Exercise 1. Let $f$ be absolutely continuous on $[a,b]$ Show that

$$T_a^b(f) = \int_a^b |f'(x)| \, dx$$

and

$$P_a^b(f) = \int_a^b [f']^+.$$

Conclude that if $f$ is in AC then it is the difference of two monotone absolutely continuous functions.

Answer:

In homework 4 exercise 5 we showed that

$$\int_a^b |f'(x)| \, dx \leq T_a^b(f)$$

for all functions $f$ of bounded variation. Since absolutely continuous functions have bounded variation then it suffices to only show the other inequality. Let $\{x_i\}_{i=0}^n$ be a partition of of $[a,b]$. Then

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f'(x) \, dx \right| \leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(x)| \, dx = \int_a^b |f'(x)| \, dx.$$

Taking supremum over all partitions of $[a,b]$ it follows that

$$T_a^b(f) \leq \int_a^b |f'(x)| \, dx.$$

Hence

$$T_a^b(f) = \int_a^b |f'(x)| \, dx.$$

For the second part, we similarly see that for any partition $\{x_i\}_{i=0}^n$ of $[a,b]$ we have

$$\sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ = \sum_{i=1}^n \left[ \int_{x_{i-1}}^{x_i} f'(x) \, dx \right]^+ \leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} [f'(x)]^+ \, dx = \int_a^b [f'(x)]^+ \, dx.$$
Taking supremum over all partitions of \([a, b]\) it follows that

\[ P^b_a(f) \leq \int_a^b [f']^+. \]

For the converse inequality note that \( f(x) = P^x_a(f) - N^x_a(f) + f(a) \), so \( f'(x) = P^x_a(f)' - N^x_a(f)' \) almost everywhere. Since \( P^x_a(f) \) and \( N^x_a(f) \) are nondecreasing functions then \( P^x_a(f)' \geq 0 \) and \( N^x_a(f)' \geq 0 \) almost everywhere. So \([f(x)]^+ = [P^x_a(f)' - N^x_a(f)']^+ \leq P^x_a(f)' \) almost everywhere. So we conclude that

\[ \int_a^b [f']^+ \leq \int_a^b P^x_a(f)' \, dx \leq P^b_a(f) - P^a_a(f) = P^b_a(f), \]

which is the inequality that we wanted. Thus

\[ \int_a^b [f']^+ = P^b_a(f). \]

Finally, since \( P^x_a(f) \) and \( T^x_a(f) \) are both integrals they are in particular absolutely continuous. Since

\[ f(x) = P^x_a(f) - N^x_a(f) + f(a) = 2P^x_a(f) - P^x_a(f) - N^x_a(f) + f(a) = 2P^x_a(f) - T^x_a(f) + f(a) \]

then \( f \) is the difference of two monotone absolutely continuous functions.

**Exercise 2.** Show that if \( f \) is in AC on \([a, b]\) and if \( f \) is never zero there, then \( g = \frac{1}{f} \) is also in AC on \([a, b]\).

**Answer:**

Since \( f \) is a continuous function on the connected set \([a, b]\) and never zero, then either \( f(x) > 0 \) for all \( x \in [a, b] \) or \( f(x) < 0 \) for all \( x \in [a, b] \). Without loss of generality we can assume that \( f(x) > 0 \) for all \( x \in [a, b] \). Since \([a, b]\) is compact then \( f \) attains its minimum on \([a, b]\), so there exists \( c > 0 \) so that \( f(x) \geq c > 0 \) for all \( x \in [a, b] \). Fix \( \varepsilon > 0 \). Since \( f \) is absolutely continuous we find \( \delta > 0 \) so that

