Large deviation principles for singular Riesz-type diffusive flows

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Abstract

We combine hydrodynamic and modulated energy techniques to study the large deviations of systems of particles with pairwise singular repulsive interactions and additive noise. In particular, we consider periodic Riesz interactions indexed by parameter $\mathbf{s} \in [0, d-2)$ for $d \geqslant 3$ on the torus. When $\mathbf{s} = 0$, that is, when the interaction potential is logarithmic, we establish a full LDP for the empirical trajectories of the particles given sufficiently strong convergence of the initial conditions. When $\mathbf{s} \in (0, d-2)$, we give an LDP upper bound and partial lower bounds. Additionally, we show a local LDP holds in a stronger distance.

1. Introduction

1.1 The Problem

We are interested in the large deviations as $N \to \infty$ of the systems of interacting particles given by

$$\begin{cases} dx_i^t = -\frac{1}{N} \sum_{1 \leqslant j \leqslant N : j \neq i} \nabla \mathbf{g}(x_i^t - x_j^t) dt + \sqrt{2\sigma} dw_i^t, \\ x_i^t|_{t=0} = x_i^0, \end{cases} \qquad i \in \{1, \cdots, N\}. \tag{1.1}$$

eq:SDE

Above, $\{w_i\}_{i=1}^N$ are N independent standard Brownian motions in the d-dimensional torus[1] $\mathbb{T}^d$, $\sigma > 0$ is the temperature of the system, and $\mathbf{g}$ is a periodic sub-Coulombic Riesz potential. Indexed by $\mathbf{s} \in [0, d)$, the periodic Riesz potentials are the unique zero average solutions to

$$(-\Delta)^{\frac{d-\mathbf{s}}{2}} \mathbf{g} = c_{d,\mathbf{s}}(\delta_0 - 1), \qquad c_{d,\mathbf{s}} := \begin{cases} \frac{4^{(d-\mathbf{s})/2} \Gamma((d-\mathbf{s})/2) \pi^{d/2}}{\Gamma(s/2)} & \mathbf{s} \in (0, d), \\ \frac{\Gamma(d/2)(4\pi)^{d/2}}{2} & \mathbf{s} = 0. \end{cases} \tag{1.2}$$

eq:Riesz

The choice of the scaling constants are made so $\mathbf{g}$ behaves like $-\log|x|$ or $|x|^{-\mathbf{s}}$ near the origin when $\mathbf{s} = 0$ or $\mathbf{s} \in (0, d)$ respectively [HSSS17]. We restrict our attention to the sub-Coulombic potentials: the sub-family corresponding to $\mathbf{s} \in [0, d-2)$ when $d \geqslant 3$. As $\mathbf{g}$ is singular, the well-posedness of the stochastic differential equation (1.1) is not immediate, but holds as long as the initial conditions are pairwise distinct.

The aggregate behaviour of (1.1) is described by the associated *empirical measure* and *empirical trajectory*

$$\mu_N^t := \frac{1}{N} \sum_{i=1}^N \delta_{x_i^t}, \qquad \mu_N : t \mapsto \mu_N^t.$$

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[1] We associate $\mathbb{T}^d$ with $[-\frac{1}{2}, \frac{1}{2}]^d$ under periodic boundary conditions.

For fixed t the empirical measure μ_N^t is a random element of the space of probability measures $\mathcal{P}(\mathbb{T}^d)$ metrized by the Wasserstein-2 distance. For a fixed time horizon $T > 0$, the empirical trajectory is naturally viewed as a random element of $C([0,T], \mathcal{P}(\mathbb{T}^d))$.

The main result of this paper gives large deviation estimates on the empirical trajectories $(\mu_N)_{N \geqslant 1}$ at speed N given strong convergence of the initial conditions. We show that for all $\mathbf{s} \in [0, d-2)$ there exists a function $I : C([0,T], \mathcal{P}(\mathbb{T}^d)) \to [0, \infty]$ with compact sublevel sets and a dense subset $\mathscr{A} \subset C([0,T], \mathcal{P}(\mathbb{T}^d))$ so that for all Borel $B \subset C([0,T], \mathcal{P}(\mathbb{T}^d))$

$$-\inf_{\mu \in B^\circ \cap \mathscr{A}} I(\mu) \leqslant \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\mu_N \in B) \leqslant \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\mu_N \in B) \leqslant -\inf_{\mu \in \overline{B}} I(\mu).$$

B° and $\overline{B}$ respectively denote the interior and closure of B in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. In particular, $\{I < \infty\} \subset \mathscr{A}$ when $\mathbf{s} = 0$, thus $(\mu_N)_{N \geqslant 1}$ satisfy an LDP with good rate function I.

When $\mu \in \mathscr{A}$ and $I(\mu) < \infty$

$$I(\mu) = \frac{1}{4\sigma} \int_0^T \|\partial_t \mu^t - \sigma \Delta \mu^t - \operatorname{div}(\mu^t \nabla \mathbf{g} * \mu^t)\|_{-1,\mu^t}^2 \, dt,$$

where $\|\cdot\|_{-1,\mu}$ is a norm on distributions that depends on μ. This is the same as the rate function in the case where $\mathbf{g}$ is regular [DG87].

1.2 Background

The system (1.1) corresponds to dissipative dynamics with respect to the energy

$$H_N(\underline{x}_N) := \frac{1}{N^2} \sum_{1 \leqslant i \neq j \leqslant N} \mathbf{g}(x_i - x_j), \quad \underline{x}_N := (x_1, \ldots, x_N) \in (\mathbb{T}^d)^N.$$

If $\nabla \mathbf{g}$ is replaced with $\mathbb{M} \nabla \mathbf{g}$ in the definition of (1.1) where $\mathbb{M}$ is an antisymmetric matrix, then the system instead corresponds to conservative dynamics. For reasons explained later, we only consider the dissipative setting.

Systems of the form (1.1) arise in many pure and applied settings where particles or individuals interact pairwise with each other. They describe the dynamics of gases [Vla68], the eigenvalues of random matrices [Dys62, AGZ09], vortices in viscous fluids [Hel67, Osa85], the collective motion of animals or bacteria [Per07, FJ17], and scaling limits for neural networks [MMN18, RVE22].

There has been significant recent interest in studying the concentration of the empirical measures as $N \to \infty$. If μ_N^t converge in a suitable topology, the system is said to satisfy a *mean-field limit*. In particular, one expects that if $\mu_N^0 \to \mu_0$, then μ_N^t converges to the deterministic solution of the McKean–Vlasov equation

$$\partial_t \mu - \sigma \Delta \mu - \operatorname{div}(\mu \nabla \mathbf{g} * \mu) = 0, \qquad \mu|_{t=0} = \mu_0. \tag{1.3}$$
eq:mve

This can be justified by a formal argument using Itô calculus.

Mean-field limits for systems with regular interactions are well understood [McK67, Dob79, Szn91]. One classical approach involves coupling solutions of (1.1) with different values of N to show that μ_N^t is Cauchy in a suitable metric [McK67]. There has been some recent success in using this strategy to show mean-field convergence of Riesz interacting systems in one dimension [GLBM22], but the proof does not generalize to larger dimensions.

Mean-field limits were first proved for deterministic sub-Coulomb Riesz interactions in the Wasserstein-∞ topology [Hau09, CCH14]. Later, in [JW18], quantitative mean-field limits were shown for singular kernels of so called $W^{-1,\infty}$ type using relative entropy techniques.

In [Due16, Ser20], mean-field limits were shown for the deterministic system of ODEs given by (1.1) with $\sigma = 0$ and Coulomb or super-Coulomb ($\mathbf{s} \in (d-2, d)$) interaction when the limiting equation satisfies some regularity conditions. These works introduced the *modulated energy*, defined for a particle configuration $\underline{x}_N \in (\mathbb{T}^d)^N$ and a probability measure $\mu \in L^\infty(\mathbb{T}^d)$ by

$$F_N(\underline{x}_N, \mu) := \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y), \qquad \Delta := \Big\{(x,y) \in (\mathbb{T}^d)^2 : x = y\Big\}. \tag{1.4}$$

eq:modulated-energ

This acts as a pseudo-distance between the empirical measure $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and μ. If $\underline{x}_N$ are a sequence of point configurations so that $\lim_{N\to\infty} F_N(\underline{x}_N, \mu) = 0$, then the associated empirical measures μ_N converge to μ weakly. The proofs of the mean-field limits in [Due16, Ser20] proceed via a Grönwall argument applied to $F_N(\underline{x}_N^t, \mu^t)$ when $\underline{x}_N^t$ is the solution to the ODE system and μ^t is a solution to the limiting equation (1.3). The crucial point of the proof is the following *commutator estimate* which allows the modulated energy to control a term in its time derivative

$$\left| \int\int_{(\mathbb{T}^d)^2 \setminus \Delta} (\phi(x) - \phi(y)) \cdot \nabla \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) \right| \leqslant C_\phi \Big(F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta}\Big), \tag{1.5}$$

eq:intro-commutato

where C_ϕ is a constant depending on the derivatives of a vector field ϕ, and $C, \beta > 0$. This inequality is also crucial for the techniques used in this paper. The conditions on the limiting equation were further relaxed in [Ros22b, Ros22a]. In [NRS22], the modulated-energy method was used to show the mean-field convergence of systems interacting via sub-Coulomb potentials.

Returning to the non-deterministic setting ($\sigma > 0$), in the works [BJW19a, BJW19b, BJW20], the authors introduced the *modulated free energy* which combines the modulated energy with relative entropy. Using an exact cancellation which occurs between terms, they demonstrated the mean-field convergence of systems of the form (1.1) with a wide class of singular interactions, including logarithmic attraction. In [RS23], global-in-time quantitative mean-field convergence was shown for sub-Coulombic interactions in $\mathbb{R}^d$ using the modulated energy method. Most recently, in [dCRS23], by combining the modulated free energy with relaxation rates of the limiting equation to equilibrium, the authors showed global-in-time mean-field convergence for periodic Riesz interactions when $\mathbf{s} \in [d-2, d)$ on the torus.

The large deviations of the empirical paths of the system (1.1) have also been studied extensively when $\mathbf{g}$ is regular. In the early work [DG87], the empirical paths of weakly interacting diffusions were shown to satisfy an LDP in $C([0,T], \mathcal{P}(\mathbb{R}^d))$. In [BAB90], this was improved to LDPs on the level of the entire process, where one instead considers the process level empirical measure

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i|_{t\in[0,T]}} \in \mathcal{P}(C([0,T], \mathbb{T}^d)),$$

which is a random measure on paths. The proof techniques in [DG87, BAB90] rely on the Cameron-Martin-Girsanov theorem to view the law of interacting diffusions as a measure transform of the laws of non-interacting processes. Using ideas from stochastic optimal control theory, in [BDF12], LDPs were shown for diffusions with regular drift and diffusion coefficients which depend on the current empirical distribution of the particles. In [FK06, Section 13.3], a strategy to prove LDPs for interacting diffusions using Hamilton-Jacobi theory was introduced.

As opposed to measure transform methods, LDPs have also been shown for regular interactions using the contraction principle [CDFM20]. This uses the construction of a continuous map from the process level empirical measure of the noise to the process level empirical measure of the process, and can also handle non-martingale noise.

Much of the study of LDPs for interacting diffusions with singular interactions has been in the context of random matrix theory. Dyson Brownian motion is a solution to (1.1) on $\mathbb{R}$ with logarithmic repulsion and N dependent temperature $\sigma = (\beta N)^{-1}$ for some inverse temperature β. A large deviation upper bound at speed N^2 and inverse temperature $\beta = 1$ was shown in [DG01]. Furthermore, a partial lower bound was given. This was expanded to a complete lower bound for $\beta = 1$ or 2 in [GZ02, GZ04]. Recently, in [GH21], LDPs were shown for Dyson Bessel processes, and the range of β for which the LDP holds for Dyson Brownian motion was extended to $\beta \geqslant 1$. In all of these papers the fact that the space is one dimensional is central to the proof techniques.

There are few results for more general singular interactions. The measure transformation technique used for regular interactions in [BAB90] has been successfully extended to interactions which are less singular than logarithmic potentials [HHMT20]. In [Fon04], using the ideas from [DG01] together with a uniqueness result for perturbed Mckean-Vlasov solutions, an LDP upper bound and a partial lower bound were shown in one dimension with $\mathbf{s} = 0$. Recently, an LDP for (1.1) with conservative dynamics was shown on the two dimensional torus with $\mathbf{s} = 0$ [CG22]. Up to now, no LDP-type results for Riesz interactions in dimensions greater than 2 or $\mathbf{s} > 0$ have been shown.

1.3 Spaces and energy functions

subseq:notation

Before introducing the main results, we define some of the relevant spaces, norms, and energies.

1.3.1. Wasserstein-2 distance and absolutely continuous curves. We endow $\mathcal{P}(\mathbb{T}^d)$ with the Wasserstein-2 distance

$$d(\mu,\nu)^2 = \sup_{\pi \in \Pi(\mu,\nu)} \int |x-y|^2 \, d\pi(x,y),$$

where $|x-y|$ is the canonical distance on $\mathbb{T}^d$ and $\Pi(\mu,\nu)$ is the set of all couplings of μ and ν. As the torus is compact, the Wasserstein-2 distance induces the same topology as weak convergence.

$\mathcal{AC}^T := \mathcal{AC}([0,T], \mathcal{P}(\mathbb{T}^d))$ then denotes the space of absolutely continuous curves from $[0,T]$ to $\mathcal{P}(\mathbb{T}^d)$ [AGS08]. Notably, $\mu \in \mathcal{AC}^T$ if and only if there exists a vector field $v \in L^2([0,T], L^2(\mu^t))$ so that μ is a weak solution to

$$\partial_t \mu + \mathrm{div}\,(v\mu) = 0.$$

We then say that $\partial_t \mu^t = -\mathrm{div}\,(v^t \mu^t)$. Given a probability measure μ, the function

$$\|T\|_{-1,\mu}^2 := \sup_{\phi \in C^\infty(\mathbb{T}^d)} \left\{ 2\langle T, \phi\rangle - \int_{\mathbb{T}^d} |\nabla \phi|^2 d\mu \right\}, \qquad T \in C^\infty(\mathbb{T}^d)',$$

defines a norm on the subspace of distributions $\{T \in C^\infty(\mathbb{T}^d)' : \|T\|_{-1,\mu} < \infty\}$. This notation is justified by the fact that $\|T\|_{-1,\mu} < \infty$, then $T = \mathrm{div}\,(v\mu)$ for some v in the closure of $\{\nabla\phi : \phi \in C^\infty(\mathbb{T}^d)\}$ with respect to the $L^2(\mu)$ norm and

$$\|T\|_{-1,\mu}^2 = \|\mathrm{div}\,(v\mu)\|_{-1,\mu}^2 = \int |v|^2 \, d\mu.$$

1.3.2. Fractional Sobolev spaces and energy functions. For $\alpha \in \mathbb{R}$, we define the semi-norm

$$\|f\|^2_{\dot{H}^\alpha(\mathbb{T}^d)} := \sum_{k\in\mathbb{Z}^d : k\neq 0} |k|^{2\alpha} |\widehat{f}(k)|^2$$

on distributions. This is a norm on the space of zero mean distributions with finite semi-norm, which we denote $\dot{H}^\alpha_0(\mathbb{T}^d)$. More generally, $|\nabla|$ denotes the operator $(-\Delta)^{1/2}$ with Fourier multiplier $2\pi|k|$.

For $\mathbf{s} \in [0, d)$, of particular relevance is the $\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)$ semi-norm. By taking the Fourier transform it holds that

$$\mathcal{E}(\mu) := \iint_{(\mathbb{T}^d)^2} \mathbf{g}(x-y)\, d\mu(x)\, d\mu(y) = c_{d,\mathbf{s}} \|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}. \tag{1.6}$$

eq:sobolevequivalen

For convenience we set

$$\mathscr{C}^T := C([0,T], \dot{H}_0^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)).$$

If $\mathbf{s} \in [0, d-2)$, then since $(-\Delta)\mathbf{g}$ is a constant multiple of the Riesz kernel corresponding to parameter $\mathbf{s}+2$, it also holds that

$$(-\Delta)\mathcal{E}(\mu) := \iint_{(\mathbb{T}^d)^2} (-\Delta)\mathbf{g}(x-y)\, d\mu(x)\, d\mu(y) = c_{d,\mathbf{s}} \|\mu\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}.$$

Given $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d))$ we define the energy

$$Q^T(\mu) := \sup_{t\in[0,T]} \left\{ \mathcal{E}(\mu^t) + 2\sigma \int_0^t (-\Delta)\mathcal{E}(\mu^\tau)\, d\tau + \int_0^t \|\mathrm{div}\,(\mu^\tau \nabla \mathbf{g} * \mu^\tau)\|^2_{-1,\mu^\tau}\, d\tau \right\}. \tag{1.7}$$

eq:energy-definition

It is not clear the last term in the supremum is well defined, but if $\mathcal{E}(\mu) < \infty$, then $\mu\nabla\mathbf{g} * \mu$ can be made sense of as a distribution (see Proposition 2.2). If $\mathrm{div}\,(\mu^t \nabla \mathbf{g} * \mu^t)$ is not well-defined for any $t \in [0,T]$, we let $Q^T(\mu) = +\infty$. This is similar to the function introduced in [CG22], and will play a similar role in the proof of the LDP.

1.4 Main results

Our main theorem states that if the initial conditions of (1.1) converge strongly to some regular probability measure μ, then an LDP upper bound and a partial lower bound hold.

thm:LDP **Theorem 1.1.** *Let $d \geqslant 3$ and $\mathbf{s} \in [0, d-2)$. Suppose $(\mu_N)_{N\geqslant 1}$ are the empirical paths associated to solutions to (1.1) with initial conditions satisfying*

$$\lim_{N\to\infty} F_N(\underline{x}^0_N, \mu_0) = 0 \tag{1.8}$$

cond:initial-converg

for some $\mu_0 \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$. Then $I : C([0,T], \mathcal{P}(\mathbb{T}^d)) \to [0,\infty]$ defined by

$$I(\mu) := \begin{cases} \frac{1}{4\sigma}\left(\int_0^T \|\partial_t \mu^t - \sigma\Delta\mu^t - \mathrm{div}\,(\mu^t \nabla\mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\, dt \vee \left(Q^T(\mu) - \mathcal{E}(\mu_0)\right)\right) & \mu \in \mathcal{AC}^T, \mu^0 = \mu_0 \\ +\infty & o.w. \end{cases}$$

is a good rate function with the following properties.

1. *For all closed* $\Gamma \subset C([0,T],\mathcal{P}(\mathbb{T}^d))$

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}(\mu_N \in \Gamma) \leqslant -\inf_{\mu\in\Gamma} I(\mu). \tag{1.9}$$
eq:LDP-upper-boun

2. *For all open* $\mathcal{O} \subset C([0,T],\mathcal{P}(\mathbb{T}^d))$

$$\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}(\mu_N \in \mathcal{O}) \geqslant -\inf_{\mu\in\mathcal{O}\cap\mathscr{A}} I(\mu), \tag{1.10}$$
eq:LDP-lower-boun

where

$$\mathscr{A} := \left\{\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d)) : \int_0^T \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu^t\|^3_{L^{\frac{6d}{3d-\mathbf{s}-1}}(\mathbb{T}^d)}\, dt < \infty\right\}.$$

In particular, when $\mathbf{s} = 0$*,* $\{I(\mu) < \infty\} \subset \mathscr{A}$*, thus* $(\mu_N)_{N\geqslant 1}$ *satisfy a large deviation principle.*

Remark 1.2. When $\mu \in \mathscr{A} \cap \mathcal{AC}^T$ it holds that

$$Q^T(\mu) - \mathcal{E}(\mu_0) \leqslant \int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\, dt,$$

thus

$$I(\mu) = \frac{1}{4\sigma}\int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\, dt.$$

As a consequence, since $\{I < \infty\} \subset \mathscr{A}$ when $\mathbf{s} = 0$, the LDP holds with the good rate function

$$\widetilde{I}(\mu) := \begin{cases} \frac{1}{4\sigma}\int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\, dt & \mu \in \mathcal{AC}^T, Q^T(\mu) < \infty, \text{ and } \mu^0 = \mu_0 \\ +\infty & \text{o.w.} \end{cases}$$

The upper and lower bounds in Theorem 1.1 also hold with I replaced with $\widetilde{I}$ for $\mathbf{s} > 0$, but $\widetilde{I}$ is not clearly lower semi-continuous. In particular, there could exist a sequence of measures μ_k converging to μ so that $Q^T(\mu) = \infty$ but

$$\limsup_{k\to\infty}\int_0^T \|\partial_t\mu_k^t - \sigma\Delta\mu_k^t - \operatorname{div}(\mu_k^\tau\nabla\mathbf{g} * \mu_k^t)\|^2_{-1,\mu}\, dt < \infty.$$

It is for this reason we've included $Q^T(\mu)$ in our definition of I.

As a secondary theorem, we show that local large deviation principle estimates hold for the distance given by the modulated energy.

thm:local-LDP **Theorem 1.3.** *Under the conditions of Theorem 1.1, if* $\mu \in L^\infty([0,T],L^\infty(\mathbb{T}^d))$*, then*

$$\begin{aligned}\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\Big(\sup_{t\in[0,T]} F_N(\underline{x}_N^t,\mu^t) < \varepsilon\Big) &= \lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\Big(\sup_{t\in[0,T]} F_N(\underline{x}_N^t,\mu^t) < \varepsilon\Big) \\ &= -\Big(I(\mu)\mathbf{1}_{(\mu-1)\in\mathscr{C}^T} + \infty\cdot\mathbf{1}_{(\mu-1)\notin\mathscr{C}^T}\Big).\end{aligned}$$

Remark 1.4. Theorem 1.3 is not a consequence of Theorem 1.1 since the convergence of $\mu_N \to \mu$ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$ does not imply that $\sup_{t\in[0,T]} F_N(\underline{x}_N^t,\mu^t) \to 0$. In particular, the equality involving the $\liminf$ is stronger than the LDP lower bound.

1.5 Overview of proof

It is natural to attempt to adapt the proof of large deviation principles for process level empirical measures given in [BAB90]. Letting $\mathbb{P}_N$ and $\mathbb{W}_N$ respectively denote the law of the process level empirical measure of $\underline{x}_N$ and the noise $(w_1, \cdots, w_N)$, one can use the Girsanov theorem to compute the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_N}{d\mathbb{W}_N}(\mu) = \exp(NF(\mu)),$$

for some function F. If F was continuous with respect to the weak topology on $\mathcal{P}(C([0,T], \mathbb{T}^d))$, then since $\mathbb{W}_N$ satisfy a LDP by Sanov's theorem, we could conclude an LDP holds by Laplace's method [DZ10, Theorem 4.3.1].

The problem is that as $\mathbf{g}$ is singular, the resulting function F is not continuous. In particular, it has a problematic term of the form

$$\int_0^T \int |\nabla \mathbf{g} * \mu^t|^2 \, d\mu^t \, dt,$$

which is difficult to analyze.

For this reason we will instead use hydrodynamic techniques as first introduced in [KOV89]. The main advantage of this strategy is that we only have to show continuity of the function

$$\mu \to \int_0^T \int \nabla \phi^t \cdot \nabla \mathbf{g} * \mu^t \, d\mu^t \, dt = \frac{1}{2} \int_0^T \iint (\nabla \phi^t(x) - \nabla \phi^t(y)) \cdot \nabla \mathbf{g}(x-y) \, d\mu^t(x) d\mu^t(y) \, dt \quad (1.11)$$

for a fixed smooth function ϕ. Although it is still challenging to show this function has the desired behaviour, tools have already been developed in the study of mean-field limits for Riesz flows which will be helpful.

Hydrodynamic techniques are also the basis of the previous LDP results for $\mathbf{s} = 0$ [DG01, GZ02, Fon04, GZ04, GH21, CG22]. In these papers, as the potential is only taken to be logarithmic, (1.11) is much easier to define and analyze.

To explain the exact technical difficulties which arise, below we sketch a proof for an LDP upper bound when $\mathbf{g}$ is smooth using the hydrodynamic argument. We then explain how this argument needs to be modified when $\mathbf{g}$ is singular, and summarize the proof of our lower bounds. Similar to as in [CG22], we will introduce an auxiliary functional which is vital to the proof. Throughout we ignore the role of the initial conditions as they only add some small technical difficulties.

1.5.1. Upper bound for smooth potentials. For the purpose of this subsection $\mathbf{g}$ is assumed to be smooth and even.

The starting point to show an LDP upper bound for the empirical trajectories μ_N is noting that their evolution is described by a differential equation. For $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d))$, $\phi \in C^\infty([0,T] \times \mathbb{T}^d)$ and $0 \leqslant s \leqslant t \leqslant T$ let

$$L^{s,t}(\mu, \phi) := \langle \mu^t, \phi^t \rangle - \langle \mu^s, \phi^s \rangle - \int_s^t \langle \mu^\tau, \partial_t \phi^\tau \rangle + \langle \sigma \Delta \mu^\tau + \operatorname{div}(\mu^\tau \nabla \mathbf{g} * \mu^\tau), \phi^\tau \rangle \, d\tau.$$

Then $L^{s,t}(\mu, \phi) = 0$ for all s, t and ϕ if and only if μ is a weak solution to the McKean-Vlasov equation (1.3).

By applying Itô's formula to $\langle \mu_N^\tau, \phi^\tau \rangle$ we find that

$$L^{s,t}(\mu_N, \phi) = \frac{\sqrt{2\sigma}}{N} \sum_{i=1}^N \int_s^t \nabla \phi^\tau(x_i^\tau) \cdot dw_i^\tau, \quad (1.12)$$

where we've written

$$\frac{1}{N}\sum_{i=1}^{N}\nabla\phi^\tau(x_i^\tau)\cdot\left(\frac{1}{N}\sum_{1\leqslant j\leqslant N; j\neq i}\nabla\mathbf{g}(x_i^\tau-x_j^\tau)\right)=\int\nabla\phi^\tau\cdot(\nabla\mathbf{g}*\mu_N^\tau)\,d\mu_N^\tau=-\langle\operatorname{div}(\mu_N^\tau\nabla\mathbf{g}*\mu_N^\tau),\phi^\tau\rangle.$$

The point is that μ_N is almost a weak solution to the limiting equation (1.3) except for the random term

$$M^t:=\frac{\sqrt{2\sigma}}{N}\sum_{i=1}^{N}\int_s^t\nabla\phi^\tau(x_i^\tau)\cdot\,dw_i^\tau.$$

M^t is a bounded continuous martingale with respect to the filtration generated by the noise for fixed s, and has quadratic variation

$$\langle M^t\rangle=\frac{2\sigma}{N}\int_s^t\int|\nabla\phi^\tau|^2\,d\mu_N^\tau\,d\tau.$$

This implies that $\exp(NM^t-\frac{N^2}{2}\langle M^t\rangle)$ is also a martingale, so has expectation 1. Setting

$$S^{s,t}(\mu,\phi):=L^{s,t}(\mu,\phi)-\sigma\int_s^t\int|\nabla\phi^\tau|^2\,d\mu^\tau\,d\tau,$$

then since

$$S^{s,t}(\mu_N,\phi)=M^t-\frac{N}{2}\langle M^t\rangle,$$

Chebyshev's inequality gives us the bound

$$\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant-\inf_{\mu_N\in B_\varepsilon(\mu)}S^{s,t}(\mu_N,\phi),$$

for any $\mu\in C([0,T],\mathcal{P}(\mathbb{T}^d))$ and any s,t and smooth ϕ. Since $\mathbf{g}$ is smooth, it is easy to show that if $\mu_k\to\mu$ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$, then

$$\lim_{k\to\infty}\int_s^t\langle\operatorname{div}(\mu_k^\tau\nabla\mathbf{g}*\mu_k^\tau),\phi^\tau\rangle\,d\tau=\int_s^t\langle\operatorname{div}(\mu^\tau\nabla\mathbf{g}*\mu^\tau),\phi^\tau\rangle\,d\tau. \tag{1.13}$$ eq:overview-slope-c

As a consequence, if $\mu_k\to\mu$, then $\lim_{k\to\infty}S^{s,t}(\mu_k,\phi)=S^{s,t}(\mu,\phi)$. This implies that

$$\begin{aligned}\lim_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)&\leqslant-\lim_{\varepsilon\to0}\liminf_{N\to\infty}\inf_{\mu_N\in B_\varepsilon(\mu)}S^{s,t}(\mu_N,\phi)\\&\leqslant-S^{s,t}(\mu,\phi).\end{aligned}$$

After optimizing over s,t and ϕ this gives us the bound

$$\lim_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant-\sup_{\phi,s,t}S^{s,t}(\mu,\phi),$$

thus $\sup_{\phi,s,t}S^{s,t}(\mu,\phi)$ is a candidate for our rate function. The law of μ_N can be shown to be exponentially tight (see [DG87]), thus this local upper bound implies a complete LDP upper bound.

Using that $L^{s,t}(\mu,\phi)$ is a linear function in ϕ while $\int_s^t\int|\nabla\phi^\tau|^2\,d\mu^\tau\,d\tau$ is quadratic, it is not difficult to show using the Riesz representation theorem that if

$$\sup_{\phi,s,t}S^{s,t}(\mu,\phi)<\infty,$$

then there must exist some b in the closure of $\{\nabla\phi : \phi \in C^\infty([0,T]\times\mathbb{T}^d)\}$ under the $L^2([0,T],L^2(\mu^t))$ norm so that for all smooth ϕ

$$L^{s,t}(\mu,\phi) = \int_s^t \int \nabla\phi^\tau \cdot b^\tau \, d\mu^\tau \, d\tau.$$

This says that μ is a weak solution to

$$\partial_t\mu - \sigma\Delta\mu - \operatorname{div}((\nabla\mathbf{g} * \mu)\mu) = -\operatorname{div}(b\mu), \tag{1.14}$$

eq:overview-mve+b

and also implies that

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \frac{1}{4\sigma}\int_0^T \int |b^\tau|^2 \, d\mu^\tau \, d\tau.$$

The fact μ solves (1.14) and both b and $\nabla\mathbf{g} * \mu$ are in $L^2([0,T],L^2(\mu^t))$ imply that $\mu \in \mathcal{AC}^T$, and we can further write

$$\frac{1}{4\sigma}\int_0^T \int |b^\tau|^2 \, d\mu^\tau \, d\tau = \frac{1}{4\sigma}\int_0^T \|\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau)\|^2_{-1,\mu^\tau} \, d\tau.$$

Using the convention that the right-hand side below is infinite if $\mu \notin \mathcal{AC}^T$, in total we've found that for any closed $\Gamma \subset C([0,T],\mathcal{P}(\mathbb{T}^d))$

$$\limsup_{N\to\infty} \frac{1}{N}\log\mathbb{P}(\mu_N \in \Gamma) \leqslant -\inf_{\mu\in\Gamma}\frac{1}{4\sigma}\int_0^T \|\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau)\|^2_{-1,\mu^\tau} \, d\tau,$$

which is the correct LDP upper bound [DG87].

