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A METHOD FOR ANALYSIS OF C^1-CONTINUITY OF SUBDIVISION SURFACES*

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Abstract. A sufficient condition for C^1-continuity of subdivision surfaces was proposed by Reif [*Comput. Aided Geom. Design*, 12 (1995), pp. 153–174.] and extended to a more general setting in [D. Zorin, *Constr. Approx.*, accepted for publication]. In both cases, the analysis of C^1-continuity is reduced to establishing injectivity and regularity of a characteristic map. In all known proofs of C^1-continuity, explicit representation of the limit surface on an annular region was used to establish regularity, and a variety of relatively complex techniques were used to establish injectivity. We propose a new approach to this problem: we show that for a general class of subdivision schemes, regularity can be inferred from the properties of a sufficiently close linear approximation, and injectivity can be verified by computing the index of a curve. An additional advantage of our approach is that it allows us to prove C^1-continuity for all valences of vertices, rather than for an arbitrarily large but finite number of valences. As an application, we use our method to analyze C^1-continuity of most stationary subdivision schemes known to us, including interpolating butterfly and modified butterfly schemes, as well as the Kobbelt's interpolating scheme for quadrilateral meshes.

Key words. stationary subdivision, subdivision surfaces, arbitrary meshes, interval arithmetics

AMS subject classifications. 65D10, 65D17, 68U05

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1. Introduction.

Subdivision is becoming increasingly popular as a surface representation in computer graphics applications. To ensure that a subdivision algorithm has the desired behavior for almost all input data, a theoretical analysis of the surface has to be performed. For subdivision on arbitrary meshes, even the analysis of the basic property of the surfaces, C^1-continuity, poses a considerable challenge; [19, 9, 15, 16, 20]. In this paper we describe a set of theoretical results and algorithms that make it possible to perform the C^1-continuity tests automatically.

The principal result allowing one to analyze C^1-continuity of most subdivision schemes is the sufficient condition of Reif [18]. This condition reduces the analysis of stationary subdivision to the analysis of a single map, called the *characteristic map*, for each valence of vertices in the mesh. The analysis of C^1-continuity is performed in three steps for each valence:

1. compute the control net of the characteristic map;
2. prove that the characteristic map is regular;
3. prove that the characteristic map is injective.

This map can be expressed in a closed form for spline-based subdivision schemes, such as Loop, Catmull–Clark, and Doo–Sabin. For these schemes, proving regularity of the characteristic map is tedious but straightforward, as the Jacobian of the map can be expressed in terms of piecewise polynomial basis functions. Proving injectivity is somewhat more difficult [19, 15].

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Our goal is to verify regularity and injectivity automatically for arbitrary (not necessarily spline-based) subdivision schemes, once the control net for the characteristic map is known. Our approach has two additional benefits:

1. With some mild assumptions on the dependence of the coefficients of the characteristic scheme on the valence, we are able to analyze C^1-continuity for *all* valences.

2. Stability of C^1-continuity with respect to perturbations of coefficients can be estimated.

Our method is based on two results, discussed in sections 3.1 and 3.2. The estimates of section 3.1 allow us to infer regularity from the properties of the linear approximations to the limit map, which can be computed explicitly. In section 3.2 we show that a regular characteristic map can be proved to be injective simply by verifying that the index of the curve, obtained by restricting the map to the boundary of its domain, is 1. The latter result follows from self-similarity of the characteristic map.[1] Computing the index of a curve is a simple procedure that can be implemented robustly. In contrast, global injectivity of a map is often difficult to prove directly.

In the second part of the paper we describe the algorithms for verification of C^1-continuity based on the theoretical results of sections 3.1 and 3.2. A crucial element of our technique is the interval computation: although in many cases all required calculations can be performed symbolically, it is much more efficient and, in fact, simpler, to obtain guaranteed bounds on the quantities of interest using interval arithmetics. As an additional benefit, we are able to prove facts not about single characteristic maps defined by exact values of the control points, but about families of maps, corresponding to the control points with interval components.