\[ \sum_{i=1}^n |f(a_i) - f(b_i)| < c^2 \varepsilon \]
whenever \(\{(a_i, b_i)\}_{i=1}^n\) is a disjoint collection of intervals such that
\[
\sum_{i=1}^n |a_i - b_i| < \delta.
\]
This implies that
\[
\sum_{i=1}^n |g(a_i) - g(b_i)| = \sum_{i=1}^n \left| \frac{1}{f(a_i)} - \frac{1}{f(b_i)} \right| = \sum_{i=1}^n \left| \frac{f(b_i) - f(a_i)}{f(a_i)f(b_i)} \right|
\leq \frac{1}{c^2} \sum_{i=1}^n |f(a_i) - f(b_i)| < \frac{1}{c^2} \epsilon^2 = \epsilon
\]
whenever \(\{(a_i, b_i)\}_{i=1}^n\) is a disjoint collection of intervals such that
\[
\sum_{i=1}^n |a_i - b_i| < \delta.
\]
Hence \(g = \frac{1}{f}\) is absolutely continuous on \([a, b]\).

**Exercise 3.** Show that there is a strictly increasing singular function on \([0, 1]\). (Recall that a monotone function \(f\) on \([a, b]\) is called singular if \(f' = 0\) almost everywhere.)

**Answer:**
We follow the suggestions given by Royden.

**Lemma 1.** Let \(f\) be a monotone increasing function. Then \(f = g + h\) where \(g\) is absolutely continuous and \(h\) is singular, and both \(g\) and \(h\) are monotone increasing.

**Proof:**
Since \(f\) is monotone then \(f'\) exists almost everywhere and is measurable, and \(f' \geq 0\) since \(f\) is increasing. We define
\[
g(x) = \int_a^x f'(t) \, dt
\]
for all \(x\). Now \(g\) is absolutely continuous as an integral and \(g\) is monotone increasing since \(f' \geq 0\). We define \(h = f - g\). Now
\[ h' = f' - g' = f' - f' = 0 \text{ almost everywhere so } h' \text{ is singular. To see that } h \text{ is monotone increasing, fix } x \leq y \text{ and observe that} \]
\[
h(y) - h(x) = f(y) - \int_a^y f'(t) \, dt - f(x) + \int_x^y f'(x) \, dx
\]
\[
= f(y) - f(x) - \int_x^y f'(t) \, dt.
\]
Since \( f \) is monotone then \( \int_x^y f'(t) \, dt \leq f(y) - f(x) \). So we conclude that \( h(x) \leq h(y) \). Finally, since \( f = g + h \) then we are done. \( \square \)

Lemma 2. Let \( f \) be a nondecreasing singular function on \([a,b]\). Then \( f \) has the following property (S): Given \( \varepsilon > 0, \delta > 0 \), there is a finite collection \( \{[y_k, x_k]\}_{k=1}^n \) of nonoverlapping intervals such that

\[
\sum_{k=1}^n |x_k - y_k| < \delta
\]

and

\[
\sum_{k=1}^n (f(x_k) - f(y_k)) > f(b) - f(a) - \varepsilon.
\]

Proof:
Fix \( \varepsilon > 0 \) and \( \delta > 0 \). Denote \( A = \{x \in [a,b] : f'(x) = 0\} \) and \( \kappa = \frac{\varepsilon}{b-a} \). Now if \( x \in A \) then

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = 0,
\]
so there exists \( h_x > 0 \) so that for all \( 0 < h < h_x \),

\[
\frac{f(x+h) - f(x)}{h} < \kappa,
\]
i.e. \( f(x+h) - f(x) < \kappa h \). Now the collection

\[
\mathcal{V} = \{[x, x+h] : x \in A, \, h < h_x \}
\]
is a Vitali cover of \( A \). Since \( m(A) < \infty \) then by Vitali covering theorem there is a finite disjoint collection \( \{[x_i, x_i + h_i]\}_{i=1}^n \subseteq \mathcal{V} \) so that