There are several places where the above proof sketch needs to be modified when $\mathbf{g}$ is a Riesz potential:

- The distribution $\mu\nabla\mathbf{g} * \mu$ is not defined for arbitrary probability measures, so $S^{s,t}(\mu,\phi)$ does not make sense for all measure trajectories.
- When $S^{s,t}(\mu,\phi)$ is well defined, its continuity with respect to the $C([0,T],\mathcal{P}(\mathbb{T}^d))$ topology is nonobvious.

1.5.2. Energy bounds. By using an energy equality the SDEs (1.1) almost solve, we will reduce the class of measure trajectories for which we need to establish a local LDP upper bound by using hydrodynamic techniques. This is vital as the admissible class has enough regularity for us to define $S^{s,t}(\mu,\phi)$.

If μ is a smooth solution to (1.3), then by taking the time derivative of $\mathcal{E}(\mu^t)$ it is easy to compute that for all t

$$\mathcal{E}(\mu^0) = \mathcal{E}(\mu^t) + 2\sigma\int_0^t (-\Delta)\mathcal{E}(\mu^\tau)\, d\tau + 2\int_0^t \int |\nabla\mathbf{g} * \mu^\tau|^2 \, d\mu^\tau \, d\tau.$$

As $\underline{x}_N$ is almost a solution to (1.3) we expect something similar. Formally taking the Itô derivative of $H_N(\underline{x}_N^t)$, we find

$$H_N(\underline{x}_N^0) + M^t = H_N(\underline{x}_N^t) + 2\sigma\int_0^t (-\Delta)H_N(\underline{x}_N^\tau))\, d\tau + 2\int_0^t D_N(\underline{x}_N^\tau)\, d\tau,$$

where

$$(-\Delta)H_N(\underline{x}_N) := \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(-\Delta)\mathbf{g}(x_i-x_j), \qquad D_N(\underline{x}_N) := \frac{1}{N}\sum_{i=1}^{N}\left|\frac{1}{N}\sum_{1\leqslant j\leqslant N; j\neq i}\nabla\mathbf{g}(x_i-x_j)\right|^2,$$

and

$$M^t := \frac{2\sqrt{2\sigma}}{N}\sum_{i=1}^{N}\int_0^t\left(\frac{1}{N}\sum_{1\leqslant j\leqslant N; j\neq i}\nabla\mathbf{g}(x_i^\tau-x_j^\tau)\right)\cdot dw_i^\tau.$$

Using that M^t is a martingale with quadratic variation $\langle M^t\rangle = \frac{8\sigma}{N}\int_0^t D_N(\underline{x}_N^\tau)\,d\tau$, it should hold that

$$\mathbb{P}\big(Q_N^T(\underline{x}_N) > L\big) \leqslant e^{-\frac{N}{4\sigma}\left(L - H_N(\underline{x}_N^0)\right)},$$

where

$$Q_N^T(\underline{x}_N) := \sup_{t\in[0,T]}\left\{H_N(\underline{x}_N^t) + 2\sigma\int_0^t(-\Delta)H_N(\underline{x}_N^\tau)\,d\tau + \int_0^t D_N(\underline{x}_N^\tau)\,d\tau\right\}, \quad \underline{x}_N \in C([0,T],(\mathbb{T}^d)^N).$$

As $\mathbf{g}$ is singular, to make this argument rigorous we need to truncate $\mathbf{g}$ and use stopping times. In fact, we show the strong (and weak) existence and uniqueness of solutions to (1.1) as a consequence of the bounds on $Q_N^T(\underline{x}_N)$. The truncation argument proceeds similarly to as the arguments in [AGZ09, Lemma 4.33] or [RS23]. For the argument to proceed, it is important that both $\mathbf{s} < d-2$ so that $(-\Delta)\mathbf{g}$ is bounded below, and that the dynamics are dissipative as otherwise the $D_N(\underline{x}_N^t)$ term disappears.

The control of $Q_N^T(\underline{x}_N)$ allows us to show that

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\Big(Q_N^T(\underline{x}_N) > L\Big) \leqslant -\frac{1}{4\sigma}\Big(L-\mathcal{E}(\mu_0)\Big), \tag{1.15}$$

eq:overview-energy-

and in particular that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big) \leqslant -\frac{1}{4\sigma}\Big(Q^T(\mu)-\mathcal{E}(\mu_0)\Big). \tag{1.16}$$

eq:overview-Q^T-b

As a consequence, to show the LDP upper bound we only have to use the hydrodynamic argument for measure trajectories such that $Q^T(\mu) < \infty$. We can also use the bounds on the probability that Q_N^T is large to show that the empirical trajectories of (1.1) are exponentially tight in $C([0,T],\mathcal{P}(\mathbb{T}^d))$.

The proof of existence of (1.1), the bounds on $Q_N^T(\underline{x}_N)$, and the exponential tightness of μ_N can be found in Section 3.

1.5.3. Definition of $S^{s,t}(\mu,\phi)$. To define $S^{s,t}(\mu,\phi)$ when $Q^T(\mu) < \infty$ we only need to define $\nu\nabla\mathbf{g}*\nu$ distributionally when $\mathcal{E}(\nu) < \infty$.

If $\mu \in C^\infty(\mathbb{T}^d)$, then for any smooth vector field ϕ it holds that

$$\begin{aligned}\int\phi\cdot\nabla\mathbf{g}*\mu\,d\mu &= \frac{1}{2}\iint(\phi(x)-\phi(y))\cdot\nabla\mathbf{g}(x-y)\mu(x)\mu(y)\\ &= \frac{1}{2}\iint(\phi(x)-\phi(y))\cdot\nabla\mathbf{g}(x-y)(\mu-1)(x)(\mu-1)(y) + \int\phi(x)\cdot\nabla\mathbf{g}(x-y)\mu(y).\end{aligned}$$

The second term in the second equality is well-defined for any measure. To define the first term when $\mathcal{E}(\mu) < \infty$, we use the following commutator estimate, which holds for any smooth mean zero functions f, g and Lipschitz vector field ϕ :

$$\left| \iint (\phi(x) - \phi(y)) \cdot \nabla \mathbf{g}(x-y) f(x) g(y) dx dy \right| \lesssim_{d,\mathbf{s}} \left(\|\nabla \phi\|_{L^\infty} + \||\nabla|^{\frac{d-\mathbf{s}}{2}} \phi\|_{L^{\frac{2d}{d-\mathbf{s}-2}}} \right) \|f\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \|g\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}. \tag{1.17}$$

eq:overview-commu

This allows us to extend the bilinear form on left hand side of (1.17) to $\dot{H}_0^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)$, and thus gives us a way to define $\mu \nabla \mathbf{g} * \mu$ distributionally. It also gives us the quantitative bound

$$\left| \langle \mu \nabla \mathbf{g} * \mu - \nu \nabla \mathbf{g} * \nu, \phi \rangle \right| \leqslant C_\phi \|\mu - \nu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} (\|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + \|\nu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + 1)^{1/2}. \tag{1.18}$$

eq:overview-commu

Accordingly, we can define $S^{s,t}(\mu, \phi)$ whenever $\mathcal{E}(\mu^\tau)$ is uniformly bounded in time. The proof of the above bounds are given at the beginning of Section 2.

1.5.4. Continuity of $S^{s,t}(\mu, \phi)$. We still need to show that if μ_N is a sequence of empirical trajectories which converge to μ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$, then $S^{s,t}(\mu_N, \phi)$ converges to $S^{s,t}(\mu, \phi)$. Due to (1.15) we actually only have to show that

$$\lim_{N\to\infty} \int_s^t \frac{1}{N} \sum_{i=1}^N \nabla \phi^\tau(x_i^\tau) \cdot \left(\frac{1}{N} \sum_{1 \leqslant j \leqslant N; j \neq i} \nabla \mathbf{g}(x_i^\tau - x_j^\tau) \right) d\tau = - \int_s^t \langle \operatorname{div} (\mu^\tau \nabla \mathbf{g} * \mu^\tau), \phi^\tau \rangle \, d\tau, \tag{1.19}$$

eq:overview-grad-co

when $\mu_N \to \mu$ and $Q_N^T(\underline{x}_N)$ are uniformly bounded.

As a heuristic, first we'll sketch a proof that for fixed smooth ϕ the function

$$\mu \mapsto \int_s^t \langle \mu^\tau \nabla \mathbf{g} * \mu^\tau, \phi^\tau \rangle \, d\tau \tag{1.20}$$

eq:overview-regular

is continuous on the sublevel sets of Q^T. Suppose $Q^T(\mu) < \infty$ and μ_k are a sequence of measure trajectories so that $Q^T(\mu_k)$ are uniformly bounded over k and $\mu_k \to \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. The bound (1.17) and Hölder's inequality imply that

$$\begin{aligned}
&\left| \int_s^t \langle \mu_k^\tau \nabla \mathbf{g} * \mu_k^\tau - \mu^\tau \nabla \mathbf{g} * \mu^\tau, \phi^\tau \rangle \, d\tau \right| \\
&\quad \leqslant C_\phi \int_s^t \|\mu_k^\tau - \mu^\tau\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} (\|\mu_k^\tau\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + \|\mu^\tau\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + 1)^{1/2} \, d\tau \\
&\quad \leqslant C_\phi \left(\sup_{\tau \in [0,T]} \|\mu_k^\tau\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + \sup_{\tau \in [0,T]} \|\mu^\tau\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + 1 \right)^{1/2} \int_s^t \|\mu_k^\tau - \mu^\tau\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \, d\tau \\
&\quad \leqslant C_{\phi,d,\mathbf{s}} \left(Q^T(\mu_k) + Q^T(\mu) + 1 \right)^{1/2} \int_s^t \|\mu_k^\tau - \mu^\tau\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \, d\tau.
\end{aligned}$$

Thus to show the continuity of (1.20) it suffices to show that

$$\lim_{k\to\infty} \int_s^t \|\mu_k^\tau - \mu^\tau\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \, d\tau = 0.$$

This then follows by interpolating between the convergence of μ_k to μ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$, and the uniform bounds on

$$\int_0^T \|\mu_k^\tau\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}} \, d\tau \text{ and } \int_0^T \|\mu^\tau\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}} \, d\tau,$$

where we note that $\frac{\mathbf{s}-d}{2} < 1 + \frac{\mathbf{s}-d}{2}$.

To adapt this proof when μ_N are empirical trajectories and $Q_N^T(\underline{x}_N) \leqslant L$ we use the modulated energy. In particular, we show an analogous bound to (1.17), namely that for any $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{i=1}^N \phi(x_i) \cdot \left(\frac{1}{N} \sum_{1 \leqslant j \leqslant N; j \neq i} \nabla \mathbf{g}(x_i - x_j) \right) - \langle \mu \nabla \mathbf{g} * \mu, \phi \rangle \right| \\
&\qquad \leqslant C_\phi \Big(F_N(\underline{x}_N, \mu) + C \|\mu\|_{L^\infty} N^{-\beta} \Big)^{1/2} \Big(H_N(\underline{x}_N) + \|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + C \Big)^{1/2}
\end{aligned} \tag{1.21}$$

eq:overview-renorm

where $C, \beta > 0$ depend on d and $\mathbf{s}$. This follows by a renormalization argument as pioneered in [Ser20], and is similar to the commutator estimate (1.5).

The rest of the proof proceeds similarly to the proof of the continuity of (1.20), but now we interpolate between the weak convergence of μ_N to μ and the uniform bounds on

$$\int_0^T (-\Delta) H_N(\underline{x}_N^t)\, dt$$

to show that

$$\lim_{N \to \infty} \int_0^T F_N(\underline{x}_N^t, \mu^t)\, dt = 0$$

when $\mu \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$. When μ is a more general measure trajectory, we adapt the argument by mollifying μ appropriately.

The complete proof of the continuity of $S^{s,t}(\mu, \phi)$ on sublevel sets of Q^T and the convergence (1.19) along with the necessary preliminary bounds are in Section 2.

1.5.5. Completing the upper bound. Using (1.16) and (1.19) we can adapt the argument sketch in Subsection 1.5.1 to find that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \leqslant - \left(\sup_{\phi, s, t} S^{s,t}(\mu, \phi) \vee \frac{1}{4\sigma} \big(Q^T(\mu) - \mathcal{E}(\mu_0) \big) \right).$$

We can use the same argument as in the smooth case to show that if both $Q^T(\mu)$ and $\sup_{\phi,s,t} S^{s,t}(\mu, \phi)$ are finite, then μ must be a weak solution to (4.2). To prove the equality

$$\sup_{\phi, s, t} S^{s,t}(\mu, \phi) = \frac{1}{4\sigma} \int_0^T \|\partial_t \mu^t - \sigma \Delta \mu^t - \operatorname{div}(\mu^t \nabla \mathbf{g} * \mu^t)\|^2_{-1, \mu^t}\, dt,$$

we still need that $\mu \in \mathcal{AC}^T$. It is here we use the last term in the definition of $Q^T(\mu)$ since if

$$\int_0^T \|\operatorname{div}(\mu^t \nabla \mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\, dt < \infty,$$

then there must exist some $E \in L^2([0,T], L^2(\mu^t))$ so that $\operatorname{div}(\mu^t \nabla \mathbf{g} * \mu^t) = \operatorname{div}(E^t \mu^t)$ distributionally for almost every t.

The complete arguments are all given in Section 4, where the lower semi-continuity of the rate function is also proved.

1.5.6. Proof of lower bound. We use the same general strategy for proving lower large deviation bounds as used in the previous papers on LDPs for Riesz flows. This proceeds in three steps:

1. First we show that if μ is a weak solution to

$$\partial_t \mu - \sigma\Delta\mu - \mathrm{div}\,(\mu\nabla\mathbf{g} * \mu) = -\mathrm{div}\,(b\mu)$$

   and μ and b are sufficiently regular, then the empirical trajectories corresponding to the SDE

$$\begin{cases} dx_i^t = -\frac{1}{N}\sum_{1\leqslant j\leqslant N: j\neq i} \nabla\mathbf{g}(x_i^t - x_j^t)dt + b^t(x_i^t)dt + \sqrt{2\sigma}dw_i^t, \\ x_i^t|_{t=0} = x_i^0, \end{cases} \qquad i\in\{1,\cdots,N\} \tag{1.22}$$

eq:SDE+b

   almost surely converge to μ. This says that (1.22) satisfy a pathwise mean-field limit.

2. We use this to show that if μ satisfies the regularity conditions of the mean-field limit, then for all $\varepsilon > 0$

$$\liminf_{N\to\infty} \frac{1}{N}\log\mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \geqslant -\int_0^T \int |b^t|^2\, d\mu^t\, dt.$$

   This uses that (1.22) is equal in law to solutions of (1.1) under the tilting

$$\frac{d\mathbb{P}}{d\mathbb{P}_b} := \exp\left(\frac{1}{\sqrt{2\sigma}}\sum_{i=1}^N \int_0^T b^t(x_i^t)\cdot dw_i^t - \frac{1}{4\sigma}\sum_{i=1}^N \int_0^T |b^t(x_i^t)|^2 dt\right)$$

   by the Girsanov theorem.

3. Finally, if $\mu \in \mathscr{A}$ and $I(\mu) < \infty$, we show that there exists a sequence of measures μ_k with drifts b_k satisfying the regularity conditions for the mean-field limit so that

$$\limsup_{k\to\infty} \int_0^T \int |b_k^t|^2\, d\mu_k^t\, dt \leqslant I(\mu),$$

   and $\mu_k \to \mu$ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$. This allows us to conclude that

$$\liminf_{N\to\infty} \frac{1}{N}\log\mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \geqslant -I(\mu).$$

   With this rough plan, our proof does diverge from the previous papers in several ways.

1.5.7. Mean-field limit. We use a modulated energy argument to show a quantitative mean-field limit as opposed to a soft argument which uses the uniqueness of solutions to (4.2). If $\mu \in L^\infty([0,T],L^\infty(\mathbb{T}^d))$ is a solution to (4.2) with drift satisfying

$$\int_0^T \|\nabla b^t\|_{L^\infty(\mathbb{T}^d)}^2 + \||\nabla|^{\frac{d-\mathbf{s}}{2}} b^t\|^2_{L^{\frac{2d}{d-2-\mathbf{s}}}(\mathbb{T}^d)}\, dt < \infty,$$

and $\underline{x}_N$ are the solutions to (4.2), then we prove that if $\lim_{N\to\infty} F_N(\underline{x}_N^0, \mu^0) = 0$, then

$$\lim_{N\to\infty} \sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) = 0$$

almost surely. It is here where we use the strong convergence of the initial conditions.

The proof of the mean-field limit is very similar to that of the main result in [RS23]. The argument follows by applying Itô's formula to $F_N(\underline{x}_N^t, \mu^t)$, and then using the commutator estimate (1.5), Grönwall's inequality, and Doob's martingale inequality. The restriction that $\mathbf{s} < d-2$ again arises to make sense of an Itô correction term of the form

$$\int_0^t \iint (-\Delta)\mathbf{g}(x-y)\, d\big(\mu_N^\tau - \mu^\tau\big)^{\otimes 2}(x,y)\, d\tau.$$

We need to be careful when applying Itô's formula to $F_N(\underline{x}_N^t, \mu^t)$ and use a truncation and stopping time argument similar to as in the proof of the existence of (1.1). The argument is given in Section 5.

1.5.8. Lower LDP bounds for regular trajectories. As it is useful for Theorem 1.1, we actually show that

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon \right) \geqslant -\frac{1}{4\sigma} \int_0^T \int |b^t|^2\, d\mu^t\, dt.$$

Since the modulated energy controls weak convergence this also implies the LDP lower bound for balls in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. The proof of this is given in Section 6.

1.5.9. Approximating sequences. We use space mollifications to approximate measure trajectories $\mu \in \mathscr{A}$ with $I(\mu) < \infty$. Letting $\mu_\varepsilon = \Phi^\varepsilon * \mu$, where Φ^t is the fundamental solution of the heat equation, it is immediate that $\mu_\varepsilon \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$ and $\mu_\varepsilon \to \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. Since μ solves (4.2) for some drift b, it is easy to verify that μ_ε is a weak solution to

$$\partial_t \mu_\varepsilon - \sigma\Delta\mu_\varepsilon - \mathrm{div}\,(\mu_\varepsilon \nabla\mathbf{g} * \mu_\varepsilon) = -\mathrm{div}\,(b_\varepsilon \mu_\varepsilon)$$

for the drift

$$b_\varepsilon = \frac{(b\mu)_\varepsilon}{\mu_\varepsilon} + \frac{(\mu\nabla\mathbf{g} * \mu)_\varepsilon}{\mu_\varepsilon} - \nabla\mathbf{g} * \mu_\varepsilon.$$

It is here where we use that our domain is the torus as it implies that μ_ε is uniformly bounded below. This allows us to show that μ_ε and b_ε satisfy the conditions of the mean-field limit.

To show that μ_ε are well behaved with respect to the rate function, we want to show

$$\limsup_{\varepsilon\to\infty} \int_0^T \int |b_\varepsilon^t|^2\, d\mu_\varepsilon^t\, dt \leqslant \int_0^T \int |b^t|^2\, d\mu^t\, dt \leqslant I(\mu).$$

Lemma 8.1.10 in [AGS08] implies that

$$\frac{1}{4\sigma} \int_0^T \int \left| \frac{(b^\tau \mu^\tau)_\varepsilon}{\mu_\varepsilon^\tau} \right|^2 d\mu_\varepsilon^\tau\, d\tau \leqslant \frac{1}{4\sigma} \int_0^T \int |b^\tau|^2\, d\mu^\tau\, d\tau,$$

thus we only need to show that

$$\lim_{\varepsilon\to 0} \int_0^T \int \left| \frac{(\mu^\tau \nabla\mathbf{g} * \mu^\tau)_\varepsilon}{\mu_\varepsilon^\tau} - \nabla\mathbf{g} * \mu_\varepsilon^\tau \right|^2 d\mu_\varepsilon^\tau\, d\tau = 0.$$

Our class $\mathscr{A}$ is exactly the largest class where we know this holds. When $\mathbf{s} = 0$, by interpolating between $\dot{H}^{-\frac{d}{2}}(\mathbb{T}^d))$ and $\dot{H}^{1-\frac{d}{2}}(\mathbb{T}^d)$, we can show that $\{Q^T(\mu) < \infty\} \subset \mathscr{A}$, but the numerology does not work when $\mathbf{s} > 0$.

The construction of the approximating sequences and the proof of the lower bound for $\mu \in \mathscr{A}$ are also in Section 6.

1.5.10. Proof of Theorem 1.3. The upper inequality in Theorem 1.3 is an immediate consequence of the upper bound of Theorem 1.1, and is given as a proposition in Section 4. The lower bound is essentially a consequence of the fact that the lower LDP bounds around regular trajectories are shown with respect to the modulated energy. The full proof is at the very end of Section 6.

1.6 Notation

We will use the following notation and conventions throughout the rest of the paper.

- $\mathcal{D}(\mathbb{T}^d)$ denotes the space of test functions $C^\infty(\mathbb{T}^d)$.
- For a set of parameters Θ, $A \lesssim_\Theta B$ means that there exists a constant $C_\Theta > 0$ depending on these parameters so that $A \leqslant C_\Theta B$. We say $A \approx_\Theta B$ if $A \lesssim_\Theta B$ and $B \lesssim_\Theta A$.
- Throughout we allow the constant β in $N^{-\beta}$ to change line to line as is common with the multiplicative constant C. β is always allowed to depend on d and $\mathbf{s}$.
- For two vector valued function f and g we abuse notation by letting
$$f * g(x) = \int f(x-y) \cdot g(y) dy.$$
- Given a distribution T, T_ε denotes $T * \Phi^\varepsilon$ where Φ is the heat kernel satisfying
$$\begin{cases} \partial_t \Phi - \sigma \Delta \Phi = 0, \\ \Phi|_{t=0} = \delta_0. \end{cases}$$
- Unless ambiguous, we drop the domain $\mathbb{T}^d$ in spaces and norms.

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1.8 Layout of paper

Let us briefly comment on the organization of the rest of the article.

In Section 2 we show that $\mu \nabla \mathbf{g} * \mu$ has a well-behaved distributional definition. We then prove a number of inequalities involving the modulated free energy before completing the proof of (1.19). In Section 3 we show that the SDE (1.1) is well-posed, prove the exponential bounds on the probability $Q_N^T(\underline{x}_N)$ is large, and argue that μ_N are exponentially tight. In Section 4 the results of the previous two sections are used to prove the upper bound of Theorem 1.1 and one half of the statement of Theorem 1.3. The goodness of the rate function is also proved in this section. In Section 5 the systems (1.22) are shown to satisfy a mean-field limit when the drift and limiting solution are sufficiently regular. In Section 6 the result of the previous section is used to prove the lower bound of Theorem 1.1 and complete the proof of Theorem 1.3.

2. Modulated energy, renormalization, and functional inequalities

odulated-energy-etc

The primary goals of this section are to:

1. Define $\mu\nabla\mathbf{g} * \mu$ when $\mathcal{E}(\mu) < \infty$.
2. Show that if $\underline{x}_N$ are a sequence of trajectories so that $\mu_N \to \mu$ and $Q_N^T(\underline{x}_N)$ is uniformly bounded, then for any $\phi \in C^\infty([0,T] \times \mathbb{T}^d)$

$$\lim_{N\to\infty} \int_0^T \frac{1}{N}\sum_{i=1}^N \nabla\phi^t(x_i^t) \cdot \left(\frac{1}{N}\sum_{\substack{1\leqslant j\leqslant N\\ j\neq i}} \nabla\mathbf{g}(x_i^t - x_j^t)\right) dt = -\int_0^T \langle \mu^t \nabla\mathbf{g} * \mu^t, \phi^t\rangle\, dt.$$

The first point is shown in Subsection 2.1. In Subsection 2.2 the modulated energy $F_N(\underline{x}_N, \mu)$ is shown to be asymptotically equivalent to the $\frac{\mathbf{s}-d}{2}$-Sobolev distance between a regularization of the empirical measure of $\underline{x}_N$ and μ. This is used in Subsection 2.3 to prove the commutator-like inequality (1.21). Finally, in Subsection 2.4, using the propositions from the previous subsections, the second point is proved.

2.1 The commutator estimate

subseq:com-first

As discussed in the overview of the proof, to define $\mu\nabla\mathbf{g} * \mu$ first we will show that we can extend

$$\iint (\phi(x) - \phi(y)) \cdot \nabla\mathbf{g}(x-y) f(x) g(y)\, dxdy$$

to arbitrary $f, g \in \dot{H}_0^{\frac{\mathbf{s}-d}{2}}$. For notational convenience, for a vector field ϕ we set

$$\mathbf{K}_\phi(x-y) := (\phi(x) - \phi(y)) \cdot \nabla\mathbf{g}(x-y).$$

The following proposition is analogous to Proposition 3.1 in [NRS22] in the periodic setting. As we only consider potentials which are exact solutions to (1.2), we give a simpler proof which essentially follows by repeated integration by parts.

ommutator-estimate

Proposition 2.1. *For any $f, g \in \mathcal{D}(\mathbb{T}^d)$ with zero mean and Lipschitz vector field ϕ*

$$\left|\iint \mathbf{K}_\phi(x,y) f(x) g(y) dxdy\right| \lesssim_{d,\mathbf{s}} \left(\|\nabla\phi\|_{L^\infty} + \||\nabla|^{\frac{d-\mathbf{s}}{2}}\phi\|_{L^{\frac{2d}{d-\mathbf{s}-2}}}\right) \|f\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \|g\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}. \qquad (2.1)$$

eq:commutator

Consequently, the integral in the left-hand side of (2.1) extends to a bounded bilinear form on $\dot{H}_0^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)$ satisfying the bound (2.1).

Proof. By approximation we may assume that ϕ is smooth. We then set

$$F = \mathbf{g} * f \text{ and } G = \mathbf{g} * g,$$

so that

$$\|F\|_{\dot{H}^{\frac{d-\mathbf{s}}{2}}(\mathbb{T}^d)} = c_{d,\mathbf{s}} \|f\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \text{ and } \|G\|_{\dot{H}^{\frac{d-\mathbf{s}}{2}}(\mathbb{T}^d)} = c_{d,\mathbf{s}} \|g\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}},$$

where $c_{d,\mathbf{s}}$ depends on d and $\mathbf{s}$. Using that $\mathbf{g}$ solves (1.2) we can then write

$$\iint \mathbf{K}_\phi(x,y) f(x) g(y) dxdy = c_{d,\mathbf{s}} \int \phi \cdot \left(\nabla F (-\Delta)^{\frac{d-\mathbf{s}}{2}} G + \nabla G (-\Delta)^{\frac{d-\mathbf{s}}{2}} F\right).$$

Letting $\frac{d-\mathbf{s}}{2} = \alpha$ it thus suffices for us to prove that for all $0 < \alpha \leqslant \frac{d}{2}$ and any smooth zero mean F and G that

$$\int \phi \cdot \big(\nabla F(-\Delta)^{\alpha} G + \nabla G(-\Delta)^{\alpha} F\big) \lesssim_{d,\alpha} \Big(\||\nabla|^{\alpha}\phi\|_{L^{\frac{2d}{2(\alpha-1)}}} \mathbf{1}_{\alpha>1} + \|\nabla\phi\|_{L^\infty}\Big)\|F\|_{\dot{H}^{\alpha}}\|G\|_{\dot{H}^{\alpha}}. \tag{2.2}$$

eq:commutator-equ

We will prove this inductively in the integer part of α, namely $m \in \mathbb{N}$ so that $\alpha = m + \beta$ for some $0 < \beta \leqslant 1$.

We will frequently use the Caffarelli-Silvestre extension on the torus [RS16]. For $0 < \beta \leqslant 1$ let $k = 1$ when $\beta < 1$ and $k = 0$ when $\beta = 1$. Then for all $H \in \mathcal{D}$ there exists an extension $\widetilde{H}$ on $\mathbb{T}^d \times \mathbb{R}^k$ so that

$$(-\Delta)^{\beta} H \delta_{\mathbb{T}^d \times \{0\}} = -\mathrm{div}\,_z(|y|^{1-2\beta}\nabla_z \widetilde{H}).$$

Above $z = (x, y) \in \mathbb{T}^d \times \mathbb{R}^k$ and $\delta_{\mathbb{T}^d}$ denotes the restriction to $\mathbb{T}^d$ viewed as a subspace of $\mathbb{T}^d \times \mathbb{R}^k$. For convenience we will set $\gamma := 1 - 2\beta$.

The base case m=0: Abusing notation so that ϕ denotes both $\phi(x)$ and $\phi(x, y) = (\phi(x), 0)$, by integrating by parts we find that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \phi \cdot \big(\nabla F(-\Delta)^{\beta} G + \nabla G(-\Delta)^{\beta} F\big) \\
&\quad = -\int_{\mathbb{T}^d \times \mathbb{R}^k} \phi \cdot \big(\nabla_z \widetilde{F} \mathrm{div}\,_z(|y|^{\gamma}\nabla_z \widetilde{G}) + \nabla_z \widetilde{G} \mathrm{div}\,_z(|y|^{\gamma}\nabla_z \widetilde{F})\big) \\
&\quad = \int_{\mathbb{T}^d \times \mathbb{R}^k} \nabla\phi : \Big(\nabla_z \widetilde{F} \otimes \nabla_z \widetilde{G} + \nabla_z \widetilde{G} \otimes \nabla_z \widetilde{F} - \mathrm{I}_d \nabla_z \widetilde{F} \cdot \nabla_z \widetilde{G}\Big)|y|^{\gamma}.
\end{aligned}$$

Applying Cauchy-Schwarz thus gives that

$$\begin{aligned}
\Big|\int_{\mathbb{T}^d} \phi \cdot \big(\nabla F(-\Delta)^{\beta} G + \nabla G(-\Delta)^{\beta} F\big)\Big| &\lesssim_d \|\nabla\phi\|_{L^\infty}\Big(\int_{\mathbb{T}^d \times \mathbb{R}^k} |\nabla_z \widetilde{F}|^2 |y|^{\gamma}\Big)^{1/2}\Big(\int_{\mathbb{T}^d \times \mathbb{R}^k} |\nabla_z \widetilde{G}|^2 |y|^{\gamma}\Big)^{1/2} \\
&= \|\nabla\phi\|_{L^\infty}\|F\|_{\dot{H}^{\beta}}\|G\|_{\dot{H}^{\beta}}.
\end{aligned}$$

This is exactly (2.2).