Using our method, we analyze interpolating triangular and quadrilateral subdivision schemes — the butterfly [6], the modified butterfly [24], and the scheme described by Kobbelt [12]. (A similar scheme was proposed earlier by Leber [13].) We also repeat the analysis for two schemes that were analyzed previously by other authors: the Loop [19] scheme and the Catmull–Clark scheme [15]. For the latter schemes we extend the analysis to all valences. It is important to emphasize that once the control points for the characteristic maps are computed, the same code is used to analyze all these schemes.

Related work. This work further extends the results presented in [20]. To the best of our knowledge, all schemes that were analyzed by most other authors admitted closed-form expressions for the characteristic maps; one notable exception is the analysis of Leber [13]. However, the analysis of [13] relies on the fact that the rule used in the regular case is a tensor product of two identical one-dimensional rules, as well as on some specific monotonicity properties of the one-dimensional rule. Our method for establishing regularity radically differs both from symbolic methods used in [19, 9, 15] and Leber's approach: it is simpler to apply and more general.

Our estimates of the errors of linear approximations rely on the work of Cavaretta, Dahmen, and Micchelli [2], and on the work of Cohen, Dyn, and Levin [3] on matrix subdivision.

Initial discrete fourier transform (DFT) analysis that we use to find the control points for the characteristic map for invariant schemes follows the well-established pattern used in [1, 9, 19, 24, 15].

[1] In general, it is not true that a map is injective if it is regular on its domain even if the domain is the plane and the map is polynomial. This statement is known as the Jacobian conjecture for dimension 2, and a counterexample was found by S. Pinchuk in [17].

Finally, we extensively use interval arithmetics (see, for example, [14]).

Overview. In section 2, we describe the notation for subdivision on complexes and state relevant results from [22] and [20].

In section 3 we present the results forming the theoretical foundation of our method. In section 3.1, we discuss the basic properties of the matrix subdivision schemes, and we derive estimates for the convergence rates of linear and piecewise constant approximations to the limit functions generated by subdivision. These estimates are used to verify regularity of the characteristic maps. In section 3.2, we prove that the index of a curve can be used to test injectivity of the characteristic map.

Section 4 provides a brief description of algorithms for verification of C^1-continuity based on the results of section 3.

In section 5 we apply our method to analyze C^1-continuity of the butterfly and the modified butterfly schemes, the Kobbelt interpolating scheme, and several other schemes.

2. Subdivision schemes.

In this section we summarize the main definitions and facts about subdivision on complexes that we use. The main theorem is the generalization of Reif's sufficient condition (Theorem 2.1). The main results of the paper, presented in the following sections, will provide a constructive way of verifying the assumptions of this theorem. We also discuss the invariant subdivision schemes, for which our algorithms can be further improved. Most commonly used schemes are invariant, or are extensions of invariant schemes. The details and proofs can be found in [22, 20].

2.1. Subdivision on complexes.

Simplicial complexes. Subdivision surfaces are naturally defined as functions on two-dimensional simplicial complexes. Recall that a simplicial complex K is a set of vertices, edges, and triangles in some Eucledean space $\mathbf{R}^N$, such that for any triangle all its sides are in K, and for any edge its endpoints are vertices of K. We assume that there are no isolated vertices or edges, that is, every vertex is an endpoint of an edge and every edge is a side of a triangle. $|K|$ denotes the union of triangles of the complex regarded as a subset of $\mathbf{R}^N$ with induced metric. We say that two complexes K_1 and K_2 are *isomorphic* if there is a homeomorphism between $|K_1|$ and $|K_2|$ that maps vertices to vertices, edges to edges, and triangles to triangles.

A *subcomplex* of a complex K is a subset of K that is a complex. A 1-neighborhood $N_1(v, K)$ of a vertex v in a complex K is the subcomplex formed by all triangles that have v as a vertex. The m-neighborhood of a vertex v is defined recursively as a union of all 1-neighborhoods of vertices in the $(m-1)$-neighborhood of v. We omit K in the notation for neighborhoods when it is clear what complex we refer to.

Recall that a *link* of a vertex is the set of edges of $N_1(v, K)$ that do not contain v. We consider only complexes with all vertices having links that are connected simple polygonal lines, open or closed. If the link of a vertex is an open polygonal line, this vertex is a boundary vertex, otherwise it is an internal vertex.

