Homework 3

Pricing and Hedging

Consider a stock $S_t$ in the Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

starting at $S_0 = 100$. For $\sigma = 20\%$ per year, and starting with $r = 0\%$ per year.

Denote by $C_{K,T}$ to be the price of a European call with strike $K$ that matures at time $T = 1$ year. In other words, the payoff is $(S_T - K)^+$. Start with $K = 100$.

1. Compute the price $C_{K,T}$ by a Monte-Carlo simulation, as well as its variance. Try it for various timesteps $dt = T/N$ (e.g. $N_t = 10, 100, 252, 1000$), and number of total paths $N = 10^k, k = 3, 4, 5$.

   In other words, denoting by $S^i_T$ the value of $S_T$ every simulated path $i$, compute

   $$C_{K,T} \approx \frac{1}{N} \sum_{i=1}^{N} (S^i_T - K)^+, \quad v \approx \frac{1}{N - 1} \sum_{i=1}^{N} \left( (S^i_T - K)^+ - \frac{1}{N} \sum_{j=1}^{N} (S^j_T - K)^+ \right)^2$$

2. Another way of simulating the price at time 0 is to use the $\Delta$-hedging strategy provided by the Black-Scholes price.

   Recall this value $\Delta_t$ of the $\Delta$-hedge function of time and the stock position, and justify the formula

   $$(S_T - K)^+ - C_{K,T} = \int_0^T \Delta_t dS_t$$

   By discretizing the integral above, compute the price $C_{K,T}$ by a Monte-Carlo simulation;

   $$C_{K,T} \approx \frac{1}{N} \sum_{i=1}^{N} \left[ (S^i_T - K)^+ - \sum_{k=0}^{N_t-1} \Delta^i_{kdt} (S^i_{(k+1)dt} - S^i_{kdt}) \right]$$

   as well as the variance of this Monte-Carlo simulation.

3. Explain the improvement of variance by the second method, and compare with the analysis in the slides.

   **Hint:** What is theoretically the expectation of the $\Delta$-hedging strategy $\int_0^T \Delta_t dS_t$? Is it highly correlated with $(S_T - K)^+$?