Induction step: Suppose that the bound (2.2) holds for m. Then integrating by parts we find that

$$\begin{aligned}
&\int_{\mathbb{T}^d} \phi \cdot \Big(\nabla F(-\Delta)^{m+1+\beta} G + \nabla G(-\Delta)^{m+1+\beta} F\Big) \\
&\quad = \sum_{i=1}^{d} \int_{\mathbb{T}^d} \phi_i \Big(\partial_i F(-\Delta)^{m+1+\beta} G + \partial_i G(-\Delta)^{m+1+\beta} F\Big) \\
&\quad = \sum_{i=1}^{d} \int_{\mathbb{T}^d} \nabla(\phi_i \partial_i F) \cdot \nabla(-\Delta)^{m+\beta} G + \nabla(\phi_i \partial_i G) \cdot \nabla(-\Delta)^{m+\beta} F \\
&\quad = \sum_{j=1}^{d} \int_{\mathbb{T}^d} \phi \cdot \nabla(\partial_j F)(-\Delta)^{m+\beta}\partial_j G + \phi \cdot \nabla(\partial_j G) \cdot (-\Delta)^{m+\beta}\partial_j F \\
&\qquad\quad + \sum_{i,j=1}^{d} \int_{\mathbb{T}^d} \partial_j \phi_i \partial_i F(-\Delta)^{m+\beta}\partial_j G + \partial_j \phi_i \partial_i G(-\Delta)^{m+\beta}\partial_j F.
\end{aligned}$$

We can bound the first term in the last line above using the inductive hypothesis and then the Sobolev inequality to find that

$$
\begin{aligned}
&\left| \int_{\mathbb{T}^d} \phi \cdot \nabla(\partial_j F)(-\Delta)^{m+\beta} \partial_j G + \phi \cdot \nabla(\partial_j G)(-\Delta)^{m+\beta} \partial_j F \right| \\
&\quad \lesssim_{d,\alpha} \left(\||\nabla|^{m+\beta}\phi\|_{L^{\frac{2d}{2(m+\beta)-2}}} \mathbf{1}_{m>1} + \|\nabla\phi\|_{L^\infty} \right) \|\partial_j F\|_{\dot{H}^{m+\beta}} \|\partial_j G\|_{\dot{H}^{m+\beta}} \\
&\quad \lesssim_{d,\alpha} \left(\||\nabla|^{m+1+\beta}\phi\|_{L^{\frac{2d}{2(m+1+\beta)-2}}} \mathbf{1}_{m>1} + \|\nabla\phi\|_{L^\infty} \right) \|F\|_{\dot{H}^{m+1+\beta}} \|G\|_{\dot{H}^{m+1+\beta}}.
\end{aligned}
$$

To bound the remaining terms we integrate by parts again to find that

$$
\begin{aligned}
\int_{\mathbb{T}^d} \partial_j \phi \partial_i F \partial_j (-\Delta)^{m+\beta} G &= \int_{\mathbb{T}^d} \nabla^m(\partial_j \phi_i \partial_j F) \cdot (-\Delta)^{\beta} \nabla^m \partial_j G \\
&= -\int_{\mathbb{T}^d} \nabla^m(\widetilde{\partial_j \phi_i} \partial_j f) \cdot \operatorname{div}_z(|y|^\gamma \nabla_z \widetilde{\nabla^m \partial_j} G) \\
&= \int_{\mathbb{T}^d \times \mathbb{R}^k} \nabla_z \nabla^m(\widetilde{\partial_j \phi_i} \partial_j F) \cdot \nabla_z \widetilde{\nabla^m \partial_j} G |y|^\gamma.
\end{aligned}
$$

Then by applying Cauchy-Schwarz we find that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^{d+k}} \nabla_z \nabla^m(\widetilde{\partial_j \phi_i \partial_j} F) \cdot \nabla_z \widetilde{\nabla^m \partial_j} G |y|^\gamma \right| \\
&\quad \leqslant \left(\int_{\mathbb{T}^d \times \mathbb{R}^k} |\nabla_z \nabla^m(\widetilde{\partial_j \phi_i \partial_j} F)|^2 |y|^\gamma \right)^{1/2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^k} |\nabla_z \widetilde{\nabla^m \partial_j} G|^2 |y|^\gamma \right)^{1/2} \\
&\quad = \|\nabla^m(\partial_j \phi_i \partial_j F)\|_{\dot{H}^\beta} \|\nabla^m \partial_j G\|_{\dot{H}^\beta} \\
&\quad \lesssim_{d,\alpha} \|\nabla^m(\partial_j \phi_i \partial_j F)\|_{\dot{H}^\beta} \|G\|_{\dot{H}^{m+1+\beta}}.
\end{aligned}
$$

The fractional Leibniz rule [Gra14, Theorem 7.6.1][2] then implies that

$$
\|\nabla^m(\partial_j \phi_i \partial_j F)\|_{\dot{H}^\beta} \lesssim_{d,\alpha} \||\nabla|^{m+1+\beta}\phi\|_{L^{\frac{2d}{2(m+1+\beta)-2}}} \|\partial_j F\|_{L^{\frac{2d}{d-2(m+\beta)}}} + \|\nabla\phi\|_{L^\infty} \|F\|_{\dot{H}^{m+1+\beta}},
$$

and Sobolev's inequality gives that

$$
\|\partial_j F\|_{L^{\frac{2d}{d-2(m+\beta)}}} \lesssim_{d,\alpha} \|F\|_{\dot{H}^{m+1+\beta}}.
$$

Together these imply that

$$
\left| \int_{\mathbb{T}^d} \partial_j \phi \partial_i F \partial_j (-\Delta)^{m+\beta} G \right| \lesssim_{d,\alpha} \left(\||\nabla|^{m+1+\beta}\phi\|_{L^{\frac{2d}{2(m+1+\beta)-2}}} + \|\nabla\phi\|_{L^\infty} \right) \|F\|_{\dot{H}^{m+1+\beta}} \|G\|_{\dot{H}^{m+1+\beta}}.
$$

A symmetric argument gives the bound

$$
\left| \int \partial_j \phi_i \partial_i G \cdot (-\Delta)^{m+\beta} \partial_j F \right| \lesssim_{d,\alpha} \left(\||\nabla|^{m+1+\beta}\phi\|_{L^{\frac{2d}{2(m+1+\beta)-2}}} + \|\nabla\phi\|_{L^\infty} \right) \|F\|_{\dot{H}^{m+1+\beta}} \|G\|_{\dot{H}^{m+1+\beta}},
$$

and we have completed the induction. □

[2]The estimates are stated for $\mathbb{R}^d$, but they carry over to mean zero functions on $\mathbb{T}^d$ as well.

Since probability measures are not zero mean, we need to modify Proposition 2.1 accordingly to define

$$\iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d\mu(x)d\mu(y),$$

for all μ so that $\mathcal{E}(\mu) < \infty$.

or:commutatorbreak **Corollary 2.2.** *Let $\mu, \nu \in \mathcal{P}(\mathbb{T}^d) \cap \mathcal{D}(\mathbb{T}^d)$ and ϕ be a Lipschitz vector field. Then*

$$\left| \iint \mathbf{K}_\phi(x,y)d(\mu^{\otimes 2} - \nu^{\otimes 2})(x,y) \right| \lesssim_{d,\mathbf{s}} C_\phi \|\mu - \nu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} \left(1 + \|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} + \|\nu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\right)^{\frac{1}{2}}, \quad (2.3)$$ eq:commutatoruu2

where $C_\phi = \|\nabla\phi\|_{L^\infty} + \||\nabla|^{\frac{d-\mathbf{s}}{2}}\phi\|_{L^{\frac{2d}{d-2-\mathbf{s}}}}$. Consequently,

$$\iint \mathbf{K}_\phi(x,y)\, d\mu(x)d\mu(y)$$

can be extended to an operator on $\{\mu \in \mathcal{P}(\mathbb{T}^d) : \mathcal{E}(\mu) < \infty\}$ which satisfies the bound

$$\left| \int_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d\mu(x)d\mu(y) \right| \lesssim_{d,\mathbf{s}} C_\phi \|\mu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \left(\|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + 1\right)^{1/2}. \quad (2.4)$$ eq:commutatoruu1

Proof. To show (2.3) we use that $\mathbf{g}$ is zero mean to write

$$\begin{aligned}
&\iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu^{\otimes 2} - \nu^{\otimes 2})(x,y) \\
&\quad = \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu-\nu)(x)d(\mu+\nu)(y) \\
&\quad = \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu-\nu)(x)d(\mu+\nu-2)(y) + 2\iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu-\nu)(x)dy \\
&\quad = \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu-\nu)(x)d(\mu+\nu-2)(y) + 2\iint\limits_{(\mathbb{T}^d)^2} (\nabla\mathbf{g} * \phi)d(\mu-\nu).
\end{aligned}$$

Proposition 2.1 with the triangle inequality imply that

$$\left| \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d(\mu-\nu)(x)d(\mu+\nu-2)(y) \right| \lesssim_{d,\mathbf{s}} C_\phi \|\mu-\nu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \left(\|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + \|\nu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}\right)^{1/2}. \quad (2.5)$$ align:commutatorcc

Using Fourier multipliers we can bound the last term by

$$\left| \iint\limits_{(\mathbb{T}^d)^2} (\nabla\mathbf{g} * \phi)d(\mu-\nu) \right| \lesssim \|\phi * \nabla\mathbf{g}\|_{\dot{H}^{\frac{d-\mathbf{s}}{2}}} \|\mu-\nu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \lesssim \|\phi\|_{\dot{H}^{\frac{\mathbf{s}-(d-2)}{2}}} \|\mu-\nu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}.$$

In fact $\|\phi\|_{\dot{H}^{\frac{\mathbf{s}-(d-2)}{2}}(\mathbb{T}^d)} \lesssim \|\nabla\phi\|_{L^\infty}$ since $\mathbf{s} < d-2$, which completes the inequality. □

Throughout the rest of the paper we will also use $\mu\nabla\mathbf{g}*\mu$ to denote the distribution

$$\langle \mu\nabla\mathbf{g}*\mu, \phi\rangle = \frac{1}{2}\iint_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y)d\mu(x)d\mu(y)$$

when $\mathcal{E}(\mu) < \infty$. We note that if μ is sufficiently regular (for example in $L^\infty(\mathbb{T}^d)$), then the distributional definition of $\mu\nabla\mathbf{g}*\mu$ agrees with the pointwise definition.

The inequality (2.3) also allows us to show that $S^{s,t}(\mu,\phi)$ as defined in the introduction is a continuous function on the sublevel sets of Q^T and that Q^T is lower semi-continuous.

prop:Aubin-Lions **Proposition 2.3.** *Suppose that $\mu_k \in C([0,T],\mathcal{P}(\mathbb{T}^d))$ and $\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d))$ are such that*

$$Q^T(\mu_k) \text{ is uniformly bounded,}$$
$$\mu_k \to \mu \text{ in } C([0,T],\mathcal{P}(\mathbb{T}^d)).$$

Then

$$\liminf_{k\to\infty} Q^T(\mu_k) \geqslant Q^T(\mu), \tag{2.6}$$ eq:Q^T-lsc

and for any smooth vector field ϕ

$$\lim_{k\to\infty}\int_0^T \langle \mu_k^t\nabla\mathbf{g}*\mu_k^t, \phi^t\rangle\,dt = \int_0^T \langle \mu^t\nabla\mathbf{g}*\mu^t,\phi^t\rangle\,dt. \tag{2.7}$$ eq:grad-convergence

Proof. Since $\mu_k^t \to \mu^t$ for all t and $\mathbf{g}(x-y)$ and $(-\Delta)\mathbf{g}(x-y)$ are lower semi-continuous functions on $(\mathbb{T}^d)^2$,

$$\liminf_{k\to\infty}\mathcal{E}(\mu_k^t) \geqslant \mathcal{E}(\mu^t),$$

and

$$\liminf_{k\to\infty}(-\Delta)\mathcal{E}(\mu_k^t) \geqslant (-\Delta)\mathcal{E}(\mu^t).$$

Fatou's lemma then implies that

$$\liminf_{k\to\infty}\int_0^t (-\Delta)\mathcal{E}(\mu_k^\tau)\,d\tau \geqslant \int_0^t(-\Delta)\mathcal{E}(\mu^\tau)\,d\tau,$$

thus

$$\liminf_{k\to\infty}\mathcal{E}(\mu_k^t) + 2\sigma\int_0^t(-\Delta)\mathcal{E}(\mu_k^\tau)\,d\tau \geqslant \mathcal{E}(\mu^t) + 2\sigma\int_0^t(-\Delta)\mathcal{E}(\mu^\tau)\,d\tau. \tag{2.8}$$ eq:one-half-lsc

If

$$\liminf_{k\to\infty}\int_0^t \|\mathrm{div}\,(\mu_k^\tau\nabla\mathbf{g}*\mu_k^\tau)\|^2_{-1,\mu_k^\tau}\,d\tau \geqslant \int_0^t\|\mathrm{div}\,(\mu^\tau\nabla\mathbf{g}*\mu^\tau)\|^2_{-1,\mu^\tau}\,d\tau$$

we have shown that indeed Q^T is lower semi-continuous. As we will need (2.7) to establish this, we move on to the proof of the functional convergence.

The bound (2.8) implies that the right-hand side of (2.7) is well defined. To show (2.7) we first use (2.3) to bound

$$\left|\int_0^T\langle\mu_k^t\nabla\mathbf{g}*\mu_k^t - \mu^t\nabla\mathbf{g}*\mu^t,\phi^t\rangle\,dt\right| \lesssim_{d,\mathbf{s},\phi}\int_0^T\|\mu_k^t-\mu^t\|_{\dot H^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\left(1+\|\mu_k^t\|^2_{\dot H^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}+\|\mu^t\|^2_{\dot H^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\right)^{\frac{1}{2}}dt.$$

Hölder's inequality then implies that

$$\begin{aligned}
&\int_0^T \|\mu_k^t - \mu^t\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} \Big(1 + \|\mu_k^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} + \|\mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\Big)^{\frac{1}{2}} dt \\
&\leqslant T^{1/2}\bigg(\int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\, dt\bigg)^{1/2}\Big(1 + \sup_{t\in[0,T]} \|\mu_k^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} + \sup_{t\in[0,T]} \|\mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\Big)^{1/2} \\
&\lesssim T^{1/2} L \bigg(\int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\, dt\bigg)^{1/2}
\end{aligned}$$

where L comes from the uniform bounds on $Q^T(\mu_k)$. It thus suffices to show that

$$\lim_{k\to\infty} \int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}\, dt = 0, \tag{2.9}$$

eq:L^2-H-convergen

to conclude that (2.7) holds.

We will interpolate between the convergence of μ_k to μ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$ and the bounds given by $Q^T(\mu_k)$. Let $\alpha > \frac{d-\mathbf{s}}{2}$ so that $\dot{H}_0^\alpha$ compactly embeds into C_0^1. Then for any two measures $\mu,\nu \in \mathcal{P}$ and $\phi \in \dot{H}_0^\alpha$

$$\int \phi d(\mu-\nu) \leqslant \|\phi\|_{C_0^1} d(\mu,\nu) \leqslant C_\alpha \|\phi\|_{\dot{H}^\alpha} d(\mu,\nu)$$

where we've used that the Wasserstein-2 distance controls the Wasserstein-1 distance. As a consequence, $\|\mu-\nu\|_{\dot{H}^{-\alpha}} \leqslant C_\alpha d(\mu,\nu)$, thus the convergence of μ_k to μ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$ implies that $(\mu_k - \mu)$ converges to 0 in $C([0,T],\dot{H}_0^{-\alpha}(\mathbb{T}^d))$.

We can interpolate between $\dot{H}^{-\alpha}$ and $\dot{H}^{1+\frac{\mathbf{s}-d}{2}}$ to find that for all δ there exists a constant C_δ [CF88] so that

$$\|\mu\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \leqslant C_\delta \|\mu\|_{\dot{H}^{-\alpha}} + \delta\|\mu\|_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}}.$$

This implies that

$$\begin{aligned}
\int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}\, dt &\lesssim C_\delta \int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{-\alpha}}\, dt + \delta \int_0^T \|\mu_k^t\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}} + \|\mu^t\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}}\, dt \\
&\lesssim C_\delta \int_0^T \|\mu_k^t - \mu^t\|^2_{\dot{H}^{-\alpha}}\, dt + \delta T L,
\end{aligned}$$

where L is again some constant depending on the uniform bound on $Q^T(\mu_k)$. Taking $k\to\infty$ and then $\delta\to 0$ implies that (2.9) holds.

We can now complete the proof of the lower semi-continuity of Q^T. We note that if $\|T\|_{-1,\mu} \leqslant \infty$, then for all $\phi \in \mathcal{D}$

$$|\langle T,\phi\rangle| \leqslant \|T\|_{-1,\mu}\|\nabla\phi\|_{L^2(\mu)}.$$

Thus, for any $\phi \in C^\infty([0,t]\times\mathbb{T}^d)$ the Cauchy-Schwarz inequality implies that

$$\int_0^t \langle \mu_k^\tau \nabla\mathbf{g} * \mu_k^\tau, \nabla\phi^\tau\rangle\, d\tau \leqslant \bigg(\int_0^t \|\mathrm{div}\,(\mu_k^\tau\nabla\mathbf{g} * \mu_k^\tau)\|^2_{-1,\mu_k^\tau}\, d\tau\bigg)^{1/2}\bigg(\int_0^t\int |\nabla\phi^\tau|^2\, d\mu_k^\tau\, d\tau\bigg)^{1/2}.$$

Taking $k\to\infty$ and using the convergence of μ_k to μ and (2.7), this implies that

$$\int_0^t \langle \mu^\tau \nabla\mathbf{g} * \mu^\tau, \nabla\phi^\tau\rangle\, d\tau \leqslant \liminf_{k\to\infty} \bigg(\int_0^t \|\mathrm{div}\,(\mu_k^\tau\nabla\mathbf{g} * \mu_k^\tau)\|^2_{-1,\mu_k^\tau}\, d\tau\bigg)^{1/2}\bigg(\int_0^t\int |\nabla\phi^\tau|^2\, d\mu^\tau\, d\tau\bigg)^{1/2}.$$

We can then use the Riesz representation theorem to find that there exists some

$$E \in \overline{\{\nabla\phi : \phi \in C([0,t] \times \mathbb{T}^d)\}}^{L^2([0,t],L^2(\mu^\tau))}$$

so that for all $\phi \in C([0,t], \mathbb{T}^d)$

$$\int_0^t \langle \mathrm{div}\,(\mu^\tau \mathrm{div}\,\mu^\tau), \phi^\tau \rangle \, d\tau = \int_0^t \int E \cdot \nabla\phi^\tau \, d\mu^\tau \, d\tau,$$

and

$$\int_0^t \int |E^\tau|^2 \, d\mu^\tau \, d\tau \leqslant \liminf_{k\to\infty} \int_0^t \|\mathrm{div}\,(\mu_k^\tau \nabla\mathbf{g} * \mu_k^\tau)\|_{-1,\mu_k^\tau}^2 \, d\tau.$$

This immediately implies that for almost every τ

$$\mathrm{div}\,(\mu^\tau \nabla\mathbf{g} * \mu^\tau) = \mathrm{div}\,(E^\tau \mu^\tau)$$

in distribution and $E^\tau \in \overline{\{\nabla\phi : \phi \in \mathcal{D}\}}^{L^2(\mu^\tau)}$. We conclude using the equivalence

$$\|\mathrm{div}\,(\mu^\tau \nabla\mathbf{g} * \mu^\tau)\|_{-1,\mu}^2 = \int |E^\tau|^2 \, d\mu^\tau,$$

and integrating over τ.

□

cor:S-continuity **Corollary 2.4.** *For all $\phi \in C^\infty([0,T] \times \mathbb{T}^d)$ and $0 \leqslant s \leqslant t \leqslant T$*

$$S^{s,t}(\mu,\phi) := \langle \mu^t, \phi^t \rangle - \langle \mu^s, \phi^s \rangle - \int_s^t \langle \mu^\tau, \partial_t\phi^\tau \rangle + \langle \sigma\Delta\mu^\tau + \mathrm{div}\,(\mu^\tau \nabla\mathbf{g} * \mu^\tau), \phi^\tau \rangle \, d\tau - \sigma \int_s^t \int |\nabla\phi^\tau|^2 \, d\mu^\tau \, dt$$

is a continuous function on sublevel sets of Q^T.

Proof. This is an immediate consequence of Proposition 2.3 since all other terms of $S^{s,t}(\mu,\phi)$ are immediately continuous in the $C([0,T], \mathcal{P}(\mathbb{T}^d))$ topology besides

$$\int_s^t \langle \mathrm{div}\,(\mu^\tau \nabla\mathbf{g} * \mu^\tau), \phi^\tau \rangle \, d\tau.$$

□

2.2 Modulated energy inequalities.

q:modulated-energy

To adapt the proof of Proposition 2.3 for empirical trajectories we need an analogous bound to (2.3) between

$$\langle \mu_N \nabla\mathbf{g} * \mu_N, \phi \rangle := \frac{1}{N} \sum_{i=1}^N \phi(x_i) \cdot \left(\frac{1}{N} \sum_{1\leqslant j\leqslant N; i\neq j} \nabla\mathbf{g}(x_i - x_j) \right)$$

for a point configuration $\underline{x}_N$, and $\langle \mu\nabla\mathbf{g} * \mu, \phi \rangle$ for some measure μ with $\mathcal{E}(\mu) < \infty$. We thus need an analogous distance between μ_N and μ to the $\dot{H}^{\frac{\mathsf{s}-d}{2}}$ norm. This is exactly the role the modulated energy

$$F_N(\underline{x}_N, \mu) := \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{g}(x-y) \, d\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} - \mu \right)^{\otimes 2} (x,y),$$

plays. In particular, we'll show that there exists $C, \beta > 0$ depending on d and $\mathbf{s}$ so that

$$F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta} \approx_{d,\mathbf{s}} \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \right\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}^2 + \|\mu\|_{L^\infty} N^{-\beta}, \tag{2.10}$$

eq:mod-energy-equi

for any configuration $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in L^\infty(\mathbb{T}^d)$ where

$$\mathsf{r}_i := \min\left\{ \frac{1}{4} \min_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} |x_i - x_j|, N^{-\frac{1}{d}}, \frac{\mathsf{r}_0}{4} \right\}, \tag{2.11}$$

eq:mol

and $\delta_x^{(\eta)}$ is a "smearing" of δ_x onto the sphere of radius η centered at x. By comparing quantities involving the empirical measure and the smeared measure directly, we will then be able to recover analogous bounds to Corollary 2.2. This equivalence will also be important for adapting the interpolation argument in Proposition 2.3 for empirical trajectories.

To prove this equivalence, we need some more properties of the periodic Riesz kernels. Letting

$$\mathbf{g}_E(x) := |x|^{-\mathbf{s}} \mathbf{1}_{\mathbf{s}>0} - \log|x| \mathbf{1}_{\mathbf{s}=0},$$

it is shown in [HSSS17] that for all $\mathbf{s} \in [0, d)$ $\mathbf{g}$ is smooth away from 0 and

$$\mathbf{g}(x) - \mathbf{g}_E(x) \in C^\infty(B(0, \tfrac{1}{4})). \tag{2.12}$$

eq:periodic-correcti

This implies that

$$|\nabla^{\otimes k} \mathbf{g}| \lesssim_{d,\mathbf{s}} \frac{1}{|x|^{\mathbf{s}+k}} + |\log|x|| \mathbf{1}_{\mathbf{s}=k=0} \qquad \text{for all } k \geqslant 0 \text{ and } x \in \mathbb{T}^d \backslash \{0\}. \tag{2.13}$$

eq:derivatives

As $\lim_{x \to 0} \mathbf{g}_E(x) = \infty$, it also implies that there exists a constant $\mathsf{r}_0 \leqslant \frac{1}{4}$ so that

$$\frac{1}{2} \mathbf{g}_E(x) \leqslant \mathbf{g}(x) \leqslant 2\mathbf{g}_E(x) \qquad \text{in } B(0, \mathsf{r}_0). \tag{2.14}$$

eq:riesz-approximat

When $\mathbf{s} < d - 2$, since $(-\Delta)\mathbf{g}$ is also a periodic Riesz potential (now corresponding to parameter $\mathbf{s} + 2$) we can use (2.12) to further restrict r_0 so that

$$\Delta \mathbf{g} \leqslant 0 \qquad \text{in } B(0, \mathsf{r}_0). \tag{2.15}$$

eq:subharmonicitiy

This allows us to control the difference between $\mathbf{g}$ and $\mathbf{g} * \delta_0^{(\eta)}$.

nollification-bounds **Proposition 2.5.** *Let $\delta_x^{(\eta)}$ denote the uniform probability measure on the sphere $\partial B(0, \eta)$ and*

$$\mathbf{g}_\eta(x) := \int_{\mathbb{T}^d} \mathbf{g}(x - y) \, d\delta_0^{(\eta)}(y).$$

Then for all $0 \leqslant \alpha < \eta$ and $x \in B(0, \mathsf{r}_0 - \eta) \backslash \{0\}$

$$\mathbf{g}_\eta(x) \leqslant \mathbf{g}_\alpha(x), \tag{2.16}$$

eq:monotonicity1

where we let $\mathbf{g}_0 = \mathbf{g}$. Additionally, for all $|x| > 2\eta > 2\alpha \geqslant 0$

$$|\mathbf{g}_\alpha(x) - \mathbf{g}_\eta(x)| \lesssim_{d,\mathbf{s}} \frac{\eta^2}{|x|^{-(\mathbf{s}+2)}}. \tag{2.17}$$

eq:monotonicity2

Proof. The inequality (2.16) is immediately implied by the subharmonicity of $\mathbf{g}$ (2.15) and the identity

$$\frac{d}{dr}\fint_{\partial B(x,r)} f d\mathcal{H}^{d-1} = \frac{c_d}{r^{d-1}}\int_{B(x,r)} \Delta f dy, \tag{2.18}$$ eq:leibniz

where $\mathcal{H}^{d-1}$ is the $d-1$ Hausdorff measure. The inequality (2.17) is implied by (2.13) and (2.18). $\square$

Remark 2.6. If φ is a radially symmetric mollifier supported in $B(0, \frac{1}{2})$ and $\varphi_\eta(x) = \eta^{-d}\varphi(\frac{x}{\eta})$, by using spherical coordinates it is easy to verify that identical estimates as in Proposition 2.5 hold for $\varphi_\eta * \mathbf{g}$ in place of $\mathbf{g}_\eta$. We could thus alternatively take

$$d\delta_x^{(\eta)} := \varphi_\eta(y-x)dy$$

and all the proofs follow almost identically.

Now that we have some understanding of how mollifying $\mathbf{g}$ affects its behaviour, we will restate Proposition 2.1 in [NRS22] in the context of the torus. This shows that the modulated energy is "monotone under mollification." That is, modulo some error terms which vanish as $N \to \infty$

$$\left\| \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \mu \right\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}^2 \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) + \frac{1}{N^2}\sum_{i=1}^N \mathbf{g}_{\eta_i}(0).$$

This is important for establishing one of the directions in the assymptotic equivalence (2.10).

nergy-monotonicity **Proposition 2.7.** *There exists a constant C depending only on d and s such that for every choice of $\underline{x}_N \in (\mathbb{T}^d)^N$ pairwise distinct, $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, and $0 < \eta_i < \frac{\mathsf{r}_0}{4}$,*

$$\begin{aligned}
&\frac{1}{N^2}\sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \frac{r_0}{2}}} (\mathbf{g}(x_i - x_j) - \mathbf{g}_{\eta_i}(x_i - x_j)) + C^{-1}\left\| \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \mu \right\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}^2 \\
&\leqslant F_N(\underline{x}_N, \mu) + \frac{1}{N^2}\sum_{i=1}^N \mathbf{g}_{\eta_i}(0) + \frac{C}{N}\|\mu\|_{L^\infty}\sum_{i=1}^N \left(\eta_i^{d-\mathsf{s}} + \eta_i^d(|\log|\eta_i||)\mathbf{1}_{\mathsf{s}=0} + \eta_i^2\right).
\end{aligned} \tag{2.19}$$ eq:modulated-energ

Proof. The proof follows essentially verbatim as in [NRS22] using (2.16), (2.17), and (1.6). $\square$

Remark 2.8. Applying (2.19) with $\eta_i = N^{-\frac{1}{d}}$ immediately implies that

$$-N^{-\beta}\|\mu\|_{L^\infty} \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) \tag{2.20}$$ eq:positivity

for some $\beta > 0$. This shows that the modulated energy is asymptotically positive.