Most of our constructions use two special types of complexes — *k-regular complexes* $\mathcal{R}_k$ and the *regular complex* $\mathcal{R}$. Each complex is simply a triangulation of the plane consisting of identical triangles. In the regular complex each vertex has exactly six neighbors. In a k-regular complex all vertices have six neighbors, except one vertex C, which has k neighbors. We call C the central vertex of a k-regular complex and identify it with the zero in the plane.

Subdivision of simplicial complexes. We can construct a new complex $D(K)$ from a complex K by subdivision, adding the midpoints of all edges to the set of vertices of the complex and replacing each old triangle with four new triangles. Note that k-regular complexes are self-similar, that is, $D(\mathcal{R}_k)$ and $\mathcal{R}_k$ are isomorphic.

We use notation K^j for j times subdivided complex $D^j(K)$ and V^j for the set of vertices of K^j. Note that the sets of vertices are nested: $V^0 \subset V^1 \subset \cdots$.

Subdivision schemes. Next, we attach values to the vertices of the complex; in other words, we consider the space of functions $V \to B$, where B is a vector space over $\mathbf{R}$. The range B is typically $\mathbf{R}^l$ or $\mathbf{C}^l$ for some l. We denote this space $\mathcal{P}(V, B)$, or $\mathcal{P}(V)$, if the choice of B is not important.

A *subdivision scheme* for any function $p^j(v)$ on vertices V^j of the complex K^j computes a function $p^{j+1}(v)$ on the vertices of the subdivided complex $D(K) = K^1$. More formally, a subdivision scheme is a collection of operators $S[K]$ defined for every complex K, mapping $\mathcal{P}(V)$ to $\mathcal{P}(V^1)$. We consider only subdivision schemes that are linear, that is, the operators $S[K]$ are linear functions on $\mathcal{P}(V)$. In this case the subdivision operators are defined by

$$p^1(v) = \sum_{w \in V} a_{vw} p^0(w)$$

for all $v \in V^1$. The coefficients a_{vw} may depend on K.

We restrict our attention to subdivision schemes which are finitely supported, locally invariant with respect to a set of isomorphisms of complexes, and affinely invariant.

A subdivision scheme is *finitely supported* if there is an integer M such that $a_{vw} \neq 0$ only if $w \in N_M(v, K^1)$ for any complex K (note that the neighborhood is taken in the complex K^1). We call the minimal possible M the *support size* of the scheme.

We assume our schemes to be *locally defined* and *invariant with respect to isomorphisms of complexes.*[2]

Together these two requirements can be defined as follows: there is a constant L such that if for two complexes K_1 and K_2 and two vertices $v_1 \in V_1$ and $v_2 \in V_2$ there is an isomorphism $\rho : N_L(v_1, K_1) \to N_L(v_2, K_2)$, such that $\rho(v_1) = v_2$, then $a_{v_1 w} = a_{v_2 \rho(w)}$. In most cases, the *localization size* $L = M$.

The final requirement that we impose on subdivision schemes is *affine invariance*: if T is an affine transformation $B \to B$, then for any v, $Tp^{j+1}(v) = \sum a_{vw} T p^j(v)$. This is equivalent to requiring that all coefficients a_{vw} for a fixed v sum up to 1.

Limit functions. For each vertex $v \in \cup_{j=0}^{\infty} V^j$ there is a sequence of values $p^i(v)$, $p^{i+1}(v), \ldots$ where i is the minimal number such that V^i contains v.

DEFINITION 2.1. *A subdivision scheme is called convergent on a complex K, if for any function $p \in \mathcal{P}(V, B)$ there is a continuous function f defined on $|K|$ with values in B, such that*

$$\lim_{j \to \infty} \sup_{v \in V^j} \left\| p^j(v) - f(v) \right\|_2 \to 0.$$

The function f is called the limit function of subdivision.

[2]In fact, we need invariance only with respect to a sufficiently large set of isomorphisms of complexes, including similarity transformations of k-regular complexes. This allows us to include schemes defined on tagged complexes; see [22, 20].

Notation: $f[p]$ is the limit function generated by subdivision from the initial values $p \in \mathcal{P}(V)$.

