Continuous Time Finance

Homework 1: Review

In all exercises, \((\Omega, \mathcal{F}, \mathbb{P})\) represents a probability space, \((W_t)_{t \geq 0}\) a standard Brownian motion on that probability space, and \((\mathcal{F}_t)_{t \geq 0}\) the filtration it generates.

Exercise 1: Brownian motion definitions

Consider the following two definitions of Brownian Motion:

**Definition 1:**

(i) \(W_0 = 0\) \(\mathbb{P}\)-almost surely

(ii) \(\forall 0 \leq r < s \leq t < u, W_u - W_t\) is independent of \(W_s - W_r\)

(iii) \(\forall s \neq t, W_t - W_s \sim \mathcal{N}(0, |t - s|)\)

(iv) \(t \mapsto W_t\) is continuous \(\mathbb{P}\)-almost surely

**Definition 2:**

(i) \(W_0 = 0\) \(\mathbb{P}\)-almost surely

(ii) \(\forall n \in \mathbb{N}^*, \text{ and } 0 \leq t_1 < \cdots < t_n, (W_{t_1}, \cdots, W_{t_n})\) is a Gaussian vector of mean 0 \(\in \mathbb{R}^n\) and covariance matrix \(\Sigma = [\min(t_i, t_j)]_{i,j=1,\cdots,n} \in \mathbb{R}^{n \times n}\)

(iii) \(t \mapsto W_t\) is continuous \(\mathbb{P}\)-almost surely

Show that both definitions are equivalent.

Exercise 2: Reflection principle for Brownian motion

Define the stochastic process for \(t \in \mathbb{R}^+\):

\[ M_t = \max_{0 \leq s \leq t} W_s \]

which is the running maximum of a Brownian motion. Also define the random variable for \(b \in \mathbb{R}\):

\[ \tau_b = \inf\{t \geq 0 : W_t = b\} \]

which gives the first time \(W\) reaches the level \(b\).

The goal is to derive their distribution. To do so, we will first compute:

\[ F(a, b) = \mathbb{P}(W_t \leq a, M_t \geq b) \]

by the so called 'Reflection principle'.
(a) Let \( 0 \leq a \leq b, \ t \in \mathbb{R}^+ \) and define the stochastic process for \( s \in [0, t] \):

\[
\tilde{W}_s = \begin{cases} 
W_s, & \text{if } s \leq \tau_b \\
2b - W_s, & \text{if } s \geq \tau_b 
\end{cases}
\]

Plot a Brownian path \( W \) on \([0, t]\), satisfying: \( \tau_b < t \) and \( W_t \leq a \). Plot the corresponding path (i.e. the same \( \omega \)) for \( \tilde{W} \) on \([0, t]\). In what interval does \( \tilde{W}_t \) end up?

(b) Let’s admit that \( \tilde{W} \) is still a Brownian motion\(^1\), and hence

\[
F(a, b) = \mathbb{P}(\tilde{W}_t \leq a, \max_{0 \leq s \leq t} \tilde{W}_s \geq b)
\]

By using the definition of \( \tilde{W} \), and by noting that:

\[
\{ \omega \in \Omega : \max_{0 \leq s \leq t} \tilde{W}_s \geq b \} = \{ \omega \in \Omega : \tau_b \leq t \},
\]

show that:

\[
F(a, b) = \mathbb{P}(W_t \geq 2b - a)
\]

(c) Deduce \( \mathbb{P}(M_t \geq b) \) (Hint\(^2\)), \( \mathbb{P}(\tau_b \leq t) \), the densities of \( M_t, \tau_b \) as well as the joint distribution of \((W_t, M_t)\).

Exercise 3: Time independent boundary value problems

Let \( D = [a, b] \) and consider the stochastic process:

\[
dX_t = \alpha(X_t)dt + \beta(X_t)dW_t
\]

Note that \( \alpha, \beta \) are deterministic functions that do not depend on time. Define

\[
u(x) = \mathbb{E} \left[ \int_0^{\tau_x} f(X_s)ds + g(X_{\tau_x}) \bigg| X_0 = x \right]
\]

for \( x \in D \), where

\[
\tau_x = \{ \inf t \geq 0 : X_t \notin D \}
\]

Note that \( \tau_x \) depends on \( x \) due to the starting point \( X_0 = x \). \( f, g \) are deterministic functions, that represent respectively a running payoff and a final time payoff.

In other words, we are playing a game where we receive (or pay) \( f(X_t)dt \) for each unit of time \( dt \) as long as \( X_t \) remains in \( D \). As soon as \( X_t \) exits \( D \), we get (or pay) \( g(X_t) \). \( u(x) \) represents our expected payoff from this game.

The goal is to show that \( u \) solves the ODE (becomes a PDE if \( x \in \mathbb{R}^n \)):

\[
\begin{cases}
\alpha(x) \frac{d}{dx} u(x) + \beta^2(x) \frac{d^2}{dx^2} u(x) + \frac{\beta^2(x)}{2} \frac{d^2}{dx^2} u(x) + f(x) = 0, & x \in D \\
u(a) = g(a), \ u(b) = g(b)
\end{cases}
\]

(a) Apply Ito’s lemma to \( u(X_t) \) and integrate both sides of the equation between 0 and \( \tau_x \).

(b) Assume that \( u \) does indeed solves the ODE above. Deduce that

\[
u(x) = \mathbb{E} \left[ \int_0^{\tau_x} f(X_s)ds + g(X_{\tau_x}) \bigg| X_0 = x \right]
\]

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\(^1\) a consequence of the independence of \( W_{\tau_b} \) and \( W_s - W_{\tau_b} \) for \( s \geq \tau_b \), and of \( u \mapsto W_{\tau_b + u} - W_{\tau_b} \) being a Brownian Motion

\(^2\) Hint: \( \mathbb{P}(M_t \geq b) = \mathbb{P}(M_t \geq b, W_t \leq b) + \mathbb{P}(M_t \geq b, W_t \geq b) \)
Hint: you can assume that $\mathbb{E}\left[\int_0^{\tau_x} h(X_t) dW_t \mid X_0 = x\right] = 0$ for any function $h$. This only holds if $\mathbb{E}[\tau_x \mid X_0 = x] < +\infty$, which is not hard to prove here (you are not asked to do this but will get bonus points of you do).

We just showed that if there exists a solution $u \in C^2(D)$ to the ODE, then it is necessarily given by (1). Existence is given by the theory of ODEs or PDEs (under some technical assumptions on $\alpha, \beta, f, g$) and is out of the scope of the class.

(c) Application 1: Let $dX_t = dW_t$ and define

$$p_x = \mathbb{P}(X_{\tau_x} = b)$$

Show that

$$p_x = \frac{x - a}{b - a}$$

Hint: $p_x = \mathbb{E}[1_{X_{\tau_x} = b} \mid X_0 = x]$

(d) Application 2: Let $dX_t = dW_t$ and define

$$\bar{t}_{[a,b]}(x) = \mathbb{E}[\tau_x \mid X_0 = x]$$

Show that

$$\bar{t}_{[a,b]}(x) = (b - x)(x - a)$$