1 Week 2: Risk Neutral Pricing

Lecture 2 was a bit dense mathematically speaking, so I will highlight the most important parts here, without insisting too much on the technical details. Results are presented in decreasing order of importance, with a grade from 1 (least important) to 10 (most important)

As always, good intuition comes from the discrete time and space model, for example a binomial tree. In that setting, all the formulas we present are easy to derive.

The cornerstone result of the lecture, and the only really important thing to remember is the following:

**Risk Neutral Pricing formula and stock Dynamics (importance: $$+\infty$$)**

Given a stock price with dynamics

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$$

where $$W_t$$ is a Brownian motion under some historical measure $${\mathbb{P}}$$, some interest rate $$R_t$$, there exists a measure $${\mathbb{Q}}$$ such that the price at time $$t$$ of the option maturing at $$T > t$$ with payoff $$V_T$$ is given by:

$$V_t = \mathbb{E}_Q \left[ e^{-\int_t^T R_s ds} V_T | \mathcal{F}_t \right]$$

The stock’s dynamics can be rewritten using another Brownian motion in this ‘risk neutral’ measure $${\mathbb{Q}}$$, $${\tilde{W}}_t$$;

$$dS_t = R_t S_t dt + \sigma_t S_t d{\tilde{W}}_t$$

All the stochastic processes involved $$\alpha_t, \sigma_t, R_t$$ are all $${\mathcal{F}}_t$$ measurable.

In class we saw the version where $$R_t$$ was constant, and the processes $$\alpha_t S_t, \sigma_t S_t$$ are replaced by $$\alpha_t, \sigma_t$$ (clearly equivalent).

How did we get the risk neutral measure?

**Risk Neutral measure (importance: 7)**

The risk neutral measure was obtained by ensuring that the discounted stock price was a martingale in that measure;

$$\mathbb{E}_Q \left[ e^{-\int_0^T R_u du} S_t | \mathcal{F}_s \right] = e^{-\int_0^s R_u du} S_s$$

In order to do so, we computed $$d(e^{-\int_0^T R_u du} S_t)$$ by Ito’s lemma and realized it was of the type

$$d(e^{-\int_0^t R_u du} S_t) = \text{stuff (drift } dt + d{\tilde{W}}_t \text{)}$$

We used Girsanov’s theorem to find a measure $${\mathbb{Q}}$$ such that $$d{\tilde{W}}_t = (\text{drift } dt + dW_t)$$ was a Brownian motion.

So why is the risk neutral pricing formula true?
Justification of the risk Neutral formula (importance: 5)
To justify that such a formula is true, we proceeded to find a strategy $\Delta_t$ such that trading according to this strategy yielded the exact payoff $V_T$ at time $T$, without ever injecting money in our trading scheme (except at initial time);

More precisely, there exists an initial wealth $X_t$ and a strategy $\Delta_t$ such that:

$$dX_t = \Delta_t dS_t + (X_t - \Delta_t S_t) R_t dt,$$

(at each time we hold $\Delta_t$ shares of stock and put the rest in cash in a self-financing manner)

$$X_T = V_T, \quad \text{almost surely}$$

This means that the value at time $t$ of our wealth $X_t$, has to agree with the value of the option at time $t$, $V_t$. Indeed, if we had $X_t < V_t$, sell the option at price $V_t$, pocket $V_t - X_t > 0$, and use $X_t$ and the strategy $\Delta_t$ until time $T$.

At time $T$, you will get $X_T$ from the strategy, and you owe $V_T$ from the sale of the option. Since $X_T = V_T$, you will always get from the strategy the exact amount you owe, hence yielding a sure profit of $V_t - X_t$ in any circumstances.

If $V_t > X_t$, do the reverse strategy.

It turns out that in that risk neutral measure, $D_t X_t$ is a martingale, and hence so is $D_t V_t$ which yields the risk neutral pricing formula.

So how do we get such a strategy $\Delta_t$?

Finding the strategy (importance: 1)
We didn’t see this in class this week, but theoretically we can prove the existence of a replicating strategy in this context;
We can use the Martingale representation theorem on a judiciously constructed quantity

$$E_Q[D_T V_T | F_t]$$