It is easy to show that if a limit function exists, it is unique. A *subdivision surface* is the limit function of subdivision on a complex K with values in $\mathbf{R}^3$. In this case we call the initial values $p^0(v)$ the *control points* of the surface.

Locally any surface generated by a subdivision scheme on an arbitrary complex can be thought of as a part of a subdivision surface defined on a k-regular complex. More specifically, a part of the subdivision surface defined on the union of triangles surrounding a vertex of valence k (a k-gonal patch) can be thought of as being defined on a 1-neighborhood of the central vertex in a k-regular complex.

Note that this fact alone does not guarantee that it is sufficient to study subdivision schemes only on k-regular complexes. Suppose the number of control points defining the limit surface on a k-gonal patch of the initial complex is less than the number of control points of the central k-gonal patch in the k-regular complex (these numbers are finite by locality of subdivision). In this case only a proper subspace of all possible configurations of control points on the subdivided complexes can be realized. We call such complexes *constrained.* Although it is unlikely, it is possible that for such complexes almost all configurations of control points will lead to nonsmooth surfaces, while the scheme is smooth on the k-regular complexes. This problem is discussed in greater detail in [22].

2.2. Subdivision matrices. The key to analysis of stationary subdivision is the idea of the *subdivision matrix.* Eigenvalues and eigenvectors of this matrix are closely related to the smoothness properties of subdivision. Consider the part of a subdivision surface $f[p](y)$ with $y \in U_1^j = |N_1(0, \mathcal{R}_k^j)|$, defined on the k-gon formed by triangles of the subdivided complex $\mathcal{R}_k^j$ adjacent to the central vertex. It is straightforward to show that the values at all dyadic points in this k-gon can be computed given the initial values $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. In particular, the control points $p^{j+1}(v)$ for $v \in N_L(0, \mathcal{R}_k^{j+1})$ can be computed using only control points $p^j(w)$ for $w \in N_L(0, \mathcal{R}_k^j)$. Let $\bar{p}^j$ be the vector of control points $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. Let $Q+1$ be the number of vertices in $N_L(0, \mathcal{R}_k)$.

As the subdivision operators are linear, $\bar{p}^{j+1}$ can be computed from $\bar{p}^j$ using a $(Q+1) \times (Q+1)$ matrix S^j: $\bar{p}^{j+1} = S^j \bar{p}^j$.

If for all j, $S^j = S$, we say that the subdivision scheme is *stationary on the k-regular complex*, or simply stationary, and call S the *subdivision matrix* of the scheme. Note that our definition in the case $k = 6$ is weaker than the standard definition of stationary schemes on regular complexes [2].

As we will see, eigenvalues and eigenvectors of the matrix have fundamental importance for smoothness of subdivision.

Eigenbasis functions. Let $\lambda_0 = 1, \lambda_i, \dots, \lambda_J$ be different eigenvalues of the subdivision matrix in nonincreasing order of magnitudes (in this list each distinct eigenvalue is included once, even if its multiplicity is greater than one). We can assume that $\lambda_0 = 1$ because for the scheme to be convergent and have nontrivial limits it is necessary that $|\lambda_i| < 1$ for all eigenvalues with $i \neq 0$ (see [18, 20]).

For every λ_i, let J_j^i, $j = 1 \dots$, be the complex cyclic subspaces corresponding to this eigenvalue.

Let n_j^i be the *orders* of these cyclic subspaces; the order of a cyclic subspace is equal to its dimension minus one.

Let b_{jr}^i, $r = 0 \dots n_j^i$, be the complex generalized eigenvectors corresponding to

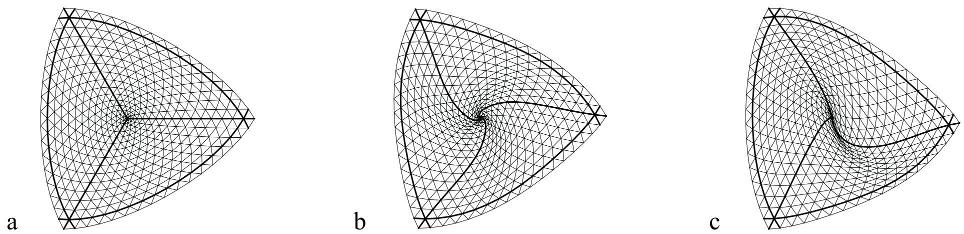

FIG. 1. *Three types of characteristic maps: control points after four subdivision steps are shown.* (a) *Two real eigenvalues.* (b) *A pair of complex-conjugate eigenvalues.* (c) *A single eigenvalue with Jordan block of size* 2.

