (A11) then results in

$$G^{\nu}(\xi,t)=2\rho\int_{\xi}^{+\infty}d\xi'\int_{-\infty}^{s_*}ds(\xi-\xi')A(s,\xi-\xi')V(s,\xi',t),\tag{A16}$$

where $s_*=-2(2\nu(\xi'-\xi))^{1/2}$, $A(s,\xi)$ was given in (11) and we used the consistency constraint

$$\int d\bar{u}\,d\xi_\mp T(\bar{u},s,\xi_-,\xi_+,t)=V_\pm(s,\xi_\pm,t).\tag{A17}$$

We also have

$$G_\xi^{\nu}(\xi,t)=2\rho\int_{\xi}^{+\infty}d\xi'\int_{-\infty}^{s_*}ds\,A(s,\xi-\xi')V(s,\xi',t)+2\nu\rho\int_{\xi}^{+\infty}d\xi'\int_{-\infty}^{s_*}ds\,\frac{\xi-\xi'}{A(s,\xi-\xi')}V(s,\xi',t).\tag{A18}$$

At statistical steady state, $V(s,\xi,t)\equiv V(s,\xi)$, and $G^{\nu}(\xi,t)\equiv G^{\nu}(\xi)$, $G_\xi^{\nu}(\xi,t)\equiv G_\xi^{\nu}(\xi)$. Then inserting (A16), (A18) in (A3) gives (10).

Note that if one neglects the $O(1)$ term in the expansion for ξ^{in} (an assumption we do *not* make), then $V(s,\xi,t)=S(s,t)\delta(\xi)$, where $S(s,t)$ is the conditional PDF of $s(y,t)$. At statistical steady state this is $V(s,\xi)=S(s)\delta(\xi)$ and (10) reduces to

$$Q^{\nu}\sim 2\nu\rho|\xi|^{-1}\int_{-\infty}^{-2(2\nu|\xi|)^{1/2}}ds\,\frac{S(s)}{(\tfrac14 s^2+2\nu\xi)^{1/2}}.\tag{A19}$$

This is the expression obtained by Gotoh and Kraichnan[5] for Q^{ν} in the viscous range. (A19) is hard to justify since it amounts to assessing the accuracy of the approximation $V(s,\xi,t)\approx S(s,t)\delta(\xi)$ for large negative ξ. Granting (A19), Q^{ν} can be further simplified if one assumes that the first inverse moment of the shock amplitude s is finite, i.e., $\langle|s|^{-1}\rangle=-\int_{-\infty}^{0}ds\,S(s)/s<+\infty$. Then, for $\xi=o(\nu^{-1})$, (A19) reduces to

$$Q^{\nu}\sim\nu C|\xi|^{-1},\tag{A20}$$

$$C=4\rho\langle|s|^{-1}\rangle.\tag{A21}$$

(A20) describes the well-known $|\xi|^{-1}$ viscous range (the -1 range). This expression can be combined with (16) to give

$$Q^{\nu}\sim C_-|\xi|^{-\alpha}+\nu C|\xi|^{-1},\tag{A22}$$

on the range $\xi=o(\nu^{-1})$, $\xi\ll-\xi_0$. Here $C_-|\xi|^{-\alpha}=|\xi|^{-3}\int_{-\infty}^{\xi}d\xi'\;\xi'F(\xi')$.

APPENDIX B: DERIVATION OF (14)

To get (15) we need to evaluate the time derivative of (using ergodicity)

$$\rho\langle s\rangle=\lim_{L\to\infty}\frac{N}{2L}\frac{1}{N}\sum_{j=1}^{N}s(y_j,t),\tag{B1}$$

where N is the number of shocks in $[-L,L]$ and the y_j's are their locations. Using (A9), clearly the time derivative of (B1) will give (15) if the time dependence of N does not make any contribution. N varies due to shock creation or shock collision. Consider the creation first, and assume a shock is created at position y_1 at time t_1. Then one has in (B1) a term like (disregarding the factor $1/2L$)

$$T_1=s(y_1,t)H(t-t_1),\tag{B2}$$

where $H(x)$ is the Heaviside function. Time differentiation of (B2) gives

$$\frac{dT_1}{dt}=\frac{d}{dt}s(y_1,t)H(t-t_1)+s(y_1,t)\delta(t-t_1). \tag{B3}$$

The second term accounts for the time dependence of N. Since the shock amplitude is zero at creation, $s(y_1,t)\delta(t-t_1)=s(y_1,t_1)\delta(t-t_1)=0$. This means that the term dN/dt makes no contribution to the time derivative of (B1) at shock creation. Consider now the merging events. Assume the shocks located at position y_2 and y_3 merge into one shock located at position y_1 at time t_1. By definition $y_1(t_1)=y_2(t_1)=y_3(t_1)$. Such an event contributes in (B1) by a term like

$$T_2=s(y_1,t)H(t-t_1)+(s(y_2,t)+s(y_3,t))H(t_1-t). \tag{B4}$$

Time differentiation of (B4) gives

$$\frac{dT_2}{dt}=\frac{d}{dt}s(y_1,t)H(t-t_1)+\frac{d}{dt}(s(y_2,t)+s(y_3,t))H(t_1-t)$$
$$+s(y_1,t)\delta(t-t_1)-(s(y_2,t)+s(y_3,t))\delta(t-t_1). \tag{B5}$$

Since shock amplitudes add up at collision:

$$\lim_{t\to 0+} s(y_1(t_1+t),t_1+t)$$
$$=\lim_{t\to 0+}(s(y_2(t_1-t),t_1-t)+s(y_3(t_1-t),t_1-t)). \tag{B6}$$

Thus the terms in (B5) involving δ functions vanish. This means that the term dN/dt makes no contribution to the time derivative of (B1) at shock collision. Hence (15).