\[
m\left(A \setminus \bigcup_{i=1}^n [x_i, x_i + h_i]\right) < \delta.
\]
Denote $I_i = [x_i, x_i + h_i]$ for all $i$. Since $m([a, b]) = m(A)$ then
\[
b - a - \sum_{i \leq n} \ell(I_i) = b - a - m(\cup_{i \leq n} I_i) = m(A \setminus \cup_{i \leq n} I_i) < \delta,
\]
so $\sum_{i \leq n} \ell(I_i) > b - a - \delta$. Denote $I'_i \subseteq I_i$ the open interval obtained by taking the endpoints of $I_i$ away. Now the complement of $\cup_{i \leq n} I'_i$ in $[a, b]$ is a finite collection of disjoint closed intervals, denoted by $\{J_i\}_{i=1}^k$, and additionally the possibility exists that the singleton endpoints $\{a, b\}$ are also in the complement in the case of $x_1 = a$ and $x_n + h_n = b$. In this case we treat $J_1 = \{a\}$ and $J_k = \{b\}$. We may assume that the intervals are listed in an increasing order, and we denote $J_i = [a_i, b_i]$ for all $i$. Note that each interval $J_i$ is an interval of the form $[x_{i_n}, x_{i_n+1} + h_{i_n+1}]$ for some $i_n$ unless we are dealing with the singleton endpoints, and $\sum_{i \leq k} \ell(J_i) < \delta$. Thus we have
\[
\sum_{i=1}^k (f(b_i) - f(a_i)) = f(b_k) - f(a_1) + \sum_{i=2}^n (f(a_i) - f(b_{i-1}))
\]
\[
= f(b) - f(a) - \sum_{i=2}^k (f(b_{i-1}) - f(a_i))
\]
\[
> f(b) - f(a) - \kappa \sum_{i=2}^k h_{i_n}
\]
\[
> f(b) - f(a) - \kappa (b - a)\]
\[
= f(b) - f(a) - \epsilon,
\]
which finishes the proof. \(\Box\)

Lemma 3. Let $f$ be a nondecreasing function on $[a, b]$ with property (S) from Lemma 2. Then $f$ is singular.

**Proof:**

From Lemma 1 it follows that $f = g + h$ where $g$ is absolutely continuous and $h$ is singular, and both $g$ and $h$ are nondecreasing. Fix $\epsilon > 0$. Since $g$ is absolutely continuous then there exists $\delta > 0$ so that
\[
\sum_{i=1}^n (g(y_i) - g(x_i)) < \frac{\epsilon}{2}
\]
for any collection of disjoint intervals $\{[x_i, y_i]\}_{i=1}^n$ with
\[ \sum_{i=1}^n |y_i - x_i| < \delta. \]

Now since $f$ has property $(S)$, for the fixed $\varepsilon > 0$ and $\delta > 0$ there is a finite collection $\{[a_i, b_i]\}_{i=1}^n$ of disjoint intervals such that
\[ \sum_{i=1}^n |a_i - b_i| < \delta \]
and
\[ \sum_{i=1}^n (f(b_i) - f(a_i)) > f(b) - f(a) - \frac{\varepsilon}{2}. \]

Since $g(x) = \int_a^x f'(t) \, dt$ for all $x$, then by denoting $b_0 = a$ and $a_{n+1} = b$, we have
\[
g(b) - g(a) = \sum_{i=0}^n (g(a_{i+1}) - g(b_i)) + \sum_{i=1}^n (g(b_i) - g(a_i))
\]
\[
< \sum_{i=0}^n \int_{b_i}^{a_{i+1}} f'(t) \, dt + \frac{\varepsilon}{2}
\]
\[
\leq \sum_{i=0}^n (f(a_{i+1}) - f(b_i)) + \frac{\varepsilon}{2}
\]
\[
= f(b) - f(a) - \sum_{i=1}^n (f(b_i) - f(a_i)) + \frac{\varepsilon}{2}
\]
\[
< f(b) - f(a) - \left(f(b) - f(a) - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \varepsilon.
\]

Since the choice of $\varepsilon > 0$ was arbitrary, it follows that $g(b) = g(a)$ for all $x$. Since $g(a) = 0$ then $g \equiv 0$. Thus $f = h$ and since $h$ was singular then $f$ is singular.