the cyclic subspace J^i_j. The vectors b^i_{jr} satisfy

$$Sb^i_{jr} = \lambda_i b^i_{jr} + b^i_{j\,r-1} \quad \text{if } r > 0, \quad Sb^i_{j0} = \lambda_i b^i_{j0}. \tag{2.1}$$

The complex *eigenbasis functions* are the limit functions defined by $f^i_{jr} = f[b^i_{jr}] : U_1 \to \mathbf{C}$. Any subdivision surface $f[p](y) : U_1 \to \mathbf{R}^3$ can be represented as

$$f[p](y) = \sum_{i,j,r} \beta^i_{jr} f^i_{jr}(y), \tag{2.2}$$

where $\beta^i_{jr} \in \mathbf{C}^3$, and if $b^i_{jr} = \overline{b^k_{lt}}$, $\beta^i_{jr} = \overline{\beta^k_{lt}}$, where the bar denotes complex conjugation.

One can show using the definition of limit functions of subdivision and (2.1) that the eigenbasis functions satisfy the following set of *scaling relations*:

$$f^i_{jr}(y/2) = \lambda_i f^i_{jr}(y) + f^i_{j\,r-1}(y) \quad \text{if } r > 0, \quad f^i_{j0}(y/2) = \lambda_i f^i_{j0}(y). \tag{2.3}$$

2.3. Sufficient condition for C^1-continuity.

C^1-continuity of surfaces. By C^1-continuous surfaces we mean two-dimensional manifolds immersed, but not necessarily embedded, in $\mathbf{R}^3$ (see [22] for more detailed discussion). It can be easily shown that no scheme can generate C^1-continuous surfaces for all possible configurations of control points. Hence, we require only that subdivision generates C^1-continuous surfaces for any choice of control points on a complex K, except a nowhere dense set of configurations. In almost all cases, for local schemes C^1-continuity for arbitrary complexes follows from C^1-continuity on k-regular complexes. As we have already mentioned, a subtle problem may occur, however, for the *constrained* complexes (see [22] for further details).

Characteristic maps.

DEFINITION 2.2. *The* characteristic map $\Phi : U_1 \to \mathbf{R}^2$ *is defined for a pair of cyclic subspaces* J^a_b, J^c_d *of the subdivision matrix as* (f^a_{b0}, f^a_{b1}) *if* $J^a_b = J^c_d$, λ_a *is real,* (f^a_{b0}, f^c_{d0}) *if* $J^a_b \neq J^c_d$, λ_a, λ_c *are real, and* $(\Re f^a_{b0}, \Im f^a_{b0})$ *if* $\lambda_a = \bar{\lambda}_c$, $b = d$.

Three types of characteristic maps are shown in Figure 1.

The domain of a characteristic map is the k-gon U_1, consisting of k triangles of the k-regular complex adjacent to the central vertex; we call these triangles *segments.* We assume that the subdivision scheme generates C^1-continuous limit functions the regular complexes, and the characteristic map is, therefore, C^1-continuous inside each segment and has continuous one-sided derivatives on the boundary of each segment. Note that the characteristic map need not be continuous across the segment boundaries, and this does not preclude smoothness of the subdivision scheme. However, the

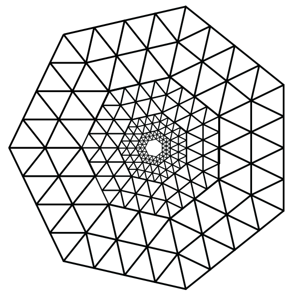

FIG. 2. *The k-gon without origin U_1 $\{0\}$ can be decomposed into similar rings, each two times smaller than the previous ring. The size of the ring is chosen in such a way that the control set of any ring does not contain the extraordinary vertex. In this figure the control set is assumed to consist out of the vertices of the triangles of the ring itself, and of a single layer of vertices