The first term on the left-hand side of the inequality above allows us to control the diagonal interactions

$$\frac{1}{N^2}\sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0).$$

by the modulated energy. This gives us one direction of the equivalence (2.10).

or:criticalbounding **Corollary 2.9.** *There exist constants $C, \beta > 0$ depending on d and s so that for all $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$*

$$\left\| \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \right\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}^2 \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty}N^{-\beta}. \tag{2.21}$$ eq:citicalbounding

Proof. Proposition 2.7 with $\eta_i = \mathsf{r}_i$ gives the inequality

$$\Big\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \Big\|_{\dot H^{\frac{\mathsf{s}-d}{2}}}^2 \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) + \frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) + C\|\mu\|_{L^\infty} N^{-\beta},$$

where we've used that $\mathsf{r}_i \leqslant N^{-\frac{1}{d}}$. To conclude all we need to show is that

$$\frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta}. \tag{2.22}$$

eq:diagonalbound

Abusing notation so that $\mathbf{g}_E$ is a function on $\mathbb{R}$ as well as $\mathbb{T}^d$, (2.14) and the definition of r_i imply that if $\mathsf{r}_i \leqslant N^{-\frac{1}{d}} \wedge \frac{\mathsf{r}_0}{4}$ and x_{j_i} is the closest other point to x_i in $\underline{x}_N$, then

$$\mathbf{g}_{\mathsf{r}_i}(0) \lesssim_{d,\mathsf{s}} \mathbf{g}_E(\mathsf{r}_i) \lesssim \mathbf{g}_E(x_i - x_{j_i}) \lesssim \mathbf{g}(x_i - x_{j_i}).$$

On the other hand, by a scaling argument, (2.14) implies that $\mathbf{g}_{N^{-1/d}}(x) \lesssim_{d,\mathsf{s}} \mathbf{g}_E(N^{-\frac{1}{d}})$ for all $x \in \mathbb{T}^d$. We thus find that

$$\begin{aligned}
\frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) &\lesssim_{d,\mathsf{s}} \frac{1}{N^2} \sum_{i=1}^N \mathbf{g}(x_i - x_{j_i}) - \mathbf{g}_{N^{-1/d}}(x_i - x_{j_i}) + N^{-\beta} \\
&\lesssim_{d,\mathsf{s}} \frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \frac{r_0}{2}}} (\mathbf{g}(x_i - x_j) - \mathbf{g}_{\eta_i}(x_i - x_j)) + N^{-\beta}.
\end{aligned}$$

The first term in the last line above is bounded using Proposition 2.7 again, but this time with $\eta_i = N^{-1/d}$ which gives that

$$\frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \frac{r_0}{2}}} (\mathbf{g}(x_i - x_j) - \mathbf{g}_{\eta_i}(x_i - x_j)) \lesssim_{d,\mathsf{s}} F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta}, \tag{2.23}$$

where we've used that $\|\mu\|_{L^\infty} \geqslant 1$ since μ is a probability measure and $|\mathbb{T}^d| = 1$. The statement is thus proved.

□

rem:weakcontrol **Remark 2.10.** Corollary 2.9 shows that the modulated energy controls weak convergence. For any Lipschitz ϕ, $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$

$$\begin{aligned}
\int_{\mathbb{R}^d} \phi \, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big) &= \int_{\mathbb{T}^d} \phi \, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\Big) + \int_{\mathbb{T}^d} \phi \, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)}\Big) \\
&\leqslant \|\phi\|_{\dot H^{\frac{d-\mathsf{s}}{2}}} \Big\| \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \Big\|_{\dot H^{\frac{\mathsf{s}-d}{2}}} + N^{-1/d} \|\nabla \phi\|_{L^\infty} \\
&\lesssim_{d,\mathsf{s}} \Big(\|\nabla\phi\|_{L^\infty(\mathbb{T}^d)} + \|\phi\|_{\dot H^{\frac{d-\mathsf{s}}{2}}(\mathbb{T}^d)} \Big) \Big(F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta} \Big)^{1/2},
\end{aligned}$$

where the first inequality follows by duality and the second follows by (2.9). In particular, if $\lim_{N \to \infty} F_N(\underline{x}_N, \mu) = 0$, then $\lim_{N\to\infty} \mu_N = \mu$ in $\mathcal{P}(\mathbb{T}^d)$.

To prove the other direction of the asymptotic equivalence (2.10), we need a mild generalization of Proposition 2.7.

prop:monotonicity **Proposition 2.11.** *Letting*

$$\mathbf{g}_{\eta,\alpha}(x) := \iint_{(\mathbb{T}^d)^2} \mathbf{g}(x-y-z)\, d\delta_0^{(\eta)}(y)\, d\delta_0^{(\alpha)}(z),$$

there exists a constant C depending only on d and $\mathbf{s}$ such that for every choice of $\underline{x}_N \in (\mathbb{T}^d)^N$ pairwise distinct, $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$, and $0 < \alpha_i < \eta_i < \frac{\mathsf{r}_0}{4}$

$$\begin{aligned}
&\frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \frac{\mathsf{r}_0}{2}}} \left(\mathbf{g}_{\alpha_i,\alpha_j}(x_i - x_j) - \mathbf{g}_{\eta_i,\alpha_j}(x_i - x_j)\right) + C^{-1} \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \mu \right\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}^2 \\
&\leqslant C^{-1} \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\alpha_i)} - \mu \right\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}^2 + \frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\eta_i}(0) + \frac{C}{N} \|\mu\|_{L^\infty} \sum_{i=1}^N \left(\eta_i^{d-\mathbf{s}} + \eta_i^d (|\log|\eta_i||) \mathbf{1}_{\mathbf{s}=0} + \eta_i^2\right).
\end{aligned} \tag{2.24}$$

eq:monotonicity

Proof. This follows exactly as Proposition 2.1 in [NRS22] except the difference between

$$\iint_{(\mathbb{T}^d)^2} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\alpha_i)} - \mu\Big)^{\otimes 2}(x,y) \text{ and } \iint_{(\mathbb{T}^d)^2} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \mu\Big)^{\otimes 2}(x,y)$$

is taken as opposed to

$$\iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) \text{ and } \iint_{(\mathbb{T}^d)^2} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \mu\Big)^{\otimes 2}(x,y).$$

□

Similar to how we used Proposition 2.7 to show the modulated energy controls $\frac{1}{N^2}\sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0)$, we will use Proposition 2.11 to show that

$$\left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \right\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}^2$$

controls the micro-scale interactions between the particles in a configuration $\underline{x}_N$.

prop:microscales **Corollary 2.12.** *There exist constants $C, \beta > 0$ depending on d and $\mathbf{s}$ such that for all $\underline{x}_N \in (\mathbb{T}^d)^N$ pairwise distinct, $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ and $\varepsilon < C^{-1}$*

$$\frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \varepsilon}} \mathbf{g}(x_i - x_j) \lesssim_{d,\mathbf{s}} \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu \right\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}^2 + \frac{\varepsilon^{-\mathbf{s}} + \mathbf{1}_{\mathbf{s}=0}|\log \varepsilon|}{N} + \|\mu\|_{L^\infty} \varepsilon^\beta. \tag{2.25}$$

eq:smeareddiagbou

Proof. First, we note that if $|x_i - x_j| \leqslant \frac{\mathsf{r}_0}{4}$, then

$$\mathbf{g}_{\mathsf{r}_i,\mathsf{r}_j}(x_i - x_j) \gtrsim_{d,\mathbf{s}} \mathbf{g}_E(|x_i - x_j| + \mathsf{r}_i + \mathsf{r}_j) \gtrsim_{d,\mathbf{s}} \mathbf{g}_E(|x_i - x_j|).$$

This means there exists some constant c so that

$$\mathbf{g}_{\mathsf{r}_i,\mathsf{r}_j}(x_i - x_j) \geqslant c^{-1}\mathbf{g}_E(|x_i - x_j|).$$

For arbitrary $\eta \leqslant \frac{\mathsf{r}_0}{2}$ we have that $\mathbf{g}_{\mathsf{r}_i,\eta}(x_i - x_j) \lesssim_{d,\mathbf{s}} \mathbf{g}_E(\eta)$. In particular, by setting η to be a large constant times ε (when $\mathbf{s} > 0$) or a large root of ε (when $\mathbf{s} = 0$) we can make it so that for all $|x_i - x_j| \leqslant \varepsilon$

$$\mathbf{g}_{\mathsf{r}_i,\eta}(x_i - x_j) \leqslant \frac{c^{-1}}{2}\mathbf{g}_E(\varepsilon) \leqslant \frac{c^{-1}}{2}\mathbf{g}_E(|x_i - x_j|),$$

as long as ε is sufficiently small. We thus find that for all $|x_i - x_j| \leqslant \varepsilon$

$$\mathbf{g}_{\mathsf{r}_i,\mathsf{r}_j}(x_i - x_j) - \mathbf{g}_{\mathsf{r}_i,\eta}(x_i - x_j) \geqslant \frac{c^{-1}}{2}\mathbf{g}_E(|x_i - x_j|) \gtrsim_{d,\mathbf{s}} \mathbf{g}(x_i - x_j).$$

Using Proposition 2.11 with the η given above we have

$$\begin{aligned}\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i - x_j|\leqslant\varepsilon}} \mathbf{g}(x_i - x_j) \leqslant& \frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i - x_j|\leqslant\frac{\mathsf{r}_0}{4}}} \mathbf{g}_{\mathsf{r}_i,\mathsf{r}_j}(x_i - x_j) - \mathbf{g}_{\mathsf{r}_i,\eta}(x_i - x_j)\\ &\leqslant \left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\right\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} + \frac{\mathbf{g}_\eta(0)}{N} + \|\mu\|_{L^\infty}\big(\eta^{d-\mathbf{s}} + \eta^d(|\log|\eta||)\mathbf{1}_{\mathbf{s}=0} + \eta^2\big).\end{aligned}$$

Since η is either a large multiple or large root of ε we have

$$\frac{\mathbf{g}_\eta(0)}{N} + \|\mu\|_{L^\infty}\big(\eta^{d-\mathbf{s}} + \eta^d(|\log|\eta||)\mathbf{1}_{\mathbf{s}=0} + \eta^2\big) \lesssim_{d,\mathbf{s}} \frac{\varepsilon^{-\mathbf{s}} + |\log\varepsilon|}{N} + \|\mu\|_{L^\infty(\mathbb{T}^d)}\varepsilon^\beta$$

for some β depending on d and $\mathbf{s}$. This completes the claim.

□

We now prove the opposite inequality as in Proposition 2.9 holds.

lated-energy-bound **Proposition 2.13.** *There exists a constant $\beta > 0$ depending on d and $\mathbf{s}$ such that for any $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$*

$$F_N(\underline{x}_N, \mu) \lesssim_{d,\mathbf{s}} \left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\right\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)} + \|\mu\|_{L^\infty}N^{-\beta}.$$

Proof. We begin by expanding

$$\begin{aligned}&\iint_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y)\\ &= \iint \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\Big)^{\otimes 2}(x,y) - \frac{2}{N}\sum_{i=1}^N \int \big(\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)\big)\, d\mu(x)\\ &\quad + \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\int \big(\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)\big)\, d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x) - \frac{1}{N^2}\sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i,\mathsf{r}_i}(0),\end{aligned}$$

namely

$$F_N(\underline{x}_N,\mu) \leqslant \iint \mathbf{g}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\Big)^{\otimes 2}(x,y) - \frac{2}{N}\sum_{i=1}^N \int \big(\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)\big)\, d\mu(x)$$
$$+ \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \int \big(\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)\big)\, d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x). \tag{2.26}$$ align:smearsplit

We note that

$$\left|\int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)) d\mu(x)\right|$$
$$\leqslant \int_{B(x_i,2\mathsf{r}_i)} |\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))| d\mu(x) + \int_{B(x_i,2\mathsf{r}_i)^c} |\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))| d\mu(x)$$
$$\lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty}\left(\int_{B(0,2\mathsf{r}_i)} \mathbf{g}(x)dx + \mathsf{r}_i^2 \int_{B(0,2\mathsf{r}_i)^c} \frac{1}{|x|^{s+2}} dx\right)$$
$$\lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty}\big(\mathsf{r}_i^{d-\mathbf{s}} + (\mathsf{r}_i^d|\log \eta|)\mathbf{1}_{\mathbf{s}=0} + \mathsf{r}_i^2\big), \tag{2.27}$$

thus since $\mathsf{r}_i \leqslant N^{-\frac{1}{d}}$, we can bound

$$\Big|\frac{2}{N}\sum_{i=1}^N \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))\, d\mu(x)\Big| \lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty} N^{-\beta}.$$

To bound the last term on the right-hand side of (2.26) we split near and far pairs of points

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))\, d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x)$$
$$= \frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}} \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))\, d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x)$$
$$+ \frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\geqslant\varepsilon}} \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i))\, d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x).$$

If $\varepsilon \leqslant \frac{\mathsf{r}_0}{2}$, then (2.16) implies that

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}} \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)) d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x) \lesssim_{d,\mathbf{s}} \frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}} \mathbf{g}(x_i-x_j).$$

When combined with (2.25) we thus find that for sufficiently small ε

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}} \int (\mathbf{g}(x-x_i) - \mathbf{g}_{\mathsf{r}_i}(x-x_i)) d(\delta_{x_j} + \delta_{x_j}^{(\mathsf{r}_j)})(x)$$
$$\lesssim_{d,\mathbf{s}} \left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)} - \mu\right\|^2_{\dot H^{\frac{\mathbf{s}-d}{2}}} + \frac{\varepsilon^{-s} + \mathbf{1}_{\mathbf{s}=0}|\log\varepsilon|}{N} + \|\mu\|_{L^\infty}\varepsilon^\beta. \tag{2.28}$$ align:nearbounds

If $\varepsilon \geqslant 2\mathsf{r}_i$, (2.17) implies that

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\geqslant\varepsilon}}\int(\mathbf{g}(x-x_i)-\mathbf{g}_{\mathsf{r}_i}(x-x_i))d(\delta_{x_j}+\delta_{x_j}^{(\mathsf{r}_j)})(x)\lesssim_{d,\mathbf{s}}\frac{1}{N}\sum_{i=1}^N\frac{\mathsf{r}_i^2}{\varepsilon^{s+2}}. \tag{2.29}$$

eq:farbounds

Thus choosing $\varepsilon > 2N^{-\frac{1}{d}}$ such that the ε dependent terms on the right-hand sides of (2.28) and (2.29) decay like $N^{-\beta}$ for some $\beta > 0$ completes the proof (for example $\varepsilon = N^{-\frac{1}{d^2}}$). □

2.3 Renormalized commutator estimates.

subseq:commutator

Now that we have shown that $F_N(\underline{x}_N,\mu)$ is a good analogue to the $\dot{H}^{\frac{\mathbf{s}-d}{2}}$ norm between μ_N and μ we will "renormalize" the commutator bounds (2.1) and (2.31). Throughout the rest of this section we'll let

$$\mu_{N,\vec{\mathsf{r}}} := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\mathsf{r}_i)}$$

for some configuration $\underline{x}_N$ with associated minimal distance vector $\vec{\mathsf{r}}$. The general idea of the proofs are to write

$$\mu_N-\mu=(\mu_N-\mu_{N,\vec{\mathsf{r}}})+(\mu_{N,\vec{\mathsf{r}}}-\mu).$$

Then $(\mu_N-\mu_{N,\vec{\mathsf{r}}})$ should be small since r_i are small, while $(\mu_{N,\vec{\mathsf{r}}}-\mu)$ is in $\dot{H}_0^{\frac{\mathbf{s}-d}{2}}$, so we can use the estimates from Subsection 2.3.

The following proposition is the renormalized analogue to (2.2).

mmutator-estimate

Proposition 2.14. *There exists a constant $C,\beta > 0$ such that for any $\underline{x}_N \in (\mathbb{T}^d)^N$ and $\mu \in \mathcal{P}(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$ and Lipschitz vector field ϕ*

$$\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\mu_N^{\otimes 2}-\mu^{\otimes 2})(x,y)$$
$$\lesssim_{d,\mathbf{s}} C_\phi\Big(F_N(\underline{x}_N,\mu)+C\|\mu\|_{L^\infty}N^{-\beta}\Big)^{\frac{1}{2}}\Big(C+H_N(\underline{x}_N)+\|\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\Big)^{\frac{1}{2}},$$

where $C_\phi = \|\nabla\phi\|_{L^\infty}+\||\nabla|^{\frac{d-\mathbf{s}}{2}}\phi\|_{L^{\frac{2d}{d-2-\mathbf{s}}}}$.

Proof. First we add and subtract by smeared point masses to break the left-hand side of (2.38) into two terms

$$\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\mu_N^{\otimes 2}-\mu^{\otimes 2})(x,y)=\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\delta_{x_i}+\delta_{x_i}^{(\mathsf{r}_i)})\otimes(\delta_{x_j}-\delta_{x_j}^{(\mathsf{r}_j)})(x,y)$$
$$+\iint_{(\mathbb{T}^d)^2}\mathbf{K}_\phi(x,y)d(\mu_{N,\vec{\mathsf{r}}}^{\otimes 2}-\mu^{\otimes 2})(x,y).$$

Using Corollary 2.2 and then Corollary 2.9 we have the bounds

$$\begin{aligned}
&\left| \iint_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y) d(\mu_{N,\vec{\mathsf{r}}}^{\otimes 2} - \mu^{\otimes 2})(x,y) \right| \\
&\lesssim_{d,\mathsf{s}} C_\phi \|\mu_{N,\vec{\mathsf{r}}} - \mu\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} (\|\mu_{N,\vec{\mathsf{r}}}\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} + \|\mu\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} + 1)^{1/2} \\
&\lesssim_{d,\mathsf{s}} C_\phi \Big(F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta} \Big)^{\frac{1}{2}} \Big(H_N(\underline{x}_N) + \|\mu\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} + C \Big)^{\frac{1}{2}} \qquad (2.30)
\end{aligned}$$

eq:renorm-commuta

where we've used that $\|\mu_{N,\vec{\mathsf{r}}}\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} = \|\mu_{N,\vec{\mathsf{r}}} - 1\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}$ and $F_N(\underline{x}_N, 1) = H_N(\underline{x}_N)$.

Next we split

$$\begin{aligned}
&\frac{1}{N^2} \sum_{1 \leqslant i,j \leqslant N} \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_\phi(x,y) d(\delta_{x_i} + \delta_{x_i}^{(\mathsf{r}_i)}) \otimes (\delta_{x_j} - \delta_{x_j}^{(\mathsf{r}_j)})(x,y) \\
&= \frac{1}{N^2} \sum_{1 \leqslant i \neq j \leqslant N} \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_\phi(x,y) d(\delta_{x_i} + \delta_{x_i}^{(\mathsf{r}_i)}) \otimes (\delta_{x_j} - \delta_{x_j}^{(\mathsf{r}_j)})(x,y) \\
&\quad - \frac{1}{N^2} \sum_{i=1}^N \int_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y) d(\delta_{x_i}^{(\mathsf{r}_i)})^{\otimes 2}(x,y). \qquad (2.31)
\end{aligned}$$

align:commutatorsp

To bound the second term on the right-hand side of (2.31) we note that (2.13) implies that if $|x-y| \leqslant \mathsf{r}_0$ then

$$|\mathbf{K}_\phi(x,y)| \lesssim_{d,\mathsf{s}} \|\nabla\phi\|_{L^\infty} \mathbf{g}(x-y) \qquad (2.32)$$

eq:commutator-deri

thus

$$\left| \int_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y) d(\delta_{x_i}^{(\mathsf{r}_i)})^{\otimes 2}(x,y) \right| \lesssim \|\nabla\phi\|_{L^\infty} \mathbf{g}_{\mathsf{r}_i}(0).$$

Together with (2.22) applied with reference measures μ and 1 this implies that

$$\begin{aligned}
\left| \frac{1}{N^2} \sum_{i=1}^N \int_{(\mathbb{T}^d)^2} \mathbf{K}_\phi(x,y) d(\delta_{x_i}^{(\mathsf{r}_i)})^{\otimes 2}(x,y) \right| &\lesssim_{d,\mathsf{s}} \frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) \\
&= \left(\frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) \right)^{1/2} \left(\frac{1}{N^2} \sum_{i=1}^N \mathbf{g}_{\mathsf{r}_i}(0) \right)^{1/2} \qquad (2.33) \\
&\lesssim_{d,\mathsf{s}} (F_N(\underline{x}_N, \mu) + C\|\mu\|_{L^\infty} N^{-\beta})^{1/2} (H_N(\underline{x}_N) + C)^{1/2}. \qquad (2.34)
\end{aligned}$$

eq:renorm-commuta

To bound the other part of (2.31) we split over near and far pairs

$$\begin{aligned}
&\left| \frac{1}{N^2} \sum_{1 \leqslant i \neq j \leqslant N}^N \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_\phi(x,y) d(\delta_{x_i} + \delta_{x_i}^{(\mathsf{r}_i)}) \otimes (\delta_{x_j} - \delta_{x_j}^{(\mathsf{r}_j)})(x,y) \right| \\
&\leqslant \frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \leqslant \varepsilon}} \left| \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_\phi(x,y) d(\delta_{x_i} + \delta_{x_i}^{(\mathsf{r}_i)}) \otimes (\delta_{x_j} - \delta_{x_j}^{(\mathsf{r}_j)})(x,y) \right| \\
&\quad + \left| \frac{1}{N^2} \sum_{\substack{1 \leqslant i \neq j \leqslant N \\ |x_i - x_j| \geqslant \varepsilon}} \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_\phi(x,y) d(\delta_{x_i} + \delta_{x_i}^{(\mathsf{r}_i)}) \otimes (\delta_{x_j} - \delta_{x_j}^{(\mathsf{r}_j)})(x,y) \right|.
\end{aligned}$$

The near pairs can be bounded using (2.32) and (2.16) to find that

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}}\left|\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\delta_{x_i}+\delta_{x_i}^{(\mathsf{r}_i)})\otimes(\delta_{x_j}-\delta_{x_j}^{(\mathsf{r}_j)})(x,y)\right|\lesssim\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}}\mathbf{g}(x_i-x_j). \quad (2.35)$$

If ε is sufficiently small, then (2.25) applied with reference measures μ and 1 implies that

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}}\mathbf{g}(x_i-x_j)\lesssim_{d,\mathbf{s}}\left(\|\mu_{N,\vec{\mathsf{r}}}-\mu\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}+\frac{\varepsilon^{-\mathbf{s}}+\mathbf{1}_{\mathbf{s}=0}|\log\varepsilon|}{N}+\|\mu\|_{L^\infty}\varepsilon^\beta\right)^{1/2}$$
$$\times\left(\|\mu_{N,\vec{\mathsf{r}}}\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}+\frac{\varepsilon^{-\mathbf{s}}+\mathbf{1}_{\mathbf{s}=0}|\log\varepsilon|}{N}+1\right)^{1/2}.$$

Corollary 2.9 in turn implies that

$$\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\leqslant\varepsilon}}\mathbf{g}(x_i-x_j)\lesssim_{d,\mathbf{s}}\left(F_N(\underline{x}_N,\mu)+C\|\mu\|_{L^\infty(\mathbb{T}^d)}(N^{-\beta}+\varepsilon^\beta)+\frac{\varepsilon^{-\mathbf{s}}+\mathbf{1}_{\mathbf{s}=0}|\log\varepsilon|}{N}\right)^{1/2}$$
$$\times\left(H_N(\underline{x}_N)+1+\frac{\varepsilon^{-\mathbf{s}}+\mathbf{1}_{\mathbf{s}=0}|\log\varepsilon|}{N}\right)^{1/2}. \quad (2.36)$$

For the far pairs, using symmetry we find that

$$\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\geqslant\varepsilon}}\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\delta_{x_i}+\delta_{x_i}^{(\mathsf{r}_i)})\otimes(\delta_{x_j}-\delta_{x_j}^{(\mathsf{r}_j)})(x,y)$$
$$=\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\geqslant\varepsilon}}\iint_{(\mathbb{T}^d)^2\setminus\Delta}(\mathbf{K}_\phi(x_i,x_j)-\mathbf{K}_\phi(x,y))d\delta_{x_i}^{(\mathsf{r}_i)}(x)\delta_{x_j}^{(\mathsf{r}_j)}(y).$$

The bounds (2.13) also imply that

$$|\nabla_x\mathbf{K}_\phi(x,y)|\lesssim\|\nabla\phi\|_{L^\infty}|x-y|^{-(\mathbf{s}+1)},$$

thus we find that

$$\left|\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |x_i-x_j|\geqslant\varepsilon}}\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\delta_{x_i}+\delta_{x_i}^{(\mathsf{r}_i)})\otimes(\delta_{x_j}-\delta_{x_j}^{(\mathsf{r}_j)})(x,y)\right|\lesssim\frac{1}{N}\sum_{i=1}^N\mathsf{r}_i\|\nabla\phi\|_{L^\infty}\varepsilon^{-(\mathbf{s}+1)}, \quad (2.37)$$

by the mean value theorem. Choosing an appropriate $\varepsilon>0$ (again $\varepsilon=N^{-\frac{1}{d^2}}$ works), using that $\mathsf{r}_i\leqslant N^{-\frac{1}{d}}$ and combining (2.30), (2.33), (2.36), and (2.37) concludes the desired bound. □

The standard renormalized commutator estimate for the torus follows very similarly. Although it is not useful for the LDP upper bound, it is for the proof of the mean-field limit in Section 5.

Proposition 2.15. *There exists constants $C,\beta>0$ such that for any $\underline{x}_N\in(\mathbb{T}^d)^N$ and $\mu\in\mathcal{P}(\mathbb{T}^d)\cap L^\infty(\mathbb{T}^d)$ and Lipschitz vector field ϕ*

$$\left|\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_\phi(x,y)d(\mu_N-\mu)^{\otimes 2}(x,y)\right|\lesssim_{d,\mathbf{s}}C_\phi\Big(F_N(\underline{x}_N,\mu)+C\|\mu\|_{L^\infty}N^{-\beta}\Big), \quad (2.38)$$

where $C_\phi=\|\nabla\phi\|_{L^\infty}+\||\nabla|^{\frac{d-\mathbf{s}}{2}}\phi\|_{L^{\frac{2d}{d-2-\mathbf{s}}}}$.

Proof. The proof follows exactly as the proof of Proposition 4.1 in [NRS22] with Propositions 2.1 and Proposition 3.1 replaced by Proposition 2.1 and Proposition 2.12 respectively. ☐

2.4 Functional convergence.

eq:grad-convergence

We have all the prerequisite inequalities to adapt the proof of Proposition 2.3 for empirical trajectories. Besides using Proposition 2.1 instead of (2.3), there is some technical difficulty in dealing with the fact that we only define $F_N(\underline{x}_N, \nu)$ when $\nu \in L^\infty(\mathbb{T}^d)$. To handle this when $\mu \notin L^\infty([0,T], L^\infty(\mathbb{T}^d))$, we mollify μ in space and use that whenever $\|\mu\|_{L^\infty}$ appears in the renormalized estimates, it is paired with a negative power of N.

There is also some additional technical difficulty in the interpolation argument between the convergence of μ_N to μ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$ and the uniform bounds on $Q_N^T(\underline{x}_N)$. We want to use the equivalence (2.10) to say that

$$\int_0^T (-\Delta) H_N(\underline{x}_N^t)\, dt + CT\|\mu\|_{L^\infty} N^{-\beta} \approx_{d,\mathbf{s}} \int_0^T \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i^t}^{(\mathsf{r}_i)} \right\|_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}}^2 dt + T\|\mu\|_{L^\infty} N^{-\beta},$$

but this only actually holds when $\mathbf{s} < d-4$ as when $\mathbf{s} \geqslant d-4$ the parameter corresponding to $(-\Delta)\mathbf{g}$ is greater than or equal to $d-2$. Instead we use the fact that if $\mathbf{s} < d-2$, then $(-\Delta)\mathbf{g}$ is more singular than any Riesz kernel with parameter between $\mathbf{s}$ and $\mathbf{s}+2$. We can then use the bound on $Q_N^T(\underline{x}_N)$ to instead control

$$\int_0^T \left\| \frac{1}{N} \sum_{i=1}^N \delta_{x_i^t}^{(\mathsf{r}_i)} \right\|_{\dot{H}^{\delta+\frac{\mathbf{s}-d}{2}}}^2 dt$$

for some $\delta > 0$. This is sufficient for the interpolation argument.

p:grad-convergence

Proposition 2.16. *Suppose that $\underline{x}_N \in C([0,T], (\mathbb{T}^d)^N)$ and $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d))$ are such that*

$$\begin{aligned}&\{Q_N^T(\underline{x}_N)\} \text{ is uniformly bounded,}\\ &\mu_N \to \mu \text{ in } C([0,T], \mathcal{P}(\mathbb{T}^d)).\end{aligned}$$

Then

$$\liminf_{N\to\infty} Q_N^T(\underline{x}_N) \geqslant Q^T(\mu), \tag{2.39}$$

eq:gamma-limit-upp

and for any $\phi \in C^\infty([0,T] \times \mathbb{T}^d)$

$$\lim_{N\to\infty} \int_0^T \iint_{(\mathbb{T}^d)^2 \setminus \Delta} \mathbf{K}_{\nabla\phi^t}(x,y)\, d\mu_N^t(x)\, d\mu_N^t(y)\, dt = \int_0^T \iint_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y)\, d\mu^t(x)\, d\mu^t(y)\, dt. \tag{2.40}$$

eq:grad-convergenc

Proof. As $Q_N^T(\underline{x}_N)$ is uniformly bounded so are

$$\left\{ \sup_{t\in[0,T]} H_N(\underline{x}_N^t) \right\}, \text{ and } \left\{ \int_0^T (-\Delta) H_N(\underline{x}_N^t)\, dt \right\}.$$

Truncating $\mathbf{g}$ and $(-\Delta\mathbf{g})$ near zero and using weak convergence, we easily find that

$$\liminf_{N\to\infty} H_N(\underline{x}_N^t) \geqslant \mathcal{E}(\mu^t)$$

and

$$\liminf_{N\to\infty} (-\Delta) H_N(\underline{x}_N^t) \geqslant (-\Delta)\mathcal{E}(\mu^t).$$

Fatou's lemma then implies that for all t

$$\liminf_{N\to\infty} H_N(\underline{x}_N^t) + 2\sigma \int_0^t (-\Delta) H_N(\underline{x}_N^\tau)\, d\tau \geqslant \mathcal{E}(\mu^t) + 2\sigma \int_0^t (-\Delta)\mathcal{E}(\mu^\tau)\, d\tau.$$

In particular, the uniform bounds on $Q_N^T(\underline{x}_N)$ guarantee that for all t

$$\mathcal{E}(\mu^t) + 2\sigma \int_0^t (-\Delta)\mathcal{E}(\mu^\tau)\, d\tau < \infty,$$

thus $\mu^t \nabla \mathbf{g} * \mu^t$ is well defined. We will again need to prove (2.40) before proving that Q_N^T satisfy the Gamma-limit upper bound (2.39).