Lemma 4. Let $(f_n)_{n=1}^\infty$ sequence of nondecreasing singular functions on $[a, b]$ so that the function
\[ f(x) = \sum_{n=1}^\infty f_n(x) \]
is everywhere finite. Show that \( f \) is also singular.

**Answer:**

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be fixed. Since

\[
\sum_{n=1}^{\infty} (f_n(b) - f_n(a)) = f(b) - f(a) < \infty,
\]

then there exists \( N \in \mathbb{N} \) such that

\[
\sum_{n=N+1}^{\infty} (f_n(b) - f_n(a)) < \frac{\varepsilon}{2}.
\]

Denote \( g(x) = \sum_{k=1}^{N} f_n(x) \). Since each \( f_n \) is nondecreasing then \( g \) is nondecreasing, and since \( g'(x) = \sum_{k=1}^{N} f'_n(x) = 0 \) for almost every \( x \), then \( g \) is singular. By Lemma 2 the function \( g \) has the property \( (S) \). So there exists a finite collection of disjoint intervals \( \{[a_i, b_i]\}_{i=1}^{m} \) so that

\[
\sum_{i=1}^{m} |b_i - a_i| < \delta
\]

and

\[
\sum_{i=1}^{m} (g(b_i) - g(a_i)) > g(b) - g(a) - \frac{\varepsilon}{2}.
\]

For this same collection of intervals we also have

\[
\sum_{i=1}^{m} (f(b_i) - f(a_i)) = \sum_{i=1}^{m} \left( \sum_{n=1}^{\infty} (f_n(b_i) - f_n(a_i)) \right)
\]

\[
\geq \sum_{i=1}^{m} \left( \sum_{n=1}^{N} (f_n(b_i) - f_n(a_i)) \right)
\]

\[
= \sum_{i=1}^{m} (g(b_i) - g(a_i)) > g(b) - g(a) - \frac{\varepsilon}{2}
\]

\[
= \sum_{n=1}^{N} (f_n(b) - f_n(a)) - \frac{\varepsilon}{2}
\]

\[
= f(b) - f(a) - \sum_{n=N+1}^{\infty} (f_n(b) - f_n(a)) - \frac{\varepsilon}{2}
\]

\[
> f(b) - f(a) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = f(b) - f(a) - \varepsilon.
\]
So we conclude that $f$ has the property $(S)$. Since $f$ is also non-decreasing then by Lemma 3 $f$ is singular.

We now return to our exercise. Since $\mathbb{Q}$ is countable then the set of all intervals with rational endpoints is countable. So we can enumerate this set, say by $\{[a_n, b_n] : n \in \mathbb{N}\}$, where $a_n, b_n \in \mathbb{Q} \cap [0, 1]$ and $a_n < b_n$. We then define $f_n : [0, 1] \to \mathbb{R}$ for each $n \in \mathbb{N}$ by setting

$$f_n(x) = 2^{-n}C\left(\frac{x-a_n}{b_n-a_n}\right),$$

where $C : \mathbb{R} \to \mathbb{R}$ is the Cantor function extended continuously to the whole real line by setting $C|_{(-\infty, 0]} \equiv 0$ and $C|_{[1, \infty)} \equiv 1$. Now note that $f_n(x) = 0$ for all $x \leq a_n$ and $f_n(x) = 2^{-n}$ for all $x \geq b_n$, $f_n'(x) = 0$ almost everywhere and $f_n$ is nondecreasing. It follows that

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

is everywhere finite and thus by Lemma 4 $f$ is singular on $[0, 1]$. It also follows that $f$ is nondecreasing because each $f_n$ is. To see that it is strictly increasing assume that $x < y$. Since the rationals are dense in the reals there exists $q \in \mathbb{Q}$ so that $x < q < y$. So any $f_n$ assigned to a rational interval with left endpoint at $q$ gives $f_n(x) = 0$ and $f_n(y) > 0$. So it follows that $f(x) < f(y)$. Hence $f$ is a strictly increasing singular function on $[0, 1]$.