APPENDIX C: GLOBAL REALIZABILITY CONSTRAINTS

Here we study (5) at steady state

$$0=\xi Q+(\xi^2 Q)_\xi+B_1 Q_{\xi\xi}+F. \tag{C1}$$

We will show that the only non-negative solution of (C1) is

$$Q_s(\xi)=\frac{1}{B_1}\int_{-\infty}^{\xi}d\xi'\ \xi' F(\xi')-\frac{\xi e^{-\Lambda}}{B_1}\int_{-\infty}^{\xi}d\xi'\ e^{\Lambda'}G(\xi'), \tag{C2}$$

where $\Lambda=\xi^3/3B_1$ and

$$G(\xi)=F(\xi)+\frac{\xi}{B_1}\int_{-\infty}^{\xi}d\xi'\ \xi' F(\xi'). \tag{C3}$$

We will also show that for large positive ξ, F decays in such a way that

$$\lim_{\xi\to+\infty}\xi^{-2}e^{\Lambda}F(\xi)=0, \tag{C4}$$

assuming the limit exists. Note that from (14), the first moment of F exists. Hence $\xi F(\xi)$ is integrable at $-\infty$.

Before proving (C2) and (C4), let us consider some simple consequences. For $\xi<0$, integrating by parts in (C2) gives

$$Q_s(\xi)=-\frac{\xi e^{-\Lambda}}{B_1^2}\int_{-\infty}^{\xi}d\xi'\frac{e^{\Lambda'}}{\xi'^2}\int_{-\infty}^{\xi'}d\xi''\ \xi'' F(\xi''). \tag{C5}$$

Similarly for $\xi>0$, we have

$$Q_s(\xi)=C_+\xi e^{-\Lambda}-\frac{\xi e^{-\Lambda}}{B_1}\int_{\xi}^{+\infty}d\xi'\frac{e_{\Lambda'}}{\xi'^2}\int_{\xi'}^{+\infty}d\xi''\ \xi'' F(\xi''), \tag{C6}$$

with

$$C_+=-\frac{1}{B_1}\int_{-\infty}^{+\infty}d\xi\, e^{\Lambda}G(\xi). \tag{C7}$$

Here we used (14) [steady state of (15)], i.e.,

$$\int_{-\infty}^{+\infty}d\xi\ \xi F=\frac{\rho}{2}(\langle s\xi_-\rangle+\langle s\xi_+\rangle)=0. \tag{C8}$$

Note that using (C8) one readily shows that C_+ is finite if (C4) holds. From (C5) and (C6) it follows that (C2) behaves asymptotically as

$$Q_s(\xi)\sim\begin{cases}|\xi|^{-3}\int_{-\infty}^{\xi}d\xi'\ \xi' F(\xi') & \text{as}\quad \xi\to-\infty\\ C_+\xi e^{-\Lambda} & \text{as}\quad \xi\to+\infty\end{cases}. \tag{C9}$$

To prove (C2) and (C4) we first note that the general solution of (C1) is

$$Q(\xi)=Q_s(\xi)+C_1Q_1(\xi)+C_2Q_2(\xi), \tag{C10}$$

where C_1 and C_2 are constants, Q_1 and Q_2 are two linearly independent solutions of the homogeneous equation associated with (C1). Two such solutions are

$$Q_1(\xi)=\xi e^{-\Lambda}, \tag{C11}$$

$$Q_2(\xi)=1-\frac{\xi e^{-\Lambda}}{B_1}\int_{-\infty}^{\xi}d\xi'\ \xi' e^{\Lambda'}. \tag{C12}$$

For $\xi<0$, after integration by part Q_2 can be expressed as

$$Q_2(\xi)=-\xi e^{-\Lambda}\int_{-\infty}^{\xi}d\xi'\frac{e^{\Lambda'}}{\xi'^2}. \tag{C13}$$

We now show that the realizability constraint requires that $C_1=C_2=0$. First, one readily checks that Q_1 grows unbounded as $\xi\to-\infty$, while $\lim_{\xi\to-\infty}Q_2=\lim_{\xi\to-\infty}Q_3=0$. Hence in order that Q be integrable we must set $C_1=0$ and the general solution of (C1) is

$$Q(\xi)=C_2Q_2(\xi)+Q_s(\xi). \tag{C14}$$

For large negative ξ, this leads to the expansion

$$Q(\xi)\sim C_2B_1|\xi|^{-3}+|\xi|^{-3}\int_{-\infty}^{\xi}d\xi'\ \xi' F(\xi'). \tag{C15}$$

For large positive ξ, we must distinguish two cases. If (C4) does not hold, then [using (C8)]

$$Q(\xi)\sim -C_2B_1\xi^{-3}+\xi^{-3}\int_{\xi}^{+\infty}d\xi'\ \xi' F(\xi'). \tag{C16}$$

In contrast, if (C4) holds, then

$$Q(\xi) \sim -C_2 B_1 \xi^{-3} + C_+ \xi e^{-\Lambda}. \tag{C17}$$

In (C15)–(C17), if nonzero the C_2 term at the right-hand side will dominate the second term. However, since the C_2 term has opposite sign as $\xi \to \pm\infty$, it must be zero, i.e., we must set $C_2 = 0$. This proves that $Q = Q_s$. Furthermore, since the F term at the right-hand side of (C16) is negative [recall that from (13), $F \leq 0$], this solution must be rejected in order that Q be non-negative. Thus (C4) must hold.

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