We let η_N be a family of mollifiers converging to δ_0 so that $\|\eta_N\|_{L^\infty} = o(N^\beta)$ for β from Proposition 2.13. First we split

$$\begin{aligned}
&\iint\limits_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu_N^t(x) d\mu_N^t(y) - \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu^t(x) d\mu^t(y) \\
&\quad = \iint\limits_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu_N^t(x) d\mu_N^t(y) - \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu^t * \eta_N(x) d\mu^t * \eta_N(y) && (2.41) \\
&\qquad + \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu^t * \eta_N(x) d\mu^t * \eta_N(y) - \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu^t(x) d\mu^t(y). && (2.42)
\end{aligned}$$

eq:commutator-spli
eq:commutator-spli

Proposition 2.14 bounds (2.41) by

$$C_\phi \Big(F_N(\underline{x}_N^t, \mu^t * \eta_N) + C\|\eta_N\|_{L^\infty} N^{-\beta} \Big)^{\frac{1}{2}} \Big(C + H_N(\underline{x}_N^t) + \|\mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \Big)^{\frac{1}{2}}, \tag{2.43}$$

eq:commutator-esti

and Proposition 2.2 bounds (2.42) by

$$C_\phi \|\mu^t * \eta_N - \mu^t\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \Big(1 + \|\mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \Big)^{\frac{1}{2}}, \tag{2.44}$$

eq:commutator-esti

where C_ϕ is some constant that depends on the derivative of ϕ and we've used that

$$\|\mu^t * \eta_N\|_{L^\infty} \leqslant \|\eta_N\|_{L^\infty} \text{ and } \|\mu^t * \eta_N\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \leqslant \|\mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)}.$$

Together (2.43) and (2.44) imply that

$$\begin{aligned}
&\left| \int_0^T \iint\limits_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu_N^t(x) d\mu_N^t(y) - \iint\limits_{(\mathbb{T}^d)^2} \mathbf{K}_{\nabla\phi^t}(x,y) d\mu^t(x) d\mu^t(y) \right| && (2.45) \\
&\quad \leqslant C_\phi \int_0^T \Big(1 + \|\mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \Big)^{\frac{1}{2}} \|\mu^t * \eta_N - \mu^t\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)}\, dt && (2.46) \\
&\qquad + C_\phi \int_0^T \Big(C + H_N(\underline{x}_N^t) + \|\mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)} \Big)^{\frac{1}{2}} \Big(F_N(\underline{x}_N^t, \mu^t * \eta_N) + C\|\eta_N\|_{L^\infty} N^{-\beta} \Big)^{\frac{1}{2}}\, dt. && (2.47)
\end{aligned}$$

Since $\mu^t * \eta_N \to \mu^t$ in $\dot{H}^{\frac{\mathbf{s}-d}{2}}$ for fixed t, and $\sup_{t\in[0,T]} \|\mu^t\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} < \infty$, the dominated convergence theorem implies that

$$\lim_{N\to\infty} \int_0^T \left(1 + \|\mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\right)^{\frac{1}{2}} \|\mu^t * \eta_N - \mu^t\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\, dt = 0.$$

Applying Hölder's inequality we have that

$$\begin{aligned}
&\int_0^T \left(C + H_N(\underline{x}_N^t) + \|\mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\right)^{\frac{1}{2}} \left(F_N(\underline{x}_N^t, \mu^t * \eta_N) + C\|\eta_N\|_{L^\infty} N^{-\beta}\right)^{\frac{1}{2}} dt \\
&\quad \leqslant T^{1/2} \sup_{t\in[0,T]} \left(C + H_N(\underline{x}_N^t) + \|\mu^t\|^2_{\dot{H}^{\frac{\mathbf{s}-d}{2}}(\mathbb{T}^d)}\right)^{\frac{1}{2}} \left(\int_0^T F_N(\underline{x}_N^t, \mu^t * \eta_N)\, dt + CT\|\eta_N\|_{L^\infty} N^{-\beta}\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\sup_{t\in[0,T]} H_N(\underline{x}_N^t)$ are uniformly bounded and

$$\lim_{N\to\infty} \|\eta_N\|_{L^\infty} N^{-\beta} = 0,$$

to conclude (2.40) it suffices to show that

$$\lim_{N\to\infty} \int_0^T F_N(\underline{x}_N^t, \mu^t * \eta_N)\, dt = 0.$$

Let $\delta > 0$ so that $\mathbf{s} < \mathbf{s} + \delta < \max\{\mathbf{s}+2, d-2\}$. Then letting $\widetilde{\mathbf{g}}$ solve

$$(-\Delta)^{\frac{d-\mathbf{s}-\delta}{2}} \widetilde{\mathbf{g}} = c_{d,\mathbf{s},\delta}(\delta_0 - 1),$$

as $-\Delta\mathbf{g}$ is more singular than $\widetilde{\mathbf{g}}$ at zero it holds that

$$\widetilde{\mathbf{g}} \lesssim_{d,\mathbf{s}} (-\Delta\mathbf{g})(x-y) + C,$$

for some constant C depending on d and s. Letting

$$\widetilde{H}_N(\underline{x}_N) := \frac{1}{N} \sum_{1\leqslant i,j\leqslant N; i\neq j} \widetilde{\mathbf{g}}(x_i - x_j),$$

then Corollary 2.9 with $\mu = 1$ implies that

$$\|\mu_{N,\vec{\mathsf{r}}}\|^2_{\dot{H}^{\frac{\delta+\mathbf{s}-d}{2}}} \lesssim \widetilde{H}_N(\underline{x}_N) + C \lesssim_{d,\mathbf{s}} (-\Delta) H_N(\underline{x}_N) + C,$$

where we note that the definition of $\vec{\mathsf{r}}$ for $\mathbf{s}$ and $\mathbf{s}+\delta$ agree when N is sufficiently large. As a consequence, we have the bound

$$\int_0^T \|\mu^t_{N,\vec{\mathsf{r}}}\|^2_{\dot{H}^{\frac{\delta+\mathbf{s}-d}{2}}}\, dt \lesssim_{d,\mathbf{s}} \int_0^T (-\Delta) H_N(\underline{x}_N^t)\, dt + CT.$$

Since $\|\mu^t\|_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}} = c_{d,\mathbf{s}}(-\Delta)\mathcal{E}(\mu^t)$ we also have that

$$\int_0^T \|\mu^t\|^2_{\dot{H}^{\frac{\delta+\mathbf{s}-d}{2}}}\, dt \lesssim_{d,\mathbf{s}} \int_0^T \|\mu^t\|^2_{\dot{H}^{1+\frac{\mathbf{s}-d}{2}}}\, dt + CT < \infty.$$

The sequence $(\mu_{N,\vec{\mathsf{r}}} - \mu * \eta_N)$ is thus uniformly bounded in $L^2([0,T], \dot{H}_0^{\frac{\delta+\mathsf{s}-d}{2}}(\mathbb{T}^d))$. Since r_i are asymptotically vanishing and η_N converges to the identity, $\mu_{N,\vec{\mathsf{r}}}$ and $\mu * \eta_N$ both converge to μ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. As in Proposition 2.3, this implies that $\mu_{N,\vec{\mathsf{r}}} - \mu * \eta_N$ converge to 0 in $C([0,T], \dot{H}_0^{-\alpha}(\mathbb{T}^d))$ for some sufficiently large $\alpha > 0$. We can then again interpolate between $H^{\frac{\delta+\mathsf{s}-d}{2}}$ and $\dot{H}^{-\alpha}$ to find that

$$\lim_{N\to\infty} \int_0^T \|\mu^t_{N,\vec{\mathsf{r}}} - \eta_N * \mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}\, dt = 0.$$

Proposition 2.13 then implies that

$$\int_0^T F_N(\underline{x}^t_N, \mu^t * \eta_N)\, dt \leqslant \int_0^T \|\mu^t_{N,\vec{\mathsf{r}}} - \mu^t * \eta_N\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}\, dt + CT\|\eta_N\|_{L^\infty} N^{-\beta},$$

thus taking the limit in N completes the proof of (2.40).

To complete the proposition, we must only finish showing the upper Gamma-limit bound (2.39). It suffices to show that for all t

$$\int_0^t \|\mathrm{div}\,(\mu^\tau \nabla \mathbf{g} * \mu^\tau)\|^2_{-1,\mu^\tau}\, d\tau \leqslant \liminf_{N\to\infty} \int_0^t D_N(\underline{x}^\tau_N)\, d\tau.$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned}\frac{1}{2}\int_0^t \iint_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{K}_{\nabla\phi^\tau}(x,y)\, d\mu^\tau_N(x)\, d\mu^\tau_N(y)\, d\tau &= \int_0^t \frac{1}{N}\sum_{i=1}^N \nabla\phi^\tau(x_i^\tau)\cdot\left(\frac{1}{N}\sum_{1\leqslant j\leqslant N; i\neq j} \nabla \mathbf{g}(x_i^\tau - x_j^\tau)\right) d\tau \\ &\leqslant \left(\int_0^t D_N(\underline{x}^\tau_N)\, d\tau\right)^{1/2}\left(\int_0^t\int |\nabla\phi^\tau|^2\, d\mu^\tau_N\, d\tau\right)^{1/2}.\end{aligned}$$

The rest of the proof follows identically to the lower semi-continuity of Q^T in Proposition 2.3 using the convergence (2.40). ☐

3. Existence, energy control and exponential tightness

sec:energy-control

In this section we show that there are unique strong solutions to the SDE (1.1), that there are exponential bounds on the probability that $Q_N^T(\underline{x}_N)$ is large, and that the empirical trajectories corresponding to the solutions of (1.1) are exponentially tight when the initial conditions satisfy (1.8).

We prove the existence of (1.1) and the exponential probability bounds on $Q^T(\underline{x}_N)$ simultaneously. Formally computing the Itô derivative of $H_N(\underline{x}^t_N)$ when $\underline{x}^t_N$ solves (1.1) we find the identity

$$H_N(\underline{x}^t_N) + 2\sigma\int_0^t (-\Delta) H_N(\underline{x}^\tau_N)\, d\tau + 2\int_0^t D_N(\underline{x}^\tau_N)\, d\tau = M^t + H_N(\underline{x}^0_N)$$

where M^t is a martingale with quadratic variation $\frac{\sqrt{2\sigma}}{N}\int_0^t D_N(\underline{x}^\tau_N)\, d\tau$. Accordingly, we should be able to bound the probability that

$$H_N(\underline{x}^t_N) + 2\sigma\int_0^t (-\Delta) H_N(\underline{x}^\tau_N)\, d\tau + \int_0^t D_N(\underline{x}^\tau_N)\, d\tau$$

is large, where we've lost the 2 in front of the $D_N(\underline{x}_N^\tau)$ term so we can write that the above is equal to $M^t - \frac{N}{8\sigma}\langle M^t\rangle + H_N(\underline{x}_N^0)$.

As this formula for $H_N(\underline{x}_N^t)$ does not rely on the form of $\mathbf{g}$, it actually holds when we replace $\mathbf{g}$ with a smooth truncation $\mathbf{g}_{(\delta)}$, equal to $\mathbf{g}(x)$ except when $|x| < \delta$. The SDE for this truncation is well defined, and more so, as long as no two particles are closer than δ, its paths should agree with those of (1.1). We can then bound the probability that any two particles are close using the above energy equation and the fact that the truncated energy is large if any pair of particles is close to show that as $\delta \to 0$ the probability any two particles get close goes to 0. This allows us to define a solution to (1.1) as the limit of these truncated processes, and will also allow us to give bounds on the probability $Q_N^T(\underline{x}_N)$ is large.

The following exponential version of Doob's martingale inequality is vital for the desired exponential bounds, and is also important for proving the mean-field limit later in the paper.

a:exponential-doobs **Lemma 3.1.** *Let M^t be a positive continuous martingale. Then for any $L \in \mathbb{R}$*

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \log M^t \geqslant L\Big) \leqslant \mathbb{E}[M^0]e^{-L}.$$

As we will use them for the lower LDP bound, we will actually show the unique existence of solutions to (1.22).

rop:energy-estimate **Proposition 3.2.** *For $b \in L^2([0,T], C_1(\mathbb{T}^d))$ the stochastic differential equations (1.22) admit unique strong (and weak) solutions if $H_N(\underline{x}_N^0) < \infty$. More so, if $b = 0$ and (1.8) holds, then*

$$\mathbb{P}\big(Q_N^T(\underline{x}_N) \geqslant L\big) \leqslant \exp\Big(-\tfrac{N}{4\sigma}\big(L - H_N(\underline{x}_N^0)\big)\Big).$$

Proof. Let χ be a smooth function so that $0 \leqslant \chi \leqslant 1$ and

$$\chi(x) = \begin{cases} 0 & |x| \geqslant 1, \\ 1 & |x| \leqslant \frac{1}{2}. \end{cases}$$

Then $\mathbf{g}_{(\delta)}(x) := (1 - \chi(\frac{x}{\delta}))\mathbf{g}(x)$ is a smooth truncation of $\mathbf{g}$ satisfying $\mathbf{g}_{(\delta)}(x) = \mathbf{g}(x)$ for $|x| \geqslant \delta$. Consequently there is a unique strong solution to the SDE

$$\begin{cases} dx_{i,\delta}^t = -\frac{1}{N}\sum_{1\leqslant j\leqslant N; j\neq i} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^t - x_{j,\delta}^t)dt + b(x_{i,\delta}^t)dt + \sqrt{2\sigma}dw_i^t, \\ x_{i,\delta}^t|_{t=0} = x_{i,0}. \end{cases} \tag{3.1}$$ eq:truncated-SDE

Let $\tau_{N,\delta}$ be the stopping time defined by

$$\tau_{N,\delta} = \inf\{t \geqslant 0 : \min_{i\neq j} |x_{i,\delta}^t - x_{j,\delta}^t| \leqslant 2\delta\}.$$

Then when $0 < t \leqslant \tau_{N,\delta}$ it also holds that for all $k \geqslant 0$

$$\nabla^k \mathbf{g}_{(\delta)}(x_{i,\delta}^t - x_{j,\delta}^t) = \nabla^k \mathbf{g}(x_{i,\delta}^t - x_{j,\delta}^t)$$

for all $i \neq j$. In particular, setting

$$H_{N,\delta}(\underline{x}_N) := \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \mathbf{g}_{(\delta)}(x_i - x_j),$$

this implies that $H_{N,\delta}(\underline{x}_{N,\delta}) = H_N(\underline{x}_{N,\delta})$ when $0 < t < \tau_{N,\delta}$.

We proceed by applying Itô's formula to $H_{N,\delta}(\underline{x}_{N,\delta})$ to find

$$\begin{aligned}
H_{N,\delta}(\underline{x}_N^t) &= H_{N,\delta}(\underline{x}_N^0) - 2\int_0^t \frac{1}{N}\sum_{i=1}^N \left|\frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau)\right|^2 d\tau \\
&\quad + 2\int_0^t \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{N}\sum_{1\leqslant j\leqslant N, i\neq j} \nabla \mathbf{g}(x_{i,\delta}^\tau - x_{j,\delta}^\tau)\right)\cdot b(x_{i,\delta}^\tau)\, d\tau \\
&\quad + 2\sigma\int_0^t \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \Delta \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau) d\tau \\
&\quad + \frac{2\sqrt{2\sigma}}{N}\sum_{i=1}^N \int_0^t \left(\frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N}^N \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau)\right) dw_i^\tau. \qquad (3.2)
\end{aligned}$$

eq:truncIto

The last term on the right-hand side of (3.2) is a martingale with respect to the filtration generated by the noise, which we will denote by M^t, which has quadratic variation

$$\langle M^t\rangle = \frac{8\sigma}{N}\int_0^t \frac{1}{N}\sum_{i=1}^N \left|\frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau)\right|^2 d\tau.$$

Applying Young's inequality to the third term on the second line of (3.2) and rearranging we find that for any parameter $\alpha < 1$

$$\begin{aligned}
&H_{N,\delta}(\underline{x}_N^t) - 2\sigma\int_0^t \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \Delta \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau) d\tau + \int_0^t \frac{1}{N}\sum_{i=1}^N \left|\frac{1}{N}\sum_{1\leqslant j\leqslant N; i\neq j}^N \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - x_{j,\delta}^\tau)\right|^2 d\tau \\
&\leqslant M^t - \frac{(1-\alpha)N}{8\sigma}\langle M^t\rangle + H_{N,\delta}(\underline{x}_N^0) + \frac{1}{\alpha}\int_0^t \frac{1}{N}\sum_{i=1}^N |b(x_{i,\delta}^\tau)|^2\, d\tau \\
&\leqslant M^t - \frac{(1-\alpha)N}{8\sigma}\langle M^t\rangle + H_{N,\delta}(\underline{x}_N^0) + \frac{1}{\alpha}\int_0^t \|b^\tau\|_{L^\infty}^2\, d\tau.
\end{aligned}$$

Setting

$$\widetilde{Q}_N^t(\underline{x}_N) := H_N(\underline{x}_{N,\delta}^t) - 2\sigma\int_0^t (-\Delta) H_N(\underline{x}_N^\tau)\, d\tau + \int_0^t D_N(\underline{x}_N^\tau)\, d\tau,$$

we found that for $0 < t \leqslant \tau_{N,\delta}$, letting $\lambda_N := \frac{(1-\alpha)N}{4\sigma}$

$$\widetilde{Q}_N^t(\underline{x}_{N,\delta}) \leqslant M^t - \tfrac{\lambda_N}{2}\langle M^t\rangle + H_N(\underline{x}_N^0) + \alpha^{-1}\|b\|_{L^2([0,T],L^\infty(\mathbb{T}^d))}^2.$$

Using Lemma 3.1 with the optional sampling theorem applied to the continuous positive martingale $\exp(\lambda_N M^t - \frac{\lambda_N^2}{2}\langle M^t\rangle)$ we find that for all $L > 0$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T\wedge\tau_{N,\delta}]} \widetilde{Q}_N^t(\underline{x}_{N,\delta}) \geqslant L\right) \\
&\leqslant \mathbb{P}\left(\sup_{t\in[0,T\wedge\tau_{N,\delta}]} \log\exp\left(\lambda_N M^t - \tfrac{\lambda_N^2}{2}\langle M^t\rangle\right) \geqslant \lambda_N(L - H_N(\underline{x}_N^0) - C_{b,\alpha})\right) \qquad (3.3) \\
&\leqslant \exp\Big(-\lambda_N(L - H_N(\underline{x}_N^0) - C_{b,\alpha})\Big)\mathbb{E}\Big[\exp\left(\lambda_N M_N^0 - \tfrac{\lambda_N^2}{2}\langle M^0\rangle\right)\Big] \\
&= \exp\Big(-\lambda_N(L - H_N(\underline{x}_N^0) - C_{b,\alpha})\Big), \qquad (3.4)
\end{aligned}$$

align:energydoobs

for a constant $C_{b,\alpha}$ depending on b and α. As $\mathbf{g}$ and $(-\Delta)\mathbf{g}$ are both bounded below there exists a constant C so that if $\tau_{N,\delta} \leqslant T$, then

$$\widetilde{Q}_N^{\tau_{N,\delta}}(\underline{x}_{N,\delta}) \geqslant H_N(\underline{x}_{N,\delta}^{\tau_{N,\delta}}) - C\sigma T \geqslant \frac{\min_{|x|\leqslant 2\delta} \mathbf{g}(x)}{N^2} - C(1+\sigma T),$$

where the second inequality follows because it must be that $|x_{i,\delta}^{\tau_{N,\delta}} - x_{j,\delta}^{\tau_{N,\delta}}| \leqslant 2\delta$ for some pair of indices i and j. Setting

$$f(\delta) := \frac{\min_{|x|\leqslant 2\delta} \mathbf{g}(x)}{N^2} - C(1+\sigma T),$$

then $f(\delta) \to \infty$ as $\delta \to 0$. Using (3.3) with $L = f(\delta)$ we thus find that

$$\mathbb{P}\big(\tau_{N,\delta} \leqslant T\big) \leqslant \mathbb{P}\Big(\widetilde{Q}_N^{\tau_{N,\delta}}(\underline{x}_{N,\delta}) \geqslant f(\delta)\Big) \leqslant \exp\Big(-\lambda_N(f(\delta) - H_N(\underline{x}_N^0) - C_b)\Big),$$

the right-hand side of which converges to 0 as $\delta \to 0$. We can thus find a sequence $\delta_k \to 0$ such that

$$\sum_{k\geqslant 1} \mathbb{P}\Big(\tau_{N,\delta_k} \leqslant T\Big) < \infty.$$

The Borell-Cantelli lemma with the monotonicty of $\tau_{N,\delta}$ in δ imply that $\lim_{\delta\to 0} \tau_{N,\delta} > T$ almost surely. Since $\underline{x}_{N,\delta}^t$ and $\underline{x}_{N,\delta'}^t$ agree on $0 < t \leqslant \tau_{N,\delta}$ when $\delta' < \delta$, this allows us to define a unique strong (and weak) solution to (1.1).

Returning to the proof of (3.5), we note that

$$Q_N^T = \sup_{t\in[0,T]} \widetilde{Q}_N^t.$$

Thus when $b = 0$, as we can take $\alpha = 0$, taking the pointwise limit in (3.3) as $\delta \to 0$ gives that

$$\mathbb{P}\big(Q_N^T(\underline{x}_N) > L\big) \leqslant \exp\Big(-\tfrac{N}{4\sigma}(L - H_N(\underline{x}_N^0)\Big),$$

as desired. $\square$

cor:energy-estimate **Corollary 3.3.** *If $\underline{x}_N$ are the unique solutions to (1.1) with initial conditions satisfying (1.8), then*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(Q_N^T(\underline{x}_N) > L\Big) \leqslant -\frac{1}{4\sigma}\Big(L - \mathcal{E}(\mu^0)\Big). \tag{3.5}$$ eq:energy-control

Proof. Our assumptions on the initial conditions (1.8) imply that

$$H_N(\underline{x}_N^0) \to \mathcal{E}(\mu_0).$$

Indeed, expanding out the definition of H_N and F_N we find that

$$\begin{aligned} H_N(\underline{x}_N^0) = \mathcal{E}(\mu_0) + F_N(\underline{x}_N^0, \mu_0) + \frac{2}{N}\sum_{i=1}^N \int \big(\mathbf{g}(x_i^0 - y) - \mathbf{g}_{\mathsf{r}_i}(x_i^0 - y)\big)d\mu_0(y) \\ + 2\int \mathbf{g}(x-y)\, d\bigg(\frac{1}{N}\sum_{i=1}^N \delta_{x_i^0}^{(\mathsf{r}_i)} - \mu_0\bigg)\, d\mu_0 \end{aligned}$$

The second and fourth terms go to zero as $N \to \infty$ since duality and Proposition 2.9 imply that

$$2\int \mathbf{g}(x-y)\, d\bigg(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i^0}^{(\mathsf{r}_i)} - \mu_0\bigg)\, d\mu \lesssim_{d,\mathbf{s}} \|\mu_{N,\vec{\mathsf{r}}}^0 - \mu_0\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}} \|\mu_0\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}$$

$$\lesssim_{d,\mathbf{s}} (F_N(\underline{x}_N^0, \mu_0) + C\|\mu_0\|_{L^\infty(\mathbb{T}^d)} N^{-\beta})^{1/2} \|\mu_0\|_{\dot{H}^{\frac{\mathbf{s}-d}{2}}}.$$

The third term is bounded by $\|\mu\|_{L^\infty} N^{-\beta}$ as in Proposition 2.11, thus also vanishes.

Using (3.1) we thus find that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(Q_N^T(\underline{x}_N) > L\big) \leqslant \limsup_{N\to\infty} -\tfrac{1}{4\sigma}(L - H_N(\underline{x}_N^0)) = -\tfrac{1}{4\sigma}(L - \mathcal{E}(\mu_0)),$$

thus the corollary holds. $\square$

The control on the probability that $H_N(\underline{x}_N^t)$ is large allows us to adapt the proof of exponential tightness in [DG87] to our setting.

Proposition 3.4. *The empirical paths $\{\mu_N\}_{N\geqslant 1}$ associated to (1.1) are exponentially tight in $C([0,T], \mathcal{P}(\mathbb{T}^d))$ if the initial conditions satisfy (1.8). That is, for all $L > 0$ there exists a compact set $K_L \subset C([0,T], \mathcal{P}(\mathbb{T}^d))$ such that*

$$\limsup_{N\to\infty} \mathbb{P}(\mu_N \in K_L^c) \leqslant -L. \tag{3.6}$$

Proof. The proof follows very similarly to the proof of exponential tightness for the measure paths of regularly interacting particles [DG87], but a small modification is required to handle the singularity which relies on the energy control (3.5).

Fixing $R > 0$, we will first show that for all $\phi \in \mathcal{D}$ and $\alpha > 0$ there exists a compact set $K_{\alpha,\phi} \subset C([0,T], \mathbb{R})$ so that

$$\mathbb{P}(\langle \mu_N^{(\cdot)},\ \phi\rangle \in K_{\alpha,\phi}^c, Q_N^T(\underline{x}_N) \leqslant R) \leqslant e^{-N\alpha}. \tag{3.7}$$

Applying Itô's formula to $\langle \mu_N^\tau, \phi^\tau\rangle$ we have that

$$\langle \mu_N^t, \phi\rangle - \langle \mu_N^s, \phi\rangle = -\int_s^t \frac{1}{N}\sum_{i=1}^N \nabla\phi(x_i^\tau)\cdot\bigg(\frac{1}{N}\sum_{1\leqslant j\leqslant N; i\neq j}^N \nabla\mathbf{g}(x_i^\tau - x_j^\tau)\bigg)d\tau + \sigma\int_s^t \langle \mu_N^\tau, \Delta\phi\rangle d\tau + M^{s,t},$$

where

$$M^{s,t} := \frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N \int_s^t \nabla\phi(x_i^\tau)dw_i^\tau, \qquad \langle M^{s,t}\rangle = \frac{2\sigma}{N}\int_s^t \langle \mu_N^\tau, |\nabla\phi|^2\rangle d\tau$$

is a martingale for fixed s. We note that for any $\underline{x}_N \in (\mathbb{T}^d)^N$

$$\frac{1}{N}\sum_{i=1}^N \nabla\phi(x_i)\cdot\bigg(\frac{1}{N}\sum_{1\leqslant j\leqslant N; i\neq j}^N \nabla\mathbf{g}(x_i - x_j)\bigg) = \frac{1}{2N^2}\sum_{1\leqslant i\neq j\leqslant N}\big(\nabla\phi(x_i) - \nabla\phi(x_j)\big)\cdot\nabla\mathbf{g}(x_i - x_j)$$

$$\lesssim_{d,\mathbf{s}} \|\nabla\phi\|_{L^\infty}\Big(H_N(\underline{x}_N) + C\Big)$$

since there exists some constant C so that $|x||\nabla\mathbf{g}(x)| \leqslant \mathbf{g}(x) + C$. When $Q_N^T(\underline{x}_N) \leqslant R$, there exists some constant $\kappa > 0$ depending on R, σ, d, and $\mathbf{s}$ and the norms of the first two derivatives of ϕ so that for all $\gamma > 0$

$$\langle \mu_N^t, \phi\rangle - \langle \mu_N^s, \phi\rangle \leqslant \kappa(1+\gamma)(s-t) + M^{s,t} - \frac{N\gamma}{2}\langle M^{s,t}\rangle.$$

Proceeding identically to as in [DG87] we thus find that for all $\delta > 0$, $s \leqslant T - 2\delta$, and $\rho > 4\kappa\delta$

$$\mathbb{P}\Bigg(\sup_{\substack{0 \leqslant s < t \leqslant T \\ |s-t| \leqslant \delta}} |\langle \mu_N^t, \phi \rangle - \langle \mu_N^s, \phi \rangle| > \rho, Q_N^T(\underline{x}_N) \leqslant R \Bigg) \leqslant \frac{2T}{\delta} \exp\left(-N \frac{(\rho - 4\kappa\delta)^2}{32\kappa\delta} \right). \tag{3.8}$$

eq.equicontinuity-b

Letting

$$\delta_k = \frac{T}{2} k^{-2}, \qquad \rho_k = 10\kappa\alpha^{1/2} k^{-1/2},$$

then since $(\delta_k, \rho_k) \to (0,0)$, the Arzelà-Ascoli theorem implies that the set

$$K_{\alpha,\phi} := \Big\{ x \in C([0,T], \mathbb{R}) : x^0 \leqslant \|\phi\|_{L^\infty}, \sup_{\substack{0 \leqslant s < t \leqslant T \\ |s-t| \leqslant \delta_k}} |x^t - x^s| \leqslant \rho_k \text{ for all } k \geqslant 1 \Big\}$$

is compact. Since $\rho_k \geqslant 4\kappa\delta_k$, (3.8) implies that

$$\begin{aligned} \mathbb{P}\big(\langle \mu_N^{(\cdot)}, \phi \rangle \in K_{\alpha,\phi}^c, Q_N^T(\underline{x}_N) \leqslant R\big) &\leqslant \sum_{k \geqslant 1} \mathbb{P}\Bigg(\sup_{\substack{0 \leqslant s < t \leqslant T \\ |s-t| \leqslant \delta_k}} |\langle \mu_N^t, \phi \rangle - \langle \mu_N^s, \phi \rangle| > \rho_k, Q_N^T(\underline{x}_N) \leqslant R \Bigg) \\ &\leqslant \sum_{k \geqslant 1} \frac{2T}{\delta_k} \exp\left(-N \frac{(\rho_k - 4\kappa\delta_k)^2}{32\kappa\delta_k} \right) \\ &\leqslant e^{-N\alpha}, \end{aligned}$$

where the last line follows by the choice of δ_k and ρ_k. We have thus shown that for all α we can find a compact set so that (3.7) holds.