**Exercise 4.** (a) Let $F$ be AC on $[c, d]$ and let $g$ be strictly increasing and AC on $[a, b]$ with $c \leq g \leq d$. Then $F \circ g$ is AC on $[a, b]$.

**Answer:**

Fix $\varepsilon > 0$. Since $F$ is in AC there is $\kappa > 0$ so that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon$$

whenever $\{[a_i, b_i]\}_{i=1}^{n}$ is a disjoint collection of intervals with

$$\sum_{i=1}^{n} |b_i - a_i| < \kappa.$$
Now use $\kappa > 0$ in the definition of the absolute continuity of $g$ to find $\delta > 0$ so that
\[
\sum_{i=1}^{n} |g(y_i) - g(x_i)| < \kappa
\]
whenever $\{[x_i, y_i]\}_{i=1}^{n}$ is a disjoint collection of intervals with
\[
\sum_{i=1}^{n} |y_i - x_i| < \delta.
\]
So fix a disjoint collection of intervals $\{[x_i, y_i]\}_{i=1}^{n}$ with
\[
\sum_{i=1}^{n} |y_i - x_i| < \delta.
\]
Then since $g$ is increasing and continuous, the intervals $[x_i, y_i]$ map to intervals under $g$. More precisely, we have $g[x_i, y_i] = [g(x_i), g(y_i)]$ for all $i$ and the intervals are disjoint as $g$ is strictly increasing. By the absolute continuity of $g$ we also have for the length of the intervals that
\[
\sum_{i=1}^{n} |g(y_i) - g(x_i)| < \kappa.
\]
So treating the disjoint collection of intervals $\{[g(x_i), g(y_i)]\}_{i=1}^{n}$ in the definition of the absolute continuity of $F$ we have that
\[
\sum_{i=1}^{n} |F(g(y_i)) - F(g(x_i))| < \varepsilon.
\]
So we have shown that there is a $\delta > 0$ such that for any disjoint collection of intervals $\{[x_i, y_i]\}_{i=1}^{n}$ with
\[
\sum_{i=1}^{n} |y_i - x_i| < \delta.
\]
we have
\[
\sum_{i=1}^{n} |(F \circ g)(y_i) - (F \circ g)(x_i)| < \varepsilon.
\]
Hence $F \circ g$ is absolutely continuous.
(b) Let \( E = \{ x : g'(x) = 0 \} \). Then \( m(g(E)) = 0 \).

**Answer:**

Fix \( \varepsilon > 0 \). The proof follows a similar argument as we did in Lemma 2 of Exercise 3. Let \( \kappa = \frac{\varepsilon}{4(b-a)} \). There now exists \( \delta > 0 \) so that \( \int_A |g'| < \frac{\varepsilon}{2} \) whenever \( A \subseteq [a,b] \) with \( m(A) < \delta \). Fix \( x \in E \). Since

\[
\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = 0,
\]

there exists \( h_x > 0 \) so that for all \( 0 < h < h_x \) we have

\[
\frac{g(x+h) - g(x)}{h} < \kappa.
\]

In other words \( g(x+h) - g(x) < \kappa h \) for all \( 0 < h < h_x \). We then define

\[ V = \{ [x, x+h] : x \in E, \ 0 < h < h_x \}, \]

which is a Vitali cover of the set \( E \). Since \( m(E) < \infty \), by Vitali covering theorem we find a finite disjoint subcollection of \( V \), denoted by \( \{ [x_i, x_i + h_i] \}_{i=1}^n \), so that

\[
m\left( E \setminus \bigcup_{i=1}^n [x_i, x_i + h_i] \right) < \frac{\delta}{2}.
\]

Denote \( U = \bigcup_{i=1}^n [x_i, x_i + h_i] \), and choose an open set \( O \supseteq E \setminus U \) with \( m(O \setminus (E \setminus U)) < \frac{\delta}{2} \). Now

\[
m(O) - m(E \setminus U) = m(O \setminus (E \setminus U)) < \frac{\delta}{2},
\]

so

\[
m(O) < \frac{\delta}{2} + m(E \setminus U) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