Now, for fixed $L > 0$, since $H_N(\underline{x}_N^0) \to \mathcal{E}(\mu^0)$, there exists $R > 0$ so that

$$\frac{1}{N} \log \mathbb{P}(Q_N^T(\underline{x}_N) > R) \leqslant -L,$$

for all N. Letting $\{\phi_\ell\}_{\ell \geqslant 1} \subset \mathcal{D}$ be dense in $C(\mathbb{T}^d)$, and $K_{L+\ell,\phi_\ell}$ the compact subset of $C([0,T], \mathbb{R})$ so that (3.7) holds with ϕ_ℓ and $\alpha = L + \ell$ we define

$$K_L := \bigcap_{\ell \geqslant 1} \Big\{ \nu \in C([0,T], \mathcal{P}(\mathbb{T}^d)) : \langle \nu, \phi_\ell \rangle \in K_{L+\ell,\phi_\ell} \Big\}.$$

Then the set K_L is compact by [Gä88, Lemma 1.3] since the Wasserstein-2 metric topologizes weak convergence. Moreover,

$$\begin{aligned} \mathbb{P}(\mu_N \in K_L^c) &\leqslant \sum_{\ell \geqslant 1} \mathbb{P}\big(\langle \mu_N, \phi_\ell \rangle \in K_{L+\ell,\phi_\ell}^c, Q_N^T(\underline{x}_N) \leqslant R\big) + \mathbb{P}\big(Q_N^T(\underline{x}_N) > R\big) \\ &\leqslant \sum_{\ell \geqslant 1} e^{-N(L+\ell)} + \mathbb{P}\big(Q_N^T(\underline{x}_N) > R\big) \\ &\lesssim e^{-NL}, \end{aligned}$$

thus indeed (3.6) holds. We have thus shown that μ_N are exponentially compact in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. $\square$

4. Upper bound

sec:upper-bound

Now that we have shown that (1.1) is well defined and we have control on $Q_N^T(\underline{x}_N)$, we can use Proposition 2.16 to give local LDP bounds. This proceeds very similarly to as in the sketch in the introduction.

prop:upper-bound **Proposition 4.1.** *For all $\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d))$, if μ_N are empirical trajectories corresponding to (1.1) with initial conditions satisfying (1.8) then*

$$\lim_{\varepsilon\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant -\left(\sup_{\phi,s,t}S^{s,t}(\mu,\phi)\vee\frac{1}{4\sigma}\big(Q^T(\mu)-\mathcal{E}(\mu^0)\big)+\infty\cdot\mathbf{1}_{\mu^0=\mu_0}\right). \quad (4.1)$$ eq:auxillary-upper-b

Proof. First we note Assumption 1.8 and Remark 2.10 imply that $\mu_N^0\to\mu_0$ in $\mathcal{P}(\mathbb{T}^d)$. Thus if $\mu\in C([0,T],\mathcal{P}(\mathbb{T}^d))$ is such that $\mu^0\neq\mu_0$, then for any $\varepsilon>0$

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)=-\infty.$$

On the other hand the upper Gamma-limit (2.39) implies that for all $\delta>0$ there exists $\varepsilon>0$ so that for all sufficiently large N

$$\Big\{\underline{x}_N:\mu_N\in B_\varepsilon(\mu)\Big\}\subset\Big\{\underline{x}_N:Q_N^T(\underline{x}_N)>Q^T(\mu)\wedge\tfrac{1}{\delta}-\delta\Big\}.$$

Corollary 3.3 thus implies that

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant-\frac{1}{4\sigma}(Q^T(\mu)\wedge\tfrac{1}{\delta}-\delta-\mathcal{E}(\mu^0)).$$

Taking $\delta\to0$ we then find that

$$\lim_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant-\frac{1}{4\sigma}\big(Q^T(\mu)-\mathcal{E}(\mu^0)\big).$$

To complete the proposition it suffices to show that $\mu\in C([0,T],\mathcal{P}(\mathbb{T}^d))$ and $Q^T(\mu)<\infty$, then

$$\lim_{\varepsilon\to0}\limsup_{N\to\infty}\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big)\leqslant-\sup_{\phi,s,t}S^{s,t}(\mu,\phi).$$

Fixing $\phi\in C^\infty([0,T]\times\mathbb{T}^d)$ and $0\leqslant s\leqslant t\leqslant T$, Itô's formula gives use the equation

$$\begin{aligned}\langle\mu_N^t,\phi^t\rangle=\langle\mu_N^s,\phi^s\rangle-\frac{1}{N}\sum_{i=1}^N\int_s^t\nabla\phi^\tau(x_i^\tau)\cdot\left(\frac{1}{N}\sum_{1\leqslant j\leqslant N;i\neq j}\nabla\mathbf{g}(x_i^\tau-x_j^\tau)\,d\tau\right)d\tau\\+\sigma\int_s^t\langle\mu_N^\tau,\partial_t\phi^\tau+\sigma\Delta\phi^\tau\rangle d\tau+\frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N\int_s^t\nabla\phi^\tau(x_i^\tau)\cdot dw_i^\tau.\end{aligned}$$

By symmetrizing we see that

$$\frac{1}{N}\sum_{i=1}^N\nabla\phi^\tau(x_i^\tau)\cdot\left(\frac{1}{N}\sum_{1\leqslant j\leqslant N;i\neq j}\nabla\mathbf{g}(x_i^\tau-x_j^\tau)\right)=\frac{1}{2}\iint_{(\mathbb{T}^d)^2\setminus\Delta}\mathbf{K}_{\nabla\phi^\tau}(x,y)\,d(\mu_N^\tau)^{\otimes2}(x,y).$$

We have just found after rearranging that

$$S^{s,t}(\mu_N,\phi) = \frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N \int_s^t \nabla\phi^\tau(x_i^\tau)\cdot dw_i^\tau - \sigma\int_s^t\int |\nabla\phi^\tau|^2\, d\mu_N^\tau\, d\tau = M^{s,t} - \frac{N}{2}\langle M^{s,t}\rangle,$$

where for fixed s

$$M^{s,t} := \frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N \int_s^t \nabla\phi^\tau(x_i^\tau)\cdot dw_i^\tau$$

is a bounded continuous martingale with respect to the filtration generated by the noise.

For any $L > 0$ we have the union bound

$$\mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \leqslant \mathbb{P}(\mu_N \in B_\varepsilon(\mu), Q_N^T(\underline{x}_N) \leqslant L) + \mathbb{P}(Q_N^T(\underline{x}_N) > L).$$

Thus letting $A_{N,\varepsilon,L} := \{\underline{x}_N \in C([0,T],(\mathbb{T}^d)^N) : \mu_N \in B_\varepsilon(\mu), Q_N^T(\underline{x}_N) \leqslant L\}$

$$\begin{aligned}
\mathbb{P}(\mu_N \in B_\varepsilon(\mu), Q_N^T(\underline{x}_N) \leqslant L) &\leqslant \mathbb{E}\Big[\exp(-NS^{s,t}(\mu_N,\phi))\exp(NM^{s,t} - \frac{N^2}{2}\langle M^{s,t}\rangle)\mathbf{1}_{A_{N,\varepsilon,L}}\Big] \\
&\leqslant \exp\Big(-N\inf_{\underline{x}_N\in A_{N,\varepsilon,L}} S^{s,t}(\mu_N,\phi)\Big)\mathbb{E}\Big[\exp(NM^{s,t} - \frac{N^2}{2}\langle M^{s,t}\rangle)\Big] \\
&= \exp\Big(-N\inf_{\underline{x}_N\in A_{N,\varepsilon,L}} S^{s,t}(\mu_N,\phi)\Big),
\end{aligned}$$

where the last line follows as $\exp(NM^{s,t} - \frac{N^2}{2}\langle M^{s,t}\rangle)$ has expectation equal to 1. This gives that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big) \\
&\quad\leqslant \Big(-\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\inf_{\underline{x}_N\in A_{N,\varepsilon,L}} S^{s,t}(\mu_N,\phi)\Big) \vee \Big(-\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(Q_N^T(\underline{x}_N) > L)\Big).
\end{aligned}$$

Proposition 2.16 immediately implies that

$$S^{s,t}(\mu,\phi) \leqslant \lim_{\varepsilon\to 0}\liminf_{N\to\infty}\inf_{\underline{x}_N\in A_{N,\varepsilon,L}} S^{s,t}(\mu_N,\phi).$$

In total we have found that

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\big(\mu_N\in B_\varepsilon(\mu)\big) \leqslant \Big(-S^{s,t}(\mu,\phi)\Big) \vee \Big(-\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}(Q_N^T(\underline{x}_N) > L)\Big).$$

Sending $L\to\infty$ and taking the infimum over all s, t and ϕ completes the proof. $\square$

To rewrite the rate function in terms of $\|\cdot\|_{-1,\mu}$ we need to show that if $\sup_{\phi,s,t} S^{s,t}(\mu,\phi) < \infty$ and $Q^T(\mu) < \infty$ then $\mu \in \mathcal{AC}^T$. We will in particular show that μ must solve the perturbed McKean–Vlasov equation (1.22) which we can rewrite as

$$\partial_t\mu - \operatorname{div}\left(\left(\frac{\nabla\mu}{\mu} + b + \nabla\mathbf{g} * \mu\right)\mu\right) = 0.$$

Both b and $\nabla\mathbf{g} * \mu$ (or a representation of $\nabla\mathbf{g} * \mu$) will be guaranteed to be in $L^2([0,T], L^2(\mu^t))$, but to show that μ is in $\mathcal{AC}^T$ we still need to show that $\operatorname{div}(\frac{\nabla\mu}{\mu}\mu) = \Delta\mu = \operatorname{div}(E\mu)$ for some $E \in L^2([0,T], L^2(\mu^t))$. The following proposition shows this using entropy.

absolute-continuity **Proposition 4.2.** *Suppose that $\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d))$ is a weak solution to*

$$\partial_t \mu - \sigma \Delta \mu + \operatorname{div}(b\mu) = 0,$$

for $b \in L^2([0,T], L^2(\mu^t))$ and $Ent(\mu^0) < \infty$ where

$$Ent(\mu) = \begin{cases} \int \mu(x)\log\mu(x)dx & \mu \ll dx \\ +\infty & otherwise. \end{cases}$$

Then $\mu^t \ll dx$ for all $t \in [0,T]$ and $\mu \in \mathcal{AC}^T$.

Proof. We will consider the sequence of mollifications $\mu_\varepsilon := \mu * \Phi^\varepsilon$. We note that μ_ε is uniformly bounded below in space and time. As well, μ_ε is a weak solution to

$$\partial_t \mu_\varepsilon - \sigma\Delta\mu_\varepsilon - \operatorname{div}((b\mu)_\varepsilon) = 0.$$

Then, differentiating by time (which can be justified by mollifying in time), we find that

$$\begin{aligned}
\mathrm{Ent}(\mu_\varepsilon^t) - \mathrm{Ent}(\mu_\varepsilon^0) &= -\sigma \int_0^t \int \left|\frac{\nabla\mu_\varepsilon^\tau}{\mu_\varepsilon^\tau}\right|^2 d\mu_\varepsilon^\tau\, d\tau + \int_0^t \int \frac{\nabla\mu_\varepsilon^\tau}{\mu_\varepsilon^\tau} \cdot \frac{(b^\tau\mu^\tau)_\varepsilon}{\mu_\varepsilon^\tau}\, d\mu_\varepsilon^\tau\, dt \\
&\leqslant -\frac{\sigma}{2}\int_0^t \int \left|\frac{\nabla\mu_\varepsilon^\tau}{\mu_\varepsilon^\tau}\right|^2 d\mu_\varepsilon^\tau\, d\tau + \frac{2}{\sigma}\int_0^t \int \left|\frac{(b^\tau\mu^\tau)_\varepsilon}{\mu_\varepsilon^t}\right|^2 d\mu_\varepsilon^t\, d\tau \\
&\leqslant -\frac{\sigma}{2}\int_0^t \int \left|\frac{\nabla\mu_\varepsilon^\tau}{\mu_\varepsilon^\tau}\right|^2 d\mu_\varepsilon^\tau\, d\tau + \frac{2}{\sigma}\int_0^t \int |b^\tau|^2\, d\mu^\tau\, d\tau
\end{aligned}$$

where the first inequality follows by Young's inequality, and the second follows by [AGS08, Lemma 8.1.10]. Rearranging, and using that $\mathrm{Ent}(\mu_\varepsilon^0) \leqslant \mathrm{Ent}(\mu^0)$, we have found that

$$\mathrm{Ent}(\mu_\varepsilon^t) + \frac{\sigma}{2}\int_0^t \int \left|\frac{\nabla\mu_\varepsilon^\tau}{\mu_\varepsilon^\tau}\right|^2 d\mu_\varepsilon^\tau\, d\tau \leqslant \mathrm{Ent}(\mu^0) + \frac{2}{\sigma}\int_0^t \int |b^\tau|^2\, d\mu^\tau\, d\tau.$$

As the entropy is lower semi-continuous, this implies that $\mathrm{Ent}(\mu^t) < \infty$ for all $t \in [0,T]$, thus μ has a density. As well, for all $\phi \in C^\infty([0,T]\times\mathbb{T}^d)$ the Cauchy-Schwarz inequality implies that

$$\int_0^T \int \Delta\phi^t d\mu_\varepsilon^t\, dt \leqslant \left(\int_0^T \int \left|\frac{\nabla\mu_\varepsilon^t}{\mu_\varepsilon^t}\right|^2 d\mu_\varepsilon^t\, d\tau\right)^{1/2} \left(\int_0^T \int |\nabla\phi^t|^2\, d\mu_\varepsilon^t\, dt\right)^{1/2}.$$

Using that $\mu_\varepsilon \to \mu$ in $C([0,T],\mathcal{P}(\mathbb{T}^d))$ and the uniform bound on the Fisher information of μ_ε, this in turn implies that

$$\int_0^T \int \Delta\phi^t d\mu^t\, dt \leqslant C\left(\int_0^T \int |\nabla\phi^t|^2\, d\mu^t\, dt\right)^{1/2}.$$

Accordingly, the Riesz representation theorem implies the existence of $E \in L^2([0,T], L^2(\mu^t))$ so that

$$\operatorname{div}(E^t\mu^t) = \Delta\mu^t$$

distributionally for almost every t. Lemma 8.3.1 in [AGS08] immediately implies that $\mu \in \mathcal{AC}^T$. □

We will now use the above proposition to show that if $Q^T(\mu) < \infty$ and $\sup_{\phi,s,t} S^{t,s}(\mu,\phi) < \infty$, then we can rewrite $\sup_{\phi,s,t} S^{t,s}(\mu,\phi)$ into a form depending on $\partial_t\mu$.

prop:variation **Proposition 4.3.** *Let $\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d))$ such that $Q^T(\mu) < \infty$ and*

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) < \infty.$$

Then there exists $b \in L^2([0,T],L^2(\mu^t))$ so that μ is a weak solution to

$$\partial_t \mu - \sigma\Delta\mu - \mathrm{div}\,(\mu\nabla \mathbf{g} * \mu) = -\mathrm{div}\,(b\mu). \tag{4.2}$$ eq:mve+b

More so $\mu \in \mathcal{AC}^T$ and

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \frac{1}{4\sigma}\int_0^T \int_{\mathbb{T}^d} |b^t|^2 d\mu^t dt = \frac{1}{4\sigma}\int_0^T \|\partial_t \mu^t - \sigma\Delta\mu^t - \mathrm{div}\,(\mu^t \nabla \mathbf{g} * \mu^t)\|_{-1,\mu^t}^2 dt. \tag{4.3}$$ eq:char

Proof. We begin by setting

$$L^{s,t}(\mu,\phi) := \langle \mu^t, \phi^t\rangle - \langle \mu^s,\phi^s\rangle - \int_s^t \langle \mu^\tau, \partial_t\phi^\tau\rangle + \langle \sigma\Delta\mu^\tau + \mathrm{div}\,(\mu^\tau\nabla\mathbf{g} * \mu^\tau), \phi^\tau\rangle\, d\tau.$$

so that

$$S^{s,t}(\mu,\phi) = L^{s,t}(\mu,\phi) - \sigma\int_s^t\int_{\mathbb{T}^d} |\nabla\phi^\tau|^2 d\mu^\tau d\tau.$$

The function $L^{s,t}(\mu,\phi)$ is thus a linear operator on $C^\infty([s,t]\times\mathbb{T}^d)$.

Using that

$$\phi \mapsto \sigma\int_s^t\int_{\mathbb{T}^d} |\nabla\phi^\tau|^2 d\mu^\tau d\tau$$

is quadratic, we find that

$$\frac{1}{4\sigma}\sup_{\phi,s,t} L^{s,t}(\mu,\phi)^2 \leqslant \sup_{\phi,s,t} S^{s,t}(\mu,\phi)\left(\int_s^t\int_{\mathbb{T}^d} |\nabla\phi^\tau|^2 d\mu^\tau d\tau\right).$$

Thus if $\sup_{\phi,s,t} S^{s,t}(\mu,\phi) < \infty$, then $L^{s,t}(\mu,\phi)$ is a bounded linear function on the Hilbert space

$$\mathcal{H}^{s,t} := \overline{\{\nabla\phi : \phi \in C([s,t]\times\mathbb{T}^d)\}}^{L^2([s,t],L^2(\mathbb{T}^d))},$$

for any s and t. The Riesz representation theorem implies the existence of $b_{s,t} \in \mathcal{H}^{s,t}$ such that for all $\phi \in C^\infty([s,t]\times\mathbb{T}^d)$,

$$L^{s,t}(\mu,\phi) = \int_s^t\int_{\mathbb{T}^d} b_{s,t}^\tau \cdot \nabla\phi^\tau d\mu^\tau d\tau.$$

Using an approximation argument, for $s \leqslant \varsigma \leqslant \tau \leqslant t$, it is easy to verify that $b_{s,t}$ and $b_{\varsigma,\tau}$ agree on $[\varsigma,\tau]$, hence we may take $b \in \mathcal{H}^{0,T}$ so that $b_{s,t} = b|_{[s,t]}$ for all $0 \leqslant s \leqslant t \leqslant T$. We have in fact found that μ weakly satisfies the partial differential equation

$$\partial_t\mu - \sigma\Delta\mu - \mathrm{div}\,(\mu\nabla\mathbf{g} * \mu) = -\mathrm{div}\,(b\mu).$$

Membership of μ in $\mathcal{AC}^T$ is then immediately implied by Proposition 4.2 and the fact that since $Q^T(\mu) < \infty$ there exists $E \in L^2([0,T],L^2(\mu^t))$ so that $\mathrm{div}\,(\mu^t\nabla\mathbf{g} * \mu^t) = \mathrm{div}\,(E^t\mu^t)$ for almost every t. This follows again by the Riesz representation theorem since for any $\phi \in C([0,T]\times\mathbb{T}^d)$

$$\int_0^T \langle \mathrm{div}\,(\mu^t\nabla\mathbf{g} * \mu^t), \phi\rangle\, dt \leqslant \left(\int_0^T\int |\nabla\phi^t|^2\, d\mu^t\, dt\right)^{1/2}\left(\int_0^T \|\mathrm{div}\,(\mu^t\nabla\mathbf{g} * \mu^t)\|_{-1,\mu^t}^2\, dt\right)^{1/2}.$$

Approximating b by elements of $\{\nabla\phi : \phi \in C^\infty([0,T],\mathbb{T}^d)\}$ we find that

$$\frac{1}{4\sigma}\int_0^T \int |b^\tau|^2 d\mu^\tau d\tau \leqslant \sup_{\phi,s,t} S^{s,t}(\mu,\phi)$$

while Hölder's inequality implies that

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \frac{1}{4\sigma}\sup_{\phi,s,t}\frac{L^{s,t}(\mu,\phi)^2}{\|\phi\|_{s,t}^2} \leqslant \frac{1}{4\sigma}\int_0^T \int |b^\tau|^2 d\mu^\tau d\tau$$

thus

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \frac{1}{4\sigma}\int_0^T \int |b^\tau|^2 d\mu^\tau d\tau.$$

Since $b^t \in \overline{\{\nabla\phi : \phi \in \mathcal{D}\}}^{L^2(\mu^t)}$ for almost every t,

$$\int_0^T \| - \operatorname{div}(b^t\mu^t)\|_{-1,\mu^t} = \int_0^T \int |b^t|^2 d\mu^t.$$

Together with the distributional definition of $\partial_t\mu^t$ this implies that

$$\int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g} * \mu^t)\|_{-1,\mu^t}^2 dt = \int_0^T \| - \operatorname{div}(b^t\mu^t)\|_{-1,\mu^t}^2 dt = \int_0^T \int_{\mathbb{T}^d} |b^t|^2 d\mu^t\, dt,$$

completing the desired claims. □

There is still some small awkwardness in our definition of I in that μ could be in $\mathcal{AC}^T$ but $\sup_{\phi,s,t} S^{s,t}(\mu,\phi)$ could still be infinite. The following proposition handles this, and proves that I is lower semi-continuous.

unction-equivalence **Corollary 4.4.** *For all $\mu \in C([0,T],\mathcal{P}(\mathbb{T}^d))$*

$$I(\mu) = \sup_{\phi,s,t} S^{s,t}(\mu,\phi) \vee \tfrac{1}{4\sigma}\big(Q^T(\mu) - \mathcal{E}(\mu^0)\big) + \infty \cdot \mathbf{1}_{\mu^0=\mu_0}, \qquad (4.4)$$ eq:rate-function-equ

and I is a good rate function.

Proof. Proposition 4.3 implies that if $\mu \notin \mathcal{AC}^T$, then

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \infty.$$

On the other hand, if $\mu \in \mathcal{AC}^T$ and $Q^T(\mu) < \infty$, then for any $\phi \in C^\infty([0,T],\mathbb{T}^d)$ and $0 \leqslant s \leqslant t \leqslant T$

$$\begin{aligned}
&\frac{1}{4\sigma}\int_s^t \|\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau)\|_{-1,\mu^\tau}^2\, d\tau\\
&\quad= \int_s^t \sup_{\varphi\in\mathcal{D}}\left\{\langle\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau),\varphi\rangle - \sigma\int |\nabla\varphi|^2 d\mu^\tau\right\}d\tau\\
&\quad\geqslant \int_s^t \langle\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau),\phi^\tau\rangle - \sigma\int |\nabla\phi^\tau|^2 d\mu^\tau\, d\tau\\
&\quad= S^{s,t}(\mu,\phi).
\end{aligned}$$

This implies that

$$\frac{1}{4\sigma}\int_0^T \|\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g}*\mu^\tau)\|_{-1,\mu^\tau}^2\,d\tau \geqslant \sup_{\phi,s,t} S^{s,t}(\mu,\phi),$$

thus

$$\sup_{\phi,s,t} S^{s,t}(\mu,\phi) = \left(\frac{1}{4\sigma}\int_0^T \|\partial_t\mu^\tau - \sigma\Delta\mu^\tau - \operatorname{div}(\mu^\tau\nabla\mathbf{g}*\mu^\tau)\|_{-1,\mu^\tau}^2\,d\tau\right)\mathbf{1}_{\mu\in\mathcal{AC}^T} + \infty\cdot\mathbf{1}_{\mu\notin\mathcal{AC}^T}.$$

This immediately implies the equivalence (4.4).

We now want to show that the sublevel sets of I are compact in $C([0,T],\mathcal{P}(\mathbb{T}^d))$. First we will show that they are closed, i.e. that I is lower semi-continuous. It suffices to prove that if μ_k are a sequence that converges to μ so that $I(\mu_k)$ is uniformly bounded by some $L>0$, then

$$\liminf_{k\to\infty} I(\mu_k) \geqslant I(\mu).$$

Using the lower semi-continuity of Q^T stated in (2.6), we have that

$$\liminf_{k\to\infty} Q^T(\mu_k) \geqslant Q^T(\mu).$$

Thus $\{\mu_k\}_{k\geqslant 1}\cup\{\mu\}$ is in the sublevel set $\{Q^T(\mu)\leqslant 4\sigma L+\mathcal{E}(\mu_0)\}$. Since

$$I(\mu_k) = \sup_{\phi,s,t} S^{s,t}(\mu_k,\phi) \vee \frac{1}{4\sigma}\big(Q^T(\mu_k) - \mathcal{E}(\mu^0)\big)$$

I is thus lower semi-continuous as long as

$$\liminf_{k\to\infty}\sup_{\phi,s,t} S^{s,t}(\mu_k,\phi) \geqslant \sup_{\phi,s,t} S^{s,t}(\mu,\phi).$$

This follows since $S^{s,t}(\phi,\mu)$ is continuous on the sublevel sets of Q^T by Corollary 2.4, thus

$$\mu \to \sup_{\phi,s,t} S^{s,t}(\mu,\phi)$$

is lower-semicontinuous as a function on the sublevel sets of Q^T.

To conclude the goodness of I, we only have to establish that the sublevel sets $\{I(\mu)\leqslant L\}$ are precompact. Using the construction of a compact set used in the proof of exponential tightness, it suffices to show that for all $\phi\in\mathcal{D}$ and $\varepsilon>0$ there exists $\delta>0$ so that

$$\{I(\mu)\leqslant L\}\subset\Big\{\mu : \sup_{0\leqslant t\leqslant s\leqslant T:|s-t|\leqslant\delta} |\langle\mu^t-\mu^s,\phi\rangle|\leqslant\varepsilon\Big\}.$$

This is easy to verify as if $I(\mu)\leqslant L$, then

$$\begin{aligned}
&|\langle\mu^t-\mu^s,\phi\rangle| \\
&\quad = \left|\int_s^t \langle\sigma\Delta\mu^\tau - \operatorname{div}(\mu\nabla\mathbf{g}*\mu^\tau) + \operatorname{div}(b^\tau\mu^\tau),\phi\rangle\,d\tau\right| \\
&\quad \leqslant \left|\int_s^t \langle\mu^\tau,\sigma\Delta\phi\rangle\,d\tau\right| + \left|\int_s^t \langle\operatorname{div}(\mu\nabla\mathbf{g}*\mu^\tau),\phi^\tau\rangle\,d\tau\right| + \left|\int_s^t\int \nabla\phi\cdot b^\tau\,d\mu^\tau\,d\tau\right|
\end{aligned}$$

We easily bound

$$\left|\int_s^t \langle \mu^\tau, \sigma\Delta\phi\rangle \, d\tau\right| \leqslant (t-s)\sigma\|\Delta\phi\|_{L^\infty},$$

while Cauchy-Schwarz implies that

$$\begin{aligned}\left|\int_s^t \langle \operatorname{div}(\mu\nabla\mathbf{g} * \mu^\tau), \phi^\tau\rangle \, d\tau\right| &\leqslant \left(\int_s^t \int |\nabla\phi^\tau|^2 \, d\mu^\tau \, d\tau\right)^{1/2} \left(\int_s^t \|\operatorname{div}(\mu^\tau\nabla\mathbf{g} * \mu^\tau)\|^2_{-1,\mu^\tau} \, d\tau\right)^{1/2} \\ &\leqslant (t-s)^{1/2}\|\nabla\phi\|_{L^\infty}\left(4\sigma(L+\mathcal{E}(\mu_0)\right)^{1/2}\end{aligned}$$

and

$$\left|\int_s^t \int \nabla\phi\cdot b^\tau \, d\mu^\tau \, d\tau\right| \leqslant \left(\int_s^t \int |\nabla\phi^\tau|^2 \, d\mu^\tau \, d\tau\right)^{1/2} \left(\int_s^t \int |b^\tau|^2 \, d\mu^\tau \, d\tau\right)^{1/2} \leqslant (t-s)^{1/2}\|\nabla\phi\|_{L^\infty}(4\sigma L)^{1/2}.$$

Thus if $(s-t)$ is taken to be sufficiently small with respect to $L, \mathcal{E}(\mu^0)$ and the norms of ϕ, it's guaranteed that $|\langle \mu^t - \mu^s, \phi\rangle| \leqslant \varepsilon$. □

We'll now prove one of the two inequalities which imply Theorem 1.3. This is a direct consequence of Proposition 4.1 and the fact that the modulated energy controls weak convergence.

al-ldp-upper-bound **Proposition 4.5.** *Suppose* $\mu \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$. *Then*

$$\lim_{\varepsilon\to\infty} \limsup_{N\to\infty} \log \mathbb{P}\left(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon\right) \leqslant -\left(I(\mu)\mathbf{1}_{(\mu-1)\in\mathscr{C}^T} + \infty\cdot\mathbf{1}_{(\mu-1)\notin\mathscr{C}^T}\right).$$

Proof. First we'll prove that if

$$\lim_{\varepsilon\to\infty} \limsup_{N\to\infty} \frac{1}{N}\log \mathbb{P}\left(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon\right) > -\infty$$

then $(\mu-1)\in\mathscr{C}^T$. Indeed if this is the case, then there must exist a sequence of trajectories $\underline{x}_{N_k} \in C([0,T], (\mathbb{T}^d)^{N_k})$ so that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} F_{N_k}(\underline{x}_{N_k}^t, \mu^t) = 0.$$

The asymptotic equivalence between $F_{N_k}(\underline{x}_{N_k}^t, \mu^t)$ and $\|\mu_{N_k,\vec{\mathsf{r}}}^t - \mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}}$ then guarantees that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \|\mu_{N_k,\vec{\mathsf{r}}}^t - \mu^t\|^2_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} = 0.$$

It is easy to verify that $(\mu_{N_k,\vec{\mathsf{r}}} - 1)$ is in $\mathscr{C}^T$, thus μ^t must be as well as the limit of continuous functions.

In particular, we only need to prove the proposition for μ so that $(\mu-1)\in\mathscr{C}^T$, which in turn implies that $\mu\in C([0,T],\mathcal{P}(\mathbb{T}^d))$. It thus suffices to prove that for all $\varepsilon>0$ there exists $\varepsilon'>0$ so that for all sufficiently large N

$$\left\{\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon'\right\} \subset \left\{\mu_N \in B_\varepsilon(\mu)\right\},$$

as then Proposition 4.1 and Corollary 4.4 complete the claim. Suppose this does not hold. Then we can find a sequence of trajectories and times $(\underline{x}_{N_k}, t_k)$ so that $\lim_{k\to\infty} t_k = t$, $d(\mu_{N_k}^{t_k}, \mu^{t_k}) \geqslant \varepsilon$ for all k and

$$\lim_{k\to\infty} F_N(\underline{x}_N^{t_k}, \mu^{t_k}) = 0.$$

First we claim that $d(\mu_{N_k}^{t_k}, \mu^t) \to 0$. Indeed, for any smooth $\phi \in \mathcal{D}$ we have that

$$\int \phi\, d(\mu_{N_k}^{t_k} - \mu^t) = \int \phi\, d(\mu_{N_k}^{t_k} - \mu^{t_k}) + \int \phi\, d(\mu^{t_k} - \mu^t).$$

By the continuity of μ in time, the second term on the right-hand side above goes to zero as $k \to \infty$. On the other hand, using identical bounds to as in Remark 2.10 we find that

$$\left| \int \phi\, d(\mu_{N_k}^{t_k} - \mu^{t_k}) \right| \lesssim_{d,\mathbf{s}} \left(\|\nabla \phi\|_{L^\infty(\mathbb{T}^d)} + \|\phi\|_{\dot{H}^{\frac{d-\mathbf{s}}{2}}(\mathbb{T}^d)} \right) \left(F_N(\underline{x}_{N_k}^{t_k}, \mu^{t_k}) + C\|\mu\|_{L^\infty}) N^{-\beta} \right)^{1/2},$$

thus the first term also converges to 0, thus $\mu_{N_k}^{t_k}$ converges to μ^t weakly. This gives us a contradiction since

$$d(\mu_{N_k}^{t_k}, \mu^t) \geqslant d(\mu_{N_k}^{t_k}, \mu^{t_k}) - d(\mu^{t_k}, \mu^t)$$

and

$$\liminf_{k\to\infty} d(\mu_{N_k}^{t_k}, \mu^{t_k}) - d(\mu^{t_k}, \mu^t) \geqslant \varepsilon.$$

□

Now that we have shown local LDP upper bounds and rewritten the rate function we can use the exponential tightness of μ_N to prove the upper bound in Theorem 1.1.