We then write \( O \) as a countable disjoint union \( O = \bigcup_{k=1}^\infty I_k \) of open intervals \( I_k \), so

\[
m(g(E \setminus U)) \leq m(g(O)) \leq \sum_{k=1}^\infty m(g(I_k)) = \sum_{k=1}^\infty m(\min_{x \in I_k} g(x), \max_{x \in I_k} g(x))
\]

\[
= \sum_{k=1}^\infty (\max_{x \in I_k} g(x) - \min_{x \in I_k} g(x)) \leq \sum_{k=1}^\infty \int_{I_k} |g'| = \int_O |g'| < \frac{\varepsilon}{2},
\]

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since \( m(O) < \delta \). Above we also used the fact that since \( g \) is continuous on the compact intervals \( I_k \) then the maximums and minimums exist. On the other hand, if \( t \in [x_i, x_i + h_i] \) then \( 0 < t - x_i < h_i \), so \( |g(t) - g(x_i)| < \kappa h_i \). Thus
\[
g[x_i, x_i + h_i] \subseteq B(g(x_i), \kappa h_i),
\]
which implies that
\[
m(g[x_i, x_i + h_i]) \leq m(B(g(x_i), \kappa h_i)) \leq 2\kappa h_i.
\]
We then observe that
\[
m(g(E \cap U)) \leq m(g(U)) \leq \sum_{i=1}^{n} m(g[x_i, x_i + h_i]) \leq \sum_{i=1}^{n} 2\kappa h_i
\]
\[
\leq 2\kappa(b-a) = \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{2}.
\]
So finally,
\[
m(g(E)) \leq m(g(E \setminus U)) + m(g(E \cap U)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\]
Since the choice of \( \varepsilon > 0 \) was arbitrary, then \( m(g(E)) = 0 \).

**Exercise 5.** Let \( g \) be an absolutely continuous monotone function on \([0, 1]\) and \( E \) a set of measure zero. Then \( g(E) \) has measure zero.

**Answer:**
Fix \( \varepsilon > 0 \). Since \( g \) is absolutely continuous there exists \( \delta > 0 \) so that
\[
\sum_{i=1}^{n} |g(b_i) - g(a_i)| < \varepsilon
\]
whenever \( \{[a_i, b_i]\}_{i=1}^{n} \) is a disjoint collection of intervals with
\[
\sum_{i=1}^{n} |b_i - a_i| < \delta.
\]
Take an open set \( O \supseteq E \) so that \( m(O) < \delta \). Such a set exists since \( m(E) = 0 \). We then write \( O \) as a countable disjoint union \( O = \bigcup_{k=1}^{\infty} I_k \) of open intervals \( I_k = (a_k, b_k) \). Let \( m \in \mathbb{N} \) be fixed. Then
\[
\sum_{k=1}^{m} |b_k - a_k| = \sum_{k=1}^{m} \ell(I_k) \leq \sum_{k=1}^{\infty} \ell(I_k) = m \left( \bigcup_{k=1}^{\infty} I_k \right) = m(O) < \delta.
\]
Since $g$ is monotone and continuous then $g(I_k)$ is the interval $[g(a_k), g(b_k)]$ or $[g(b_k), g(a_k)]$, so by absolute continuity of $g$ we have
\[
\sum_{k=1}^{m} m(g(I_k)) = \sum_{k=1}^{m} |g(b_k) - g(a_k)| < \varepsilon.
\]
Since this applies for all $m \in \mathbb{N}$ then
\[
\sum_{k=1}^{\infty} \ell(g(I_k)) < \varepsilon.
\]
In particular, it follows that
\[
m(g(E)) \leq m(g(O)) \leq \sum_{k=1}^{\infty} m(g(I_k)) < \varepsilon.
\]
Since this holds for all $\varepsilon > 0$, then $m(g(E)) = 0$.

**Exercise 6.** Let $g$ be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c$ and $g(b) = d$.

(a) Show that for any open set $O \subseteq [c, d]$, $m(O) = \int_{g^{-1}(O)} g'(x) \, dx$.

**Answer:**
We first write the open set $O$ as a countable disjoint union $O = \bigcup_{k=1}^{\infty} I_k$ of open intervals $I_k = (a_k, b_k)$. Since $g$ is continuous then $g^{-1}(a_k, b_k)$ is an open subset of $[a, b]$ for all $k$, and since $g$ is monotone increasing then $g^{-1}(a_k, b_k)$ is an interval $J_k = (x_k, y_k)$ with $g(x_k) = a_k$ and $g(y_k) = b_k$. Moreover the intervals $J_k$ are disjoint since $I_k$ are and the preimages of disjoint sets are disjoint. Now $g^{-1}(O)$ is the countable disjoint union $g^{-1}(O) = \bigcup_{k=1}^{\infty} J_k$, so
\[
\int_{g^{-1}(O)} g' = \sum_{k=1}^{\infty} \int_{J_k} g' = \sum_{k=1}^{\infty} (g(y_k) - g(x_k)) = \sum_{k=1}^{\infty} (b_k - a_k)
\]
\[
= \sum_{k=1}^{\infty} m(I_k) = m(O),
\]
as required.
(b) Let $H = \{ x : g'(x) \neq 0 \}$. If $E$ is a subset of $[c, d]$ with $m(E) = 0$, then $F = g^{-1}(E) \cap H$ has measure zero.

**Answer:**
Let $E$ be a subset of $[c, d]$ with $m(E) = 0$, and denote $F = g^{-1}(E) \cap H$. Assume towards contradiction that $m(F) > 0$. Since $g$ is increasing then $g' \geq 0$ almost everywhere, so $g'(x) > 0$ for all $x \in F$ and thus $\int_F g' > 0$. Fix an open set $O \supseteq E$. Since $F = (g^{-1}(E) \cap H) \subseteq g^{-1}(O)$, then by part (a) we have

$$m(O) = \int_{g^{-1}(O)} g' \geq \int_F g' .$$

This lower bound holds uniformly for all open sets $O \supseteq E$, so by taking infimum we get $m(E) \geq \int_F g' > 0$. This is a contradiction with the assumption $m(E) = 0$, so we conclude that $m(F) = 0$.

(c) If $E$ is a measurable subset of $[c, d]$, then $F = g^{-1}(E) \cap H$ is measurable and

$$m(E) = \int_F g' = \int_a^b \chi_E(g(x))g'(x) \, dx .$$

**Answer:**
Since $E$ is measurable then by homework 1 exercise 1 we can write $E = A \cup N$, where $A$ is a $F_\sigma$-set and $N$ has measure zero, and $A \cap N = \emptyset$. Then

$$F = g^{-1}(E) \cap H = g^{-1}(A \cup N) \cap H = (g^{-1}(A) \cup g^{-1}(N)) \cap H = (g^{-1}(A) \cap H) \cup (g^{-1}(N) \cap H) .$$

Now $H = (g')^{-1}(\mathbb{R} \setminus \{0\})$ is measurable since $g'$ is, and thus $g^{-1}(A) \cap H$ is measurable as $A$ was a Borel set. Also, by part (b) the set $g^{-1}(N) \cap H$ has measure zero and is thus measurable. Hence $F$ is measurable. Note then that

$$\int_F g' = \int_{g^{-1}(E)} g' = \int_a^b \chi_{g^{-1}(E)}g' = \int_a^b \chi_E(g)g' ,$$

since $\chi_{g^{-1}(E)} = \chi_E(g)$. So we have to show that $m(E)$ equals the above value. We will prove it in several steps.
(i) We first assume that $E$ is an open set. By part (a) we have

$$m(E) = \int_{g^{-1}(E)} g' = \int_a^b \chi_{g^{-1}(E)} g' = \int_a^b \chi_E g'. $$

So the statement is true if $E$ is open.