Proof of (1.9). Propositions 4.1 and Corollary 4.4 imply that for all $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d))$

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \leqslant -I(\mu).$$

Given a compact set $K \subset C([0,T], \mathcal{P}(\mathbb{T}^d))$ and any $\delta > 0$ we can thus cover K with a finite number of balls $B_{\varepsilon_k}(\mu_k)$ where $\mu_k \in K$ so that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_{\varepsilon_k}(\mu_k)\big) \leqslant -\Big(I(\mu) \wedge \tfrac{1}{\delta} - \delta\Big).$$

This implies that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in K\big) \leqslant -\Big(\inf_{\mu\in K} I(\mu) \wedge \tfrac{1}{\delta} - \delta\Big)$$

Thus taking $\delta \to 0$ implies that μ_N satisfy a weak LDP upper bound. As μ_N are also exponentially tight due to Proposition 3.4, this implies the strong LDP upper bound (1.9). □

5. Law of large numbers

sec:LLN

In this section we show that the empirical measures associated to solutions to (1.22) satisfy a mean-field limit when b and the solution to the McKean–Vlasov equation (4.2) are sufficiently regular. The argument is very similar to as in [RS23], but we will make use of an emergent quadratic variation term to give exponential bounds on the path.

Formally, by applying Itô's formula to $F_N(\underline{x}_N^t, \mu^t)$ when $\underline{x}_N$ is a solution to (1.22) and μ^t is a solution to (4.2), we compute that there exists some martingale M^t so that

$$\begin{aligned}
&M^t - \frac{N}{8\sigma}\langle M^t\rangle + F_N(\underline{x}_N^0, \mu^0) \\
&\quad = F_N(\underline{x}_N^t, \mu^t) + \int_0^t \iint_{(\mathbb{T}^d)^2\setminus\Delta} (-\Delta)\mathbf{g}(x-y)d(\mu_N^\tau - \mu^\tau)^{\otimes 2}\, d\tau \\
&\qquad + \int_0^t \iint_{(\mathbb{T}^d)^2\setminus\Delta} \big((\nabla\mathbf{g} * \mu^\tau - b^\tau)(x) - (\nabla\mathbf{g} * \mu^\tau - b^\tau)(y)\big)\cdot\nabla\mathbf{g}(x-y)d(\mu_N^\tau - \mu^\tau)^{\otimes 2}\, d\tau.
\end{aligned}$$

Since $(-\Delta)\mathbf{g}$ is also a Riesz-potential, the second term on the right-hand side has the form of a modulated energy, so is asymptotically positive. The third term is exactly in the form of (2.15), so we have the bound

$$\begin{aligned}
&\int_0^t \iint_{(\mathbb{T}^d)^2\setminus\Delta} \big((\nabla\mathbf{g} * \mu^\tau - b^\tau)(x) - (\nabla\mathbf{g} * \mu^\tau - b^\tau)(y)\big)\cdot\nabla\mathbf{g}(x-y)d(\mu_N^\tau - \mu^\tau)^{\otimes 2}\, d\tau \\
&\qquad \lesssim_{d,\mathbf{s}} \int_0^t C(\tau)\big(F_N(\underline{x}_N^\tau, \mu^\tau) + C\|\mu\|_{L^\infty} N^{-\beta}\big)\, d\tau
\end{aligned}$$

where $C(\tau)$ depends on the derivatives of $\nabla\mathbf{g} * \mu^\tau$ and b^τ. The conditions we require for μ and b guarantee that $C(\tau)$ is L^1 in time.

Putting the above together, we have that

$$M^t - \frac{\lambda}{N}\langle M^t\rangle + F_N(\underline{x}_N^0, \mu^0) + o_N(1) \geqslant |F_N(\underline{x}_N^t, \mu^t)| - \int_0^t C(\tau)|F_N(\underline{x}_N^\tau, \mu^\tau)|\, d\tau,$$

thus Lemma 3.1 should control the probability the right-hand side is ever large. The contrapositive of Grönwall's inequality says that if

$$|F_N(\underline{x}_N^t, \mu^t)| > \varepsilon$$

for some t then it must be the case that

$$|F_N(\underline{x}_N^t, \mu^t)| - \int_0^T C(\tau)|F_N(\underline{x}_N^\tau, \mu^\tau)|\,\tau \geqslant \varepsilon e^{-\int_0^t C(\tau)\, d\tau},$$

for some t. We thus find using Lemma 3.1 that

$$\mathbb{P}\bigg(\sup_{t\in[0,T]} |F_N(\underline{x}_N^t, \mu^t)| > \varepsilon\bigg) \leqslant \exp(-N(C^{-1}\varepsilon - CF_N(\underline{x}_N^0, \mu^0) - o_1(N)),$$

for some constant C depending on λ, μ, and b.

To make this argument rigorous we need to justify our use of Itô's formula. We will use the same truncated process $\underline{x}_{N,\delta}$ as defined in Proposition 3.2, and analogously define the truncated modulated energy

$$F_{N,\delta}(\underline{x}_N,\mu) := \iint_{(\mathbb{T}^d)^2\setminus\Delta} \mathbf{g}_{(\delta)}(x-y)\, d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y),$$

Before proceeding, the following proposition justifies the fact that the Itô correction term is almost positive.

prop:laplowerbound **Proposition 5.1.** *For $0 \leqslant \mathbf{s} < d-2$ there exists a constant $\beta > 0$ so that for every choice of $\underline{x}_N \in (\mathbb{T}^d)^N$ pairwise distinct, $\mu \in \mathcal{P}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$*

$$-\|\mu\|_{L^\infty} N^{-\beta} \lesssim_{d,\mathbf{s}} \iint_{(\mathbb{T}^d)^2\setminus\Delta} (-\Delta \mathbf{g})(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y).$$

Proof. When $0 \leqslant \mathbf{s} < d-4$ this is an immediate consequence of (2.20). When $d-4 \leqslant \mathbf{s} < d-2$, it is a consequence of Proposition 5.6. in [dCRS23]. $\square$

The following lemma is the consequence of applying Ito's formula to $F_{N,\delta}(\underline{x}_N^t, \mu^t)$ and rearranging appropriately.

Lemma 5.2. *Let $b \in L^2([0,T], C^1(\mathbb{T}^d))$, $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d)) \cap L^\infty([0,T], L^\infty(\mathbb{T}^d))$ be a weak solution to (4.2), and $\underline{x}_{N,\delta}$ be the solution to (3.1). Then*

$$\begin{aligned}
&F_{N,\delta}(\underline{x}_{N,\delta}^t, \mu^t) - F_{N,\delta}(\underline{x}_{N,\delta}^0, \mu^0) \\
&= -\frac{2}{N}\sum_{i=1}^N \int_0^t \Big| \int_{\mathbb{T}^d\setminus\{x_{i,\delta}^\tau\}} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - y)\, d(\mu_{N,\delta}^\tau - \mu^\tau)(y)\Big|^2 d\tau \\
&\quad + \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} \big((u_\delta^\tau + b^\tau)(x) - (u_\delta^\tau + b^\tau)(y)\big)\cdot \nabla \mathbf{g}_{(\delta)}(x-y) d(\mu_{N,\delta}^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau \\
&\quad + 2\sigma \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} \Delta \mathbf{g}_{(\delta)}(x-y) d(\mu_{N,\delta}^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau \\
&\quad + \frac{2\sqrt{2\sigma}}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{T}^d\setminus\{x_{i,\delta}^\tau\}} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - y), d(\mu_{N,\delta}^\tau - \mu^\tau)(y)\cdot dw_i^t \\
&\quad + 2\int_0^t \int_{\mathbb{T}^d} \mathbf{g}_{(\delta)} * \operatorname{div}\big((u^\tau - u_\delta^\tau)\mu^\tau\big)\, d(\mu_{N,\delta}^\tau - \mu^\tau)\, d\tau,
\end{aligned} \tag{5.1}$$
eq:modulated-energ

where $\mu_{N,\delta}$ is the empirical measure associated to $\underline{x}_{N,\delta}$, $u^t := -\nabla \mathbf{g} * \mu^t$ and $u_\delta^t := \nabla \mathbf{g}_{(\delta)} * \mu^t$.

Proof. The proof follows identically to that of Lemma 6.1 and Lemma 6.2 in [RS23]. The only difference are the additional terms which appear due to the drift b. Splitting

$$\begin{aligned}
F_N(\underline{x}_{N,\delta}^t, \mu) &= \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \mathbf{g}_{(\delta)}(x_i - x_j) - \frac{2}{N}\sum_{i=1}^N \mathbf{g}_{(\delta)} * \mu(x_i) + \iint \mathbf{g}_{(\delta)}(x-y) d\mu(x) d\mu(y) \\
&=: \mathrm{Term}_1 + \mathrm{Term}_2 + \mathrm{Term}_3,
\end{aligned}$$

then the drift b contributes the following additional components to each term in the Itô/differential expansion.

$$\text{Term}_1 : \frac{1}{N^2} \sum_{1 \leqslant i \neq j \leqslant N} \int_0^t \nabla \mathbf{g}_{(\delta)}((x_{i,\delta}^\tau - x_{j,\delta}^\tau) \cdot (b(x_{i,\delta}^\tau) - b(x_{j,\delta}^\tau))\, d\tau,$$

$$\text{Term}_2 : -\frac{2}{N} \int_0^t \int_{\mathbb{T}^d} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - y) \cdot b^\tau(y)\, d\mu^\tau(y)\, d\tau - \frac{2}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^d} \nabla \mathbf{g}(x_{i,\delta}^\tau - y) \cdot b(x_{i,\delta}^\tau)\, d\mu^\tau(y)\, d\tau,$$

$$\text{Term}_3 : 2 \int_0^t \int \nabla \mathbf{g}_{(\delta)} * \mu^\tau \cdot b^\tau\, d\mu^\tau.$$

These can be easily rearranged to

$$\int_0^t \iint_{(\mathbb{T}^d)^2 \setminus \Delta} (b^\tau(x) - b^\tau(y)) \cdot \nabla \mathbf{g}_{(\delta)}(x-y) d(\mu_{N,\delta}^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau, \tag{5.2}$$ eq:bterm

which completes the claim. □

With the above Itô expansion for $F_{N,\delta}(\underline{x}_N^\tau, \mu^\tau)$ we can now make the argument sketch given at the beginning of this section rigorous.

prop:LLN **Proposition 5.3.** *Suppose that $\mu \in C([0,T], \mathcal{P}(\mathbb{T}^d)) \cap L^\infty([0,T], L^\infty(\mathbb{T}^d))$ is a weak solution to (4.2) with*

$$C_b := \int_0^T \|\nabla b^t\|_{L^\infty(\mathbb{T}^d)}^2 + \||\nabla|^{\frac{d-\mathbf{s}}{2}} b^t\|^2_{L^{\frac{2d}{d-2-\mathbf{s}}}(\mathbb{T}^d)} dt < \infty,$$

and $\underline{x}_N^t$ is the solution to (1.22). Then there exists a constant C depending on $d, \mathbf{s}, b$ and μ so that

$$\mathbb{P}\Big(\sup_{t \in [0,T]} F_N(\underline{x}_N^t, \mu^t) > \varepsilon \Big) \leqslant \exp\Big(-N\big(C^{-1}\varepsilon - F_N(\underline{x}_N^0, \mu^0) - N^{-\beta}\big)\Big). \tag{5.3}$$

Proof. First we will show that the last term in (5.1) converges uniformly to 0 as $\delta \to 0$. For any $x \in \mathbb{T}^d$ we can bound

$$|\mathbf{g}_{(\delta)} * \operatorname{div}((u^\tau - u_\delta^\tau)\mu^\tau)|(x) \leqslant \|\nabla \mathbf{g}_{(\delta)}\|_{L^1} \|(u^\tau - u_\delta^\tau)\mu^\tau\|_{L^\infty} \lesssim_{d,\mathbf{s}} \|\mu^\tau\|_{L^\infty} \|u^\tau - u_\delta^\tau\|_{L^\infty}.$$

Using that $\mathbf{g}_{(\delta)} = (1 - \chi(\cdot/\delta))\mathbf{g}(x)$ we find that for any $x \in \mathbb{T}^d$

$$|u^\tau - u_\delta^\tau|(x) \leqslant \|\mu\|_{L^\infty} \int |\nabla(\mathbf{g} - \mathbf{g}_{(\delta)})|(x-y) dy \lesssim_\chi \|\mu\|_{L^\infty} \int_{B_\delta(0)} (|\nabla \mathbf{g}| + \delta^{-1}|\mathbf{g}|) dy \lesssim_{d,\mathbf{s},\chi} \|\mu\|_{L^\infty} \delta^{d-1-\mathbf{s}},$$

where the last line follows due to (2.13). Together these show that

$$|\mathbf{g}_{(\delta)} * \operatorname{div}((u^\tau - u_\delta^\tau)\mu^\tau)|(x) \lesssim_{d,\mathbf{s},\chi} \|\mu^\tau\|_{L^\infty}^2 \delta^{d-1-\mathbf{s}},$$

thus

$$\left| 2 \int_0^t \int_{\mathbb{T}^d} \mathbf{g}_{(\delta)} * \operatorname{div}\big((u^\tau - u_\delta^\tau)\mu^\tau\big)\, d(\mu_{N,\delta}^\tau - \mu^\tau)\, d\tau \right| \lesssim_{d,\mathbf{s},T,\chi} \|\mu\|_{L^\infty}^2 \delta^{d-1-\mathbf{s}}. \tag{5.4}$$ eq:mollification-erro

We will once again take advantage of the fact that the term involving the Brownian motion

$$M^t := \frac{2\sqrt{2\sigma}}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^d \setminus \{x_{i,\delta}^\tau\}} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - y) d(\mu_{N,\delta}^\tau - \mu^\tau)(y) \cdot dw_i^\tau$$

is a continuous martingale with quadratic variation

$$\langle M^t\rangle = \frac{8\sigma}{N^2}\sum_{i=1}^N \int_0^t \left| \int_{\mathbb{T}^d\setminus\{x_{i,\delta}^\tau\}} \nabla \mathbf{g}_{(\delta)}(x_{i,\delta}^\tau - y) d(\mu_{N,\delta}^\tau - \mu^\tau)(y)\right|^2 d\tau.$$

Letting $\lambda_N = \frac{N}{4\sigma}$ and rearranging (5.1) we have that there exists a constant C so that

$$\begin{aligned}\mathcal{F}_{N,\delta}^t :&= F_{N,\delta}(\underline{x}_{N,\delta}^t, \mu^t) - 2\sigma \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} \Delta \mathbf{g}_{(\delta)}(x-y) d(\mu_{N,\delta}^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau \\ &\quad - \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} (v_\delta^\tau(x) - v_\delta^\tau(y)) \cdot \nabla \mathbf{g}_{(\delta)}(x-y) d(\mu_{N,\delta}^\tau - \mu^\tau)^{\otimes 2}(x,y), d\tau \\ &\leqslant M^t - \frac{\lambda_N}{2}\langle M^t\rangle + F_{N,\delta}(\underline{x}_N^0, \mu^0) + C\delta^{d-1-\mathbf{s}},\end{aligned}$$

where the final term in the second line above comes from the bound (5.4).

Since $\exp(\lambda_N M^t - \frac{\lambda_N^2}{2}\langle M^t\rangle)$ is a continuous martingale, Lemma 3.1 implies that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \mathcal{F}_{N,\delta}^t > \varepsilon\Big) \leqslant \exp\Big(-\frac{N}{4\sigma}\big(\varepsilon - C\delta^{d-1-\mathbf{s}} - F_{N,\delta}(\underline{x}_N^0, \mu^0)\big)\Big). \tag{5.5}$$ eq:molexpdoobbour

We will show that

$$\lim_{\delta\to\infty} \sup_{t\in[0,T]} \mathcal{F}_{N,\delta}^t = \sup_{t\in[0,T]} \mathcal{F}_N^t, \quad \mathbb{P}-\text{a.s.} \tag{5.6}$$ eq:truncated-path-c

where

$$\begin{aligned}\mathcal{F}_N^t := F_N(\underline{x}_N^t, \mu^t) &- \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} \big((u^\tau + b^\tau)(x) - (u^\tau + b^\tau)(y)\big) \cdot \nabla \mathbf{g}(x-y) d(\mu_N^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau \\ &- 2\sigma \int_0^t \int_{(\mathbb{T}^d)^2\setminus\Delta} \Delta \mathbf{g}(x-y) d(\mu_N^\tau - \mu^\tau)^{\otimes 2}(x,y)\, d\tau.\end{aligned}$$

In Proposition 3.5 we showed that there exists a stopping time $\tau_{N,\delta}$ so that $\lim_{\delta\to\infty}\tau_{N,\delta} = \infty$ almost surely and

$$\underline{x}_{N,\delta} = \underline{x}_N, \qquad \min_{1\leqslant i\neq j\leqslant N} |x_i^t - x_j^t| \geqslant 2\delta,$$

when $0 < t < \tau_{N,\delta}$.

Thus if $\tau_{N,\delta} > T$, expanding out the definition of F_N and $F_{N,\delta}$ for all $t \in [0,T]$

$$\begin{aligned}\Big|F_{N,\delta}(\underline{x}_N^t, \mu^t) - F_N(\underline{x}_N^t, \mu^t)\Big| &= \Big|\int_{(\mathbb{T}^d)^2} (\mathbf{g}_{(\delta)} - \mathbf{g})(x-y)\, d\mu(x)\, d(\mu_N^t - \mu) dt\Big| \\ &\leqslant 2\|\mu\|_{L^\infty} \int_{\mathbb{T}^d} |\mathbf{g}_{(\delta)} - \mathbf{g}|(x) dx \\ &\lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty}\delta^{d-\mathbf{s}},\end{aligned} \tag{5.7}$$ eq:molmodcon

where the last line also follows by (2.13). Similarly we can bound the Itô correction term as follows

$$\begin{aligned}&\Big|\int_{(\mathbb{T}^d)^2\setminus\Delta} \Delta(\mathbf{g}_{(\delta)} - \mathbf{g})(x-y)\, d(\mu_N^t - \mu^t)^{\otimes 2}(x,y)\Big| \\ &= \Big|\int_{(\mathbb{T}^d)^2\setminus\Delta} \Delta(\mathbf{g}_{(\delta)} - \mathbf{g})(x-y)\, d\mu(x)\, d(2\mu_N^t - \mu^t)(y)\Big| \\ &\lesssim \|\mu\|_{L^\infty} \int_{\mathbb{T}^d} |\Delta(\mathbf{g}_{(\delta)} - \mathbf{g})| dx \\ &\lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty}\delta^{d-2-\mathbf{s}}.\end{aligned} \tag{5.8}$$ eq:molitocon

Next we see that letting $v^t := u^t + b^t$ and $v^t_\delta := u^t_\delta + b^t$

$$\begin{aligned}&\int_{(\mathbb{T}^d)^2\setminus\Delta} \big(v^t_\delta(x) - v^t_\delta(y)\big)\cdot\nabla\mathbf{g}_{(\delta)}(x-y)\,d(\mu^t_N)^{\otimes 2}(x,y)\\ &\qquad= \int_{(\mathbb{T}^d)^2\setminus\Delta} \big(v^t_\delta(x) - v^t_\delta(y)\big)\cdot\nabla\mathbf{g}(x-y)\,d(\mu^t_N)^{\otimes 2}(x,y).\end{aligned}$$

Using that $|x||\nabla\mathbf{g}(x)| \lesssim_{d,\mathbf{s}} \mathbf{g}(x) + C$ and the mean value theorem we find that

$$\begin{aligned}&\bigg|\int_{(\mathbb{T}^d)^2\setminus\Delta} \big(v^t_\delta(x) - v^t_\delta(y)\big)\cdot\nabla\mathbf{g}_{(\delta)}(x-y)\,d(\mu^t_N)^{\otimes 2}(x,y)\\ &\quad - \int_{(\mathbb{T}^d)^2\setminus\Delta} \big(v^t(x) - v^t(y)\big)\cdot\nabla\mathbf{g}(x-y)\,d(\mu^t_N)^{\otimes 2}(x,y)\bigg|\\ &\quad\leqslant \int_{(\mathbb{T}^d)^2\setminus\Delta} \big|(u^t_\delta - u^t)(x) - (u^t_\delta - u^t)(y)\big|\cdot|\nabla\mathbf{g}(x-y)|\,d(\mu^t_{N,\delta})^{\otimes 2}(x,y)\\ &\quad\lesssim_{d,\mathbf{s}} \|\nabla(u^t_\delta - u^t)\|_{L^\infty}(H_N(\underline{x}^t_N) + C)\\ &\quad\leqslant \|\mu\|_{L^\infty}(H_N(\underline{x}^t_N) + C)\delta^{d-2-\mathbf{s}}. \end{aligned} \tag{5.9}$$

eq:molcomdif

The last line of the above follows from the bound

$$\|\nabla(u^t_\delta - u^t)\|_{L^\infty} \leqslant \|\mu\|_{L^\infty}\int|\nabla^2(\mathbf{g}-\mathbf{g}_{(\delta)})|dx \leqslant \|\mu\|_{L^\infty}\delta^{d-2-\mathbf{s}}.$$

On the other hand using the triangle inequality we can bound

$$\begin{aligned}&\bigg|\int_{(\mathbb{T}^d)^2} \big(v^t_\delta(x) - v^t_\delta(y)\big)\cdot\nabla\mathbf{g}_{(\delta)}(x-y)\,d\mu^t(x)\,d(2\mu^t_N - \mu^t)(y)\\ &\quad - \int_{(\mathbb{T}^d)^2} \big(v^t(x) - v^t(y)\big)\cdot\nabla\mathbf{g}(x-y)\,d\mu^t(x)\,d(2\mu^t_N - \mu^t)(y)\bigg|\\ &\quad\leqslant \bigg|\int_{(\mathbb{T}^d)^2} \big(v^t_\delta(x) - v^t_\delta(y)\big)\cdot\nabla(\mathbf{g}_{(\delta)}-\mathbf{g})(x-y)\,d\mu^t(x)\,d(2\mu^t_N - \mu^t)(y)\bigg|\\ &\qquad + \bigg|\int_{(\mathbb{T}^d)^2} \big((u^t_\delta - u)(x) - (u^t_\delta - u)^t(y)\big)\cdot\nabla\mathbf{g}(x-y)\,d\mu^t(x)\,d(2\mu^t_N - \mu^t)(y)\bigg|. \end{aligned} \tag{5.10}$$

eq:molcomdiftriang

The first term on the left-hand side of the above is easily bounded by

$$2\|v^t\|_{L^\infty}\|\mu\|_{L^\infty}\int|\nabla(\mathbf{g}_{(\delta)}-\mathbf{g})|dx \lesssim_{d,\mathbf{s},\chi} (\|\mu\|_{L^\infty} + \|b^t\|_{L^\infty})\|\mu\|_{L^\infty}\delta^{d-1-\mathbf{s}}. \tag{5.11}$$

eq:molcomdif2

We can similarly bound the second term of (5.10) by

$$2\|\nabla\mathbf{g}\|_{L^1}\|\mu\|_{L^\infty}\|u^t_\delta - u^t\|_{L^\infty} \lesssim_{d,\mathbf{s},\chi} \|\mu\|_{L^\infty}\delta^{d-1-\mathbf{s}}. \tag{5.12}$$

eq:molcomdif3

Combining (5.7)-(5.12) and that $\lim_{\delta\to 0}\tau_\delta \geqslant T$ and $\sup_{t\in[0,T]} H_N(\underline{x}^t_N) < \infty$ almost surely, and using the time integrability of b we find that (5.6) holds.

The convergence (5.6), (5.7) with (5.5) immediately imply that

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\mathcal{F}^t_N > \varepsilon\Big) \leqslant \exp\Big(-\frac{N}{4\sigma}\big(\varepsilon - F_N(\underline{x}^0_N,\mu^0)\big)\Big). \tag{5.13}$$

eq:doob-bound-afte

Equation (2.20) implies that

$$F_N(\underline{x}_N^t,\mu^t) \geqslant |F_N(\underline{x}_N^t,\mu^t)| - C\|\mu\|_{L^\infty}N^{-\beta}, \tag{5.14}$$

eq:positivity2

while Proposition 2.15 gives that

$$\begin{aligned}&\left|\int_{(\mathbb{T}^d)^2\setminus\Delta}(v^\tau(x)-v^\tau(y))\cdot\nabla\mathbf{g}(x-y)d(\mu_N^\tau-\mu^\tau)^{\otimes 2}(x,y)\right|\\ &\qquad\lesssim_{d,\mathbf{s}} C_{v^\tau}\left(|F_N(\underline{x}_N^\tau,\mu^\tau)| + C\|\mu\|_{L^\infty}N^{-\beta}\right).\end{aligned} \tag{5.15}$$

eq:rencombound

where $C_{v^\tau} := \|\nabla v^\tau\|_{L^\infty} + \||\nabla|^{\frac{d-\mathbf{s}}{2}}v^\tau\|_{L^{\frac{2d}{d-\mathbf{s}-2}}}$. Using the definition of $\mathcal{F}_N^t$, (5.14), (5.15) and Proposition 5.1 imply that

$$|F_N(\underline{x}_N^t,\mu^t)| - \int_0^t C_{v^\tau}|F_N(\underline{x}_N^\tau,\mu^\tau)|\,d\tau \leqslant \mathcal{F}_N^t + C(1+T)\|\mu\|_{L^\infty}N^{-\beta} \tag{5.16}$$

eq:modulated-absol

We note that

$$\|\nabla u^\tau\|_{L^\infty} + \||\nabla|^{\frac{d-\mathbf{s}}{2}}u^\tau\|_{L^{\frac{2d}{d-\mathbf{s}-2}}} \lesssim_{d,\mathbf{s}} \|\mu\|_{L^\infty}, \tag{5.17}$$

where we bound the first term using Young's inequality and that $\nabla^2\mathbf{g}$ is integrable and for the second we use fourier multipliers. This with our conditions on b imply that

$$\int_0^T C_{v^\tau}\,d\tau := C < \infty,$$

for C depending on $d,\mathbf{s},b$ and μ. Grönwall's inequality implies that if

$$\sup_{t\in[0,T]}|F_N(\underline{x}_N^t,\mu^t)| - \int_0^t C_{v^\tau}|F_N(\underline{x}_N^\tau,\mu^\tau)|\,d\tau \leqslant \epsilon e^{-C} \Rightarrow \sup_{t\in[0,T]}|F_N(\underline{x}_N^t,\mu^t)| \leqslant \varepsilon$$

The contrapositive of this with (5.16) imply that

$$\sup_{t\in[0,T]}|F_N(\underline{x}_N^t,\mu^t)| > \varepsilon \Rightarrow \mathcal{F}_N^t + C(1+T)\|\mu\|_{L^\infty}N^{-\beta} > \varepsilon e^{-C}.$$

Combining this with (5.13) concludes the proposition. □

Remark 5.4. Proposition 5.3 actually implies that if $F_N(\underline{x}_N^0,\mu^0)\to 0$, then

$$\sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)\to 0$$

almost surely. This says a strong pathwise law of large numbers holds for μ_N with respect to the modulated energy.

6. Lower bound

sec:lower-bound

We'll now use the mean-field limit from Section 5 to prove the LDP lower bounds. This proceeds in three steps.

1. We show that if μ satisfies the conditions of Proposition 5.3, then

$$\lim_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right)\geqslant -\frac{1}{4\sigma}\int_0^T\int|b^t|^2\,d\mu^t\,dt.$$

2. We show that for any $\mu\in\mathscr{A}$ we can construct a sequence μ_k converging to μ which satisfy the conditions of Proposition 5.3 with respect to a drift b_k so that

$$\limsup_{k\to\infty}\int_0^T\int|b_k^t|^2\,d\mu_k^t\,dt\leqslant\int_0^T\int|b^t|^2\,d\mu^t\,dt.$$

3. We use the first two points to complete the claimed LDP lower bounds.

We begin with the proof of the first point.

prop:lower-bound **Proposition 6.1.** *Suppose that $\underline{x}_N$ solves (1.1) with initial conditions satisfying (1.8) and (μ,b) solve (4.2) and the conditions of Proposition 5.3 with $\mu|_{t=0}=\mu_0$. Then*

$$\lim_{\varepsilon\to\infty}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(\sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right)\geqslant -\frac{1}{4\sigma}\int_0^T\int_{\mathbb{T}^d}|b^t|^2\,d\mu^t\,dt. \tag{6.1}$$ eq:LB

Proof. We will use the change of measure

$$\frac{d\mathbb{P}}{d\mathbb{P}_b}:=\exp\left(\frac{1}{\sqrt{2\sigma}}\sum_{i=1}^N\int_0^T b^t(x_i^t)dw_i^t-\frac{1}{4\sigma}\sum_{i=1}^N\int_0^T|b^t(x_i^t)|^2dt\right).$$

Our conditions on b ensure that $b(x_i^t)$ satisfy Novikov's condition, thus we can use the Girsanov theorem to see that $\underline{x}_N^t$ is a solution to (1.22) under $\mathbb{P}_b$ [KS98].