(ii) Assume then that $E$ is a $G_\delta$ set. So there exists a nested sequence of open sets $\{O_n\}_{n=1}^\infty$ so that $O_{n+1} \subseteq O_n$ for all $n$ and $E = \bigcap_{n=1}^\infty O_n$. Now \(\lim_{n \to \infty} \chi_{O_n} = \chi_E\) pointwise and by dominated convergence theorem and part (i) we have

$$m(E) = \lim_{n \to \infty} m(O_n) = \lim_{n \to \infty} \int_a^b \chi_{O_n} g' = \int_a^b \lim_{n \to \infty} \chi_{O_n} g' = \int_a^b \chi_E g', $$

so the statement holds for all $G_\delta$ sets.

(iii) Assume then that $E$ is any measurable set. Then by exercise 1 of problem set 1 we can find a $G_\delta$-set $G$ and a set $N$ with $m(N) = 0$ so that $E \cup N = G$ and $E \cap N = \emptyset$. Thus by part (ii) we have

$$m(E) = m(E \cup N) = m(G) = \int_a^b \chi_G g' = \int_{g^{-1}(G) \cap H} g'$$

$$= \int_{g^{-1}(E \cup N) \cap H} g' = \int_{g^{-1}(E) \cap H} g' + \int_{g^{-1}(N) \cap H} g'$$

$$= \int_{g^{-1}(E) \cap H} g' = \int_a^b \chi_E g'. $$

So the statement is true for all measurable $E$.

(d) If $f$ is a non-negative measurable function on $[c, d]$, then $(f \circ g)g'$ is measurable on $[a, b]$ and

$$\int_c^d f(x) \, dx = \int_a^b f(g(x))g'(x) \, dx.$$

Answer:
Since $g$ is a monotone increasing absolutely continuous function and $f$ is measurable then $(f \circ g)$ is measurable. Since $g'$ is measurable then so is $(f \circ g)g'$. We prove the statement in several steps by using the standard approximation argument.

(i) Assume first that $f = \chi_E$ is a characteristic function of some measurable set $E \subseteq [c, d]$. Then by part (c) we have

$$\int_c^d f(x) \, dx = \int_c^d \chi_E = m(E) = \int_a^b \chi_E(g(x))g'(x) \, dx$$
$$= \int_a^b f(g(x))g'(x) \, dx,$$

so the statement is true for all characteristic functions.

(ii) Assume then that $f$ is a simple function $f = \sum_{k=1}^n a_k \chi_{E_k}$ for some constants $a_k$ and measurable sets $E_k \subseteq [c, d]$. Then by part (i) and linearity of the integral we have

$$\int_c^d f(x) \, dx = \int_c^d \sum_{k=1}^n a_k \chi_{E_k} = \sum_{k=1}^n a_k \int_c^d \chi_{E_k}$$
$$= \sum_{k=1}^n a_k \int_a^b \chi_{E_k}(g(x))g'(x) \, dx$$
$$= \int_a^b \sum_{k=1}^n a_k \chi_{E_k}(g(x))g'(x) \, dx$$
$$= \int_a^b f(g(x))g'(x) \, dx,$$

so the statement holds for all simple functions.

(iii) Let $f$ then be any non-negative measurable function, and take a nondecreasing sequence $(\varphi)^\infty_{n=1}$ of simple functions so that $\varphi_n \leq f$ for all $n$ and $\varphi_n \rightharpoonup f$ pointwise. We use monotone
convergence theorem and part (ii) to conclude that

\[
\int_c^d f(x) \, dx = \int_c^d \lim_{n \to \infty} \varphi_n(x) \, dx = \lim_{n \to \infty} \int_c^d \varphi_n(x) \, dx \\
= \lim_{n \to \infty} \int_c^d \varphi_n(g(x))g'(x) \, dx = \int_c^d \lim_{n \to \infty} \varphi_n(g(x))g'(x) \, dx \\
= \int_a^b f(g(x))g'(x) \, dx.
\]

So this proves the statement.