Letting $A_{N,\varepsilon}:=\{\sup_{t\in[0,T]}F_N(\underline{x}_N,\mu)<\varepsilon\}$ we have that

$$\mathbb{P}\left(\sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right)=\mathbb{E}_b\left[\mathbf{1}_{A_{N,\varepsilon}}\frac{d\mathbb{P}}{d\mathbb{P}_b}\right]=\mathbb{P}_b(A_{N,\varepsilon})\mathbb{E}_b\left[\frac{\mathbf{1}_{A_{N,\varepsilon}}}{\mathbb{P}_b(A_{N,\varepsilon})}\frac{d\mathbb{P}}{d\mathbb{P}_b}\right].$$

Jensen's inequality thus implies that

$$\frac{1}{N}\log\mathbb{P}\left(\sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right)\geqslant\frac{1}{N}\log\mathbb{P}_b(A_{N,\varepsilon})+\frac{1}{\mathbb{P}_b(A_{N,\varepsilon})}\mathbb{E}_b\left[N^{-1}\log\frac{d\mathbb{P}}{d\mathbb{P}_b}\mathbf{1}_{A_{N,\varepsilon}}\right]. \tag{6.2}$$ eq:jensen

Proposition 5.3 implies that $\lim_{N\to\infty}\mathbb{P}_b(A_{N,\varepsilon})=1$, thus the first term on the right-hand side of (6.2) converges to 0, while the denominator in the second term converges to 1. It thus suffices to show that

$$\lim_{\varepsilon\to\infty}\liminf_{N\to\infty}\mathbb{E}_b\left[\frac{1}{N}\log\frac{d\mathbb{P}}{d\mathbb{P}_b}\mathbf{1}_{A_{N,\varepsilon}}\right]\geqslant-\frac{1}{4\sigma}\int_0^T\int|b^t|^2\,d\mu^t\,dt, \tag{6.3}$$ eq:intermediate-low

to conclude.

Using the definition of $\frac{d\mathbb{P}}{d\mathbb{P}_b}$ and $A_{N,\varepsilon}$ we have that

$$\begin{aligned}\mathbb{E}_b\left[\frac{1}{N}\log\frac{d\mathbb{P}}{d\mathbb{P}_b}\mathbf{1}_{A_{N,\varepsilon}}\right]&=\mathbb{E}_b\left[\frac{1}{N\sqrt{2\sigma}}\sum_{i=1}^N\int_0^T b^t(x_i^t)\cdot dw_i^t\ ;\ \sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right]\\&\quad-\mathbb{E}_b\left[\frac{1}{4\sigma}\int_0^T\int_{\mathbb{R}^d}|b^t(x)|^2\,d\mu_N(x)\,dt\ ;\ \sup_{t\in[0,T]}F_N(\underline{x}_N^t,\mu^t)<\varepsilon\right].\end{aligned}$$

Hölder's inequality with the Itô isometry imply that

$$\mathbb{E}_b\left[\frac{1}{N\sqrt{2\sigma}}\sum_{i=1}^{N}\int_0^T b^t(x_i^t)\cdot dw_i^t \ ; \sup_{t\in[0,T]} F(\underline{x}_N^t,\mu^t)<\varepsilon\right]\leqslant \frac{1}{\sqrt{2N\sigma}}\left(\int_0^T \|b^t\|^2_{L^\infty(\mathbb{T}^d)}\,dt\right)^{1/2},$$

the right-hand side of which goes to 0 as $N\to\infty$. On the other hand

$$\begin{aligned}
&\left|\mathbb{E}_b\left[\frac{1}{4\sigma}\int_0^T\int_{\mathbb{R}^d}|b^t(x)|^2\,d\mu_N(x)\,dt \ ; \sup_{t\in[0,T]}F(\underline{x}_N^t,\mu^t)<\varepsilon\right]-\frac{1}{4\sigma}\int_0^t\int_{\mathbb{T}^d}|b^t|^2\,d\mu^t\,dt\right|\\
&\leqslant \mathbb{E}_b\left[\frac{1}{4\sigma}\int_0^T\left|\int_{\mathbb{T}^d}|b^t(x)|^2\,d(\mu_N^t-\mu^t)(x)\right|dt \ ; \sup_{t\in[0,T]}F(\underline{x}_N^t,\mu^t)<\varepsilon\right]\\
&\quad+\mathbb{P}_b\left(\sup_{t\in[0,T]}F(\underline{x}_N^t,\mu^t)\geqslant\varepsilon\right)\frac{1}{4\sigma}\int_0^t\int_{\mathbb{T}^d}|b^t|^2\,d\mu^t\,dt.
\end{aligned}$$

The second term on the right-hand side above goes to zero as $N\to\infty$ again using Proposition 5.3. Using the computation in Remark 2.10

$$\begin{aligned}
&\mathbb{E}_b\left[\frac{1}{4\sigma}\int_0^T\left|\int_{\mathbb{T}^d}|b^t(x)|^2 d(\mu_N^t-\mu^t)(x)\right|dt \ ; \sup_{t\in[0,T]}F(\underline{x}_N^t,\mu^t)<\varepsilon\right]\\
&\lesssim_{d,\mathsf{s}}\frac{1}{4\sigma}\int_0^T\left(\|\nabla|b^t|^2\|_{L^\infty}+\||b^t|^2\|_{\dot H^{\frac{d-\mathsf{s}}{2}}(\mathbb{T}^d)}\right)\left(\varepsilon+C\|\mu\|_{L^\infty(\mathbb{T}^d)}N^{-\beta}\right)dt.
\end{aligned}$$

Clearly $\|\nabla|b^t|^2\|_{L^\infty}\leqslant\|b^t\|^2_{C^1}$ while the fractional Leibniz rule implies that

$$\||b^t|^2\|_{\dot H^{\frac{d-\mathsf{s}}{2}}(\mathbb{T}^d)}\leqslant\|b^t\|_{C^1(\mathbb{T}^d)}\|b^t\|_{\dot H^{\frac{d-\mathsf{s}}{2}}(\mathbb{T}^d)}.$$

As $\frac{2d}{d-2-\mathsf{s}}>2$ we can further bound $\|b^t\|_{\dot H^{\frac{d-\mathsf{s}}{2}}(\mathbb{T}^d)}\leqslant\||\nabla|^{\frac{d-\mathsf{s}}{2}}b^t\|_{L^{\frac{2d}{d-2-\mathsf{s}}}}$. Using our conditions on b and taking $N\to\infty$ and then $\varepsilon\to\infty$ this proves (6.3), and thus the claimed lower bound. $\square$

To expand this lower bound to $\mu\in\mathscr{A}$ we need to approximate by regular measure trajectories in way that is well behaved with respect to the rate function. To do this, we need a particular commutator involving $\mu\nabla\mathbf{g}*\mu$ to disappear. The following functional inequality will help us show this for measure trajectories in $\mathscr{A}$.

lem:gradcon **Lemma 6.2.** *For any $\rho\in\mathcal{D}(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$ and $\mu,\nu\in\mathcal{D}(\mathbb{T}^d)$ it holds that*

$$\left|\int\nabla\mathbf{g}*\mu\cdot\nabla\mathbf{g}*\nu d\rho\right|\lesssim_{\mathsf{s},d}\left(\||\nabla|^{\frac12+\frac{\mathsf{s}-d}{2}}\rho\|_{L^{\frac{6d}{3d-\mathsf{s}-1}}}+1\right)\||\nabla|^{\frac12+\frac{\mathsf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathsf{s}-1}}}\||\nabla|^{\frac12+\frac{\mathsf{s}-d}{2}}\nu\|_{L^{\frac{6d}{3d-\mathsf{s}-1}}}. \quad (6.4)$$

eq:cubic-form-boun

Consequently the operator on the right-hand side of (6.4) extends to an operator on $\{\mu\in\mathcal{P}(\mathbb{T}^d):\||\nabla|^{\frac12+\frac{\mathsf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathsf{s}-1}}}<\infty\}$.

Proof. For the sake of convenience, let $f=\nabla\mathbf{g}*\mu$ and $g=\nabla\mathbf{g}*\nu$. First we split

$$\int f\cdot g\rho=\int f\cdot g(\rho-1)+\int f\cdot g$$

so that $\rho - 1$ has zero mean. Then using Fourier multipliers and Hölder's inequality

$$\left|\int f \cdot g(\rho - 1)\right| \leqslant \||\nabla|^{\frac{d-\mathbf{s}}{2}-\frac{1}{2}}(fg)\|_{L^{\frac{6d}{3d+\mathbf{s}+1}}} \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\rho\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

The fractional Leibniz rule in turn implies that

$$\||\nabla|^{\frac{d-\mathbf{s}}{2}-\frac{1}{2}}(fg)\|_{L^{\frac{6d}{3d+\mathbf{s}+1}}} \leqslant \||\nabla|^{\frac{d-\mathbf{s}}{2}-\frac{1}{2}}f\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}\|g\|_{L^{\frac{3d}{\mathbf{s}+1}}} + \|f\|_{L^{\frac{3d}{\mathbf{s}+1}}}\||\nabla|^{\frac{d-\mathbf{s}}{2}-\frac{1}{2}}g\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

By the definition of f

$$\||\nabla|^{\frac{d-\mathbf{s}}{2}-\frac{1}{2}}f\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}} \approx_{d,\mathbf{s}} \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}$$

while the Sobolev embedding theorem implies that

$$\|f\|_{L^{\frac{3d}{\mathbf{s}+1}}} \lesssim_{d,\mathbf{s}} \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

Using identical arguments for g we find that

$$\left|\int f \cdot g(\rho - 1)\right| \lesssim_{d,\mathbf{s}} \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\rho\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\nu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

Finally Hölder's inequality and Sobolev embedding in turn imply that

$$\int f \cdot g \lesssim_{\mathbf{s},d} \||\nabla|^{1+\mathbf{s}-d}\mu\|_{L^2}\||\nabla|^{1+\mathbf{s}-d}\nu\|_{L^2} \lesssim_{\mathbf{s},d} \||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\nu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}},$$

where we've used that $\mathbb{T}^d$ has finite measure. This completes the claim. □

We will now show that we can approximate measures in $\mathscr{A}$ by measures which satisfy the requirements of Proposition 6.1.

prop:recovery **Proposition 6.3.** *Suppose μ is a solution to (1.22) with $I(\mu) < \infty$ and $\mu \in \mathscr{A}$. Then there exists a sequence $(\mu_k)_{k\geqslant 1} \subset C([0,T], \mathcal{P}(\mathbb{T}^d))$ satisfying the following properties:*

item:1 *1. $\mu_k|_{t=0} = \mu_0$.*

item:2 *2. $\lim_{k\to\infty} \mu_k = \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$.*

item:3 *3. $\mu_k \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$ for all $k \geqslant 1$ and is a weak solution to (1.22) with drift b_k satisfying*

$$\int_0^T \|b_k^t\|_{C^1}^2 + \||\nabla|^{\frac{d-\mathbf{s}}{2}}b_k^t\|^2_{L^{\frac{2d}{d-2-\mathbf{s}}}}\,dt < \infty.$$

item:4 *4. Letting $b \in L^2([0,T], L^2(\mu^t))$ so that*

$$\int_0^T \int |b^t|^2\,d\mu^t\,dt = \int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g} * \mu^t)\|^2_{-1,\mu^t}\,dt,$$

it holds that

$$\limsup_{k\to\infty} \int_0^T \int |b_k^t|^2\,d\mu_k^t\,dt \leqslant \int_0^T \int |b^t|^2 d\mu^t\,dt.$$

Proof. Since $I(\mu) < \infty$, it must be the case that $Q^T(\mu) < \infty$ and there exists $b \in L^2([0,T], L^2(\mu^t))$ so that (μ, b) are a weak solution to (4.2) and

$$I(\mu) > \frac{1}{4\sigma}\int_0^T \int |b^t|^2 \, d\mu^t \, dt.$$

First we will show that any subsequence of $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies Properties 2-4, and then we will appropriately modify them to satisfy Property 1.

It is immediate that $\mu_\varepsilon \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$ and that $\mu_\varepsilon \to \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$ as $\varepsilon \to 0$.

On the other hand μ_ε is also a weak solution to

$$\partial_t \mu_\varepsilon - \sigma\Delta\mu_\varepsilon - \operatorname{div}(\mu_\varepsilon \nabla \mathbf{g} * \mu_\varepsilon) = -\operatorname{div}(b_\varepsilon \mu_\varepsilon),$$

where

$$b_\varepsilon := \frac{(b\mu)_\varepsilon}{\mu_\varepsilon} + \nabla \mathbf{g} * \mu_\varepsilon - \frac{(\mu\nabla \mathbf{g} * \mu)_\varepsilon}{\mu_\varepsilon}.$$

Vitally, we will use that since μ is a probability measure and Φ^ε is lower bounded on the torus for all $\varepsilon > 0$, μ^t_ε is uniformly lower bounded. As a consequence $(\mu_\varepsilon)^{-1} \in L^\infty([0,T], C^k(\mathbb{T}^d))$ for any $k \geqslant 1$.

We note that $b\mu \in L^2([0,T], TV(\mathbb{T}^d))$ since

$$\int_{\mathbb{T}^d} \phi \cdot b^t d\mu^t \leqslant \left(\int |b^t|^2 d\mu^t\right)^{1/2} \left(\int |\phi^t|^2 d\mu\right)^{1/2} \leqslant \left(\int |b^t|^2 d\mu^t dt\right)^{1/2} \|\phi\|_{C^0}.$$

Combined with the regularity of $(\mu_\varepsilon)^{-1}$ this immediately implies that

$$\int_0^T \left\|\frac{(b\mu)_\varepsilon}{\mu_\varepsilon}\right\|^2_{C^1} + \left\||\nabla|^{\frac{d-\mathbf{s}}{2}} \frac{(b\mu)_\varepsilon}{\mu_\varepsilon}\right\|_{L^{\frac{2d}{d-2-\mathbf{s}}}} dt < \infty. \tag{6.5}$$ eq:b-mol-bound

The bound (2.4) then implies that $\mu\nabla \mathbf{g} * \mu \in L^\infty([0,T], C^k(\mathbb{T}^d)')$ for any $k > \frac{d-\mathbf{s}}{2}$. Again using the regularity of $(\mu_\varepsilon)^{-1}$ this implies that

$$\int_0^T \left\|\frac{(\mu^t\nabla \mathbf{g} * \mu^t)_\varepsilon}{\mu_\varepsilon}\right\|^2_{C^1} + \left\||\nabla|^{\frac{d-\mathbf{s}}{2}} \frac{(\mu^t\nabla \mathbf{g} * \mu^t)_\varepsilon}{\mu_\varepsilon}\right\|_{L^{\frac{2d}{d-2-\mathbf{s}}}} dt < \infty. \tag{6.6}$$ eq:slope-mol-bound

Since $\nabla \mathbf{g}$ is integrable it is trivial that

$$\int_0^T \|\nabla \mathbf{g} * \mu_\varepsilon\|^2_{C^1} + \||\nabla|^{\frac{d-\mathbf{s}}{2}} \nabla \mathbf{g} * \mu_\varepsilon\|^2_{L^{\frac{2d}{d-2-\mathbf{s}}}} dt < \infty$$

thus with (6.5) and (6.6) we indeed have that Property 3 holds.

To show Property 4, since

$$\int_0^T \int |b^t_\varepsilon|^2 \, d\mu^t_\varepsilon \, dt = \int_0^T \left|\frac{(b^t\mu^t)_\varepsilon}{\mu^t_\varepsilon} + \nabla \mathbf{g} * \mu^t_\varepsilon - \frac{(\mu^t\nabla \mathbf{g} * \mu^t)_\varepsilon}{\mu^t_\varepsilon}\right|^2 d\mu^t_\varepsilon \, dt,$$

it suffices to show that

$$\int_0^T \int_{\mathbb{T}^d} \left|\frac{(b^t\mu^t)_\varepsilon}{\mu^t_\varepsilon}\right|^2 d\mu^t_\varepsilon \, dt \leqslant \int_0^T \int_{\mathbb{T}^d} |b^t|^2 d\mu^t \, dt, \tag{6.7}$$ eq:b-jensen

and

$$\limsup_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d}\left|\frac{(\mu^t\nabla\mathbf{g}*\mu^t)_\varepsilon}{\mu^t_\varepsilon}-\nabla\mathbf{g}*\mu^t_\varepsilon\right|^2 d\mu^t_\varepsilon\,dt=0. \tag{6.8}$$

eq:commutator-van

The inequality (6.7) follows by [AGS08, Lemma 8.1.10].

To prove (6.8) we expand out the integral

$$\begin{aligned}&\int_{\mathbb{T}^d}\left|\frac{(\mu^t\nabla\mathbf{g}*\mu^t)_\varepsilon}{\mu^t_\varepsilon}-\nabla\mathbf{g}*\mu^t_\varepsilon\right|^2\mu^t_\varepsilon\\ &=\int\left|\frac{(\mu^t\nabla\mathbf{g}*\mu^t)_\varepsilon}{\mu^t_\varepsilon}\right|^2\mu^t_\varepsilon-2\int(\mu^t\nabla\mathbf{g}*\mu^t)_\varepsilon\cdot\nabla\mathbf{g}*\mu_\varepsilon+\int|\nabla\mathbf{g}*\mu^t_\varepsilon|^2d\mu^t_\varepsilon.\end{aligned} \tag{6.9}$$

eq:square-expansior

Since $\mu\in\mathscr{A}$, for almost every $t\in[0,T]$,

$$\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|_{L^{\frac{6d}{3d-\mathbf{s}-1}}}<\infty.$$

For such t, for fixed $\varepsilon>0$ since $\mu_\delta\nabla\mathbf{g}*\mu_\delta$ converges to $\mu\nabla\mathbf{g}*\mu$ in distribution,

$$\lim_{\delta\to\infty}(\mu^t_\delta\nabla\mathbf{g}*\mu^t_\delta)_\varepsilon=(\mu^t\nabla\mathbf{g}*\mu)_\varepsilon$$

pointwise. The dominated convergence theorem thus implies that

$$\int\left|\frac{(\mu^t\nabla\mathbf{g}*\mu^t)_\varepsilon}{\mu^t_\varepsilon}\right|^2\mu^t_\varepsilon=\lim_{\delta\to 0}\int\left|\frac{(\mu^t_\delta\nabla\mathbf{g}*\mu^t_\delta)_\varepsilon}{(\mu_\delta)^t_\varepsilon}\right|^2(\mu_\delta)^t_\varepsilon.$$

Lemma 8.1.10 in [AGS08] and Lemma 6.2 imply that

$$\int\left|\frac{(\mu^t_\delta\nabla\mathbf{g}*\mu^t_\delta)_\varepsilon}{(\mu_\delta)^t_\varepsilon}\right|^2\mu^t_\varepsilon\leqslant\lim_{\delta\to\infty}\int|\nabla\mathbf{g}*\mu^t_\delta|^2\mu^t_\delta=\int|\nabla\mathbf{g}*\mu^t|^2d\mu^t\lesssim_{d,\mathbf{s}}1+\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|^3_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

Lemma 6.2 also implies that

$$\lim_{\varepsilon\to\infty}\int(\mu\nabla\mathbf{g}*\mu^t)_\varepsilon\cdot\nabla\mathbf{g}*\mu^t_\varepsilon=\int|\nabla\mathbf{g}*\mu^t|^2\,d\mu^t\text{ and }\lim_{\varepsilon\to\infty}\int|\nabla\mathbf{g}*\mu^t_\varepsilon|^2\mu^t_\varepsilon=\int|\nabla\mathbf{g}*\mu^t|^2\,d\mu^t.$$

Using that mollification is contractive it also holds that

$$\left|\int(\mu\nabla\mathbf{g}*\mu^t)_\varepsilon\cdot\nabla\mathbf{g}*\mu^t_\varepsilon\right|=\left|\int\mu\nabla\mathbf{g}*\mu^t\cdot\nabla\mathbf{g}*(\mu_\varepsilon)^t_\varepsilon\right|\lesssim_{d,\mathbf{s}}1+\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|^3_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

and

$$\left|\int|\nabla\mathbf{g}*\mu^t_\varepsilon|^2\mu^t_\varepsilon\right|\lesssim_{d,\mathbf{s}}1+\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|^3_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

Altogether, we have shown that (6.9) converges to 0 almost surely in t, and is bounded above by a constant times

$$1+\||\nabla|^{\frac{1}{2}+\frac{\mathbf{s}-d}{2}}\mu\|^3_{L^{\frac{6d}{3d-\mathbf{s}-1}}}.$$

Since $\mu\in\mathscr{A}$, the dominated convergence theorem thus implies that (6.8) holds, and in turn μ_ε satisfy Property 3.

All that remains is to modify the mollifications μ_ε to have the proper initial conditions. Accordingly we define

$$\tilde{\mu}_\varepsilon:=\begin{cases}\mu^0*\Phi_{\sigma t} & 0\leqslant t\leqslant\epsilon,\\ \mu^{t-\varepsilon}*\Phi_{\sigma\epsilon} & \varepsilon<t\leqslant T.\end{cases} \tag{6.10}$$

eq:approximation-d

Since $\mu_0 \in L^\infty$, $\mu_\varepsilon \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$. As well, since $(\mu_0)_\varepsilon \to \mu_0$ in $\mathcal{P}(\mathbb{T}^d)$ and $\mu_\varepsilon \to \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$, it also holds that $\widetilde{\mu}_\varepsilon \to \mu$ in $C([0,T], \mathcal{P}(\mathbb{T}^d))$. Thus $(\widetilde{\mu}_\varepsilon)_{\varepsilon>0}$ satisfy Properties 1 and 2.

It is easy to verify that $\widetilde{\mu}_\varepsilon$ is a weak solution to

$$\partial_t \widetilde{\mu}_\varepsilon - \sigma \Delta \widetilde{\mu}_\varepsilon - \operatorname{div}\left((\nabla \mathbf{g} * \widetilde{\mu}_\varepsilon)\widetilde{\mu}_\varepsilon\right) = -\operatorname{div}\left(\widetilde{b}_\varepsilon \widetilde{\mu}_\varepsilon\right)$$

where now

$$\widetilde{b}_\varepsilon = \begin{cases} \nabla \mathbf{g} * (\mu^0)_\varepsilon & 0 < t < \varepsilon, \\ b_\varepsilon^{t-\varepsilon} & \varepsilon < t < T, \end{cases}$$

for b_ε as defined for μ_ε. Since $\mu_0 \in L^\infty$ it is clear that Property 3 holds as well.

To show $\widetilde{\mu}_\varepsilon$ satisfy Property 4, it suffices to note that

$$\lim_{\varepsilon \to \infty} \int_0^\varepsilon |\nabla \mathbf{g} * (\mu_0)_t|^2 (\mu_0)_t \, dt = 0,$$

which again holds due to the regularity of μ_0. $\square$

ace-approximations **Corollary 6.4.** *Suppose $\mu \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$ so that $(\mu - 1) \in \mathscr{C}^T$ and $I(\mu) < \infty$. Then there exists a sequence μ_k satisfying properties 1, 3, and 4 in Proposition 6.3 so that $(\mu_k - \mu) \to 0$ in $\mathscr{C}^T$, and μ_k is uniformly bounded in $L^\infty([0,T], L^\infty(\mathbb{T}^d))$.*

Proof. It is easy to verify that if $\mu \in L^\infty([0,T], L^\infty(\mathbb{T}^d))$, then $\mu \in \mathscr{A}$. We can thus use the construction in Proposition 6.3 to find that $\widetilde{\mu}_\varepsilon$ as defined in (6.10) satisfy properties 1, 3, and 4. As it is immediate that $\widetilde{\mu}_\varepsilon$ are uniformly bounded in $L^\infty([0,T], L^\infty(\mathbb{T}^d))$, we only need to show that $\widetilde{\mu}_\varepsilon - \mu$ converges to zero in $\mathscr{C}^T$.

First we note that $\mu_\varepsilon - \mu$ converges to 0 in $\mathscr{C}^T$. This follows by the Arzelã-Ascoli theorem since $(\mu_\varepsilon - \mu)$ converges to 0 pointwise in time in $\dot{H}_0^{\frac{\mathsf{s}-d}{2}}$, while

$$\|\mu_\varepsilon^t - \mu_\varepsilon^s\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} \leqslant \|\mu^t - \mu^s\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}},$$

thus $\{\mu_\varepsilon - 1\}_{\varepsilon>0}$ is a uniformly continuous family.

Since $\lim_{\varepsilon \to 0} (\mu_0)_\varepsilon - \mu_0 = 0$ in $\dot{H}_0^{\frac{\mathsf{s}-d}{2}}(\mathbb{T}^d)$, and $\mu - 1 \in \mathscr{C}^T$, it immediately follows that $\widetilde{\mu}_\varepsilon - \mu$ converges to 0 as well. $\square$

We'll now prove the LDP lower bound in Theorem 1.1.

Proof of (1.10). It suffices to prove that if $\mu \in \mathscr{A}$ and $I(\mu) < \infty$, then for all $\varepsilon > 0$

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(\mu_N \in B_\varepsilon(\mu)) \geqslant -I(\mu).$$

Since μ satisfies the conditions of Proposition 6.3, there exists a sequence $(\mu_k)_{k \geqslant 1}$ satisfying Properties 1-4 with drifts b_k.

For all k sufficiently large that $B_{\frac{\varepsilon}{2}}(\mu_k) \subset B_\varepsilon(\mu)$, the argument in Proposition 4.5 implies that for all sufficiently small ε'

$$\left\{ \sup_{t \in [0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon' \right\} \subset \left\{ \mu_N \in B_{\frac{\varepsilon}{2}}(\mu_k) \right\},$$

thus

$$\mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \geqslant \mathbb{P}(\mu_N \in B_{\frac{\varepsilon}{2}}(\mu_k)) \geqslant \mathbb{P}\Bigg(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon'\Bigg).$$

Proposition 6.1 then implies that

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) &\geqslant \lim_{\varepsilon'\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Bigg(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon'\Bigg)\\ &\geqslant -\frac{1}{4\sigma}\int_0^T \int |b_k^t|^2 \, d\mu_k \, dt.\end{aligned}$$

Since

$$\limsup_{k\to\infty} \frac{1}{4\sigma}\int_0^T \int |b_k^t|^2 \, d\mu_k \, dt \leqslant \frac{1}{4\sigma}\int_0^T \int |b^t|^2 \, d\mu \, dt \leqslant I(\mu),$$

we immediately find that

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) \geqslant -I(\mu)$$

as desired.

To prove the full lower LDP bound when $\mathbf{s} = 0$, it suffices to prove that $\{Q^T(\mu) < \infty\} \subset \mathscr{A}$. The Gagliardo-Nirenberg-Sobolev inequality implies that

$$\||\nabla|^{\frac{1}{2}-\frac{d}{2}}\mu\|_{L^{\frac{6d}{3d-1}}} \leqslant \||\nabla|^{-\frac{d}{2}}\mu\|_{L^2}^{1/3} \||\nabla|^{1-\frac{d}{2}}\mu\|_{L^2}^{2/3},$$

namely

$$\int_0^T \||\nabla|^{\frac{1}{2}-\frac{d}{2}}\mu\|^2_{L^{\frac{6d}{3d-1}}} \, dt \leqslant \sup_{t\in[0,T]} \|\mu^t\|_{\dot{H}^{-\frac{d}{2}}} \int_0^T \|\mu^t\|^2_{H^{1-\frac{d}{2}}} \, dt.$$

The right-hand is finite when $Q^T(\mu) < \infty$, therefore $\mu \in \mathscr{A}$. □

As an immediate corollary, when $\mu \in \mathscr{A}$ we can compare $Q^T(\mu)$ with

$$\int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g}(\mu^t))\|^2_{-1,\mu^t} \, dt.$$

Corollary 6.5. *If $\mu \in \mathscr{A}$ and $I(\mu) < \infty$, then*

$$Q^T(\mu) - \mathcal{E}(\mu) \leqslant \int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g}(\mu^t))\|^2_{-1,\mu^t} \, dt.$$

Proof. The upper bound (1.9) and the computation used to prove the lower bound (1.10) together imply that

$$\begin{aligned}-\frac{1}{4\sigma}\int_0^T \int |b^t|^2 \, d\mu^t \, dt \leqslant \lim_{\varepsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big) &\leqslant \lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\mu_N \in B_\varepsilon(\mu)\big)\\ &\leqslant -\frac{1}{4\sigma}(Q^T(\mu) - \mathcal{E}(\mu_0)),\end{aligned}$$

thus since

$$\int_0^T \int |b^t|^2 \, d\mu^t = \int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g}(\mu^t))\|^2_{-1,\mu^t} \, dt,$$

this says that

$$Q^T(\mu) - \mathcal{E}(\mu_0) \leqslant \int_0^T \|\partial_t\mu^t - \sigma\Delta\mu^t - \operatorname{div}(\mu^t\nabla\mathbf{g}(\mu^t))\|^2_{-1,\mu^t} \, dt,$$

as claimed. □

We will now prove Theorem 1.3 which follows very similarly to as above.

Proof of Theorem 1.3. Due to Proposition 4.5, it suffices to prove that if $(\mu - 1) \in \mathscr{C}^T$ and $I(\mu) < \infty$, then for all $\varepsilon > 0$

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon \Big) \geqslant -I(\mu).$$

Using Corollary 6.4 we have that there exists a sequence $\mu_k \to \mu$ satisfying properties 1, 2, and 4 so that $\mu_k - \mu$ converges to 0 in $\mathscr{C}^T$.

First we'll show that for every $\varepsilon > 0$, there exists $\varepsilon' > 0$ so that for all sufficiently large k and N

$$\Big\{ \sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu_k^t) < \varepsilon' \Big\} \subset \Big\{ \sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon \Big\}.$$

This follows by Corollary 2.9 and Proposition 2.13 since

$$\begin{aligned} F_N(\underline{x}_N^t, \mu^t) &\lesssim_{d,\mathsf{s}} \|\mu_{N,\vec{\mathsf{r}}}^t - \mu^t\| + \|\mu\|_{L^\infty} N^{-\beta} \\ &\lesssim \|\mu_{N,\vec{\mathsf{r}}}^t - \mu_k^t\| + \|\mu^t - \mu_k^t\| + \|\mu\|_{L^\infty} N^{-\beta} \\ &\lesssim_{d,\mathsf{s}} F_N(\underline{x}_N^t, \mu_k^t) + \|\mu^t - \mu_k^t\|_{\dot{H}^{\frac{\mathsf{s}-d}{2}}} + (\|\mu^t\|_{L^\infty} + C\|\mu_k^t\|_{L^\infty}) N^{-\beta}. \end{aligned}$$

We thus find for every $\varepsilon > 0$ that

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu^t) < \varepsilon \Big) &\geqslant \liminf_{k\to\infty} \lim_{\varepsilon'\to 0} \liminf_{N\to\infty} \mathbb{P}\Big(\sup_{t\in[0,T]} F_N(\underline{x}_N^t, \mu_k^t) < \varepsilon' \Big) \\ &\geqslant -\frac{1}{4\sigma} \int_0^T \int |b^t|^2 \, d\mu^t \, dt \\ &\geqslant -I(\mu), \end{aligned}$$

and conclude. $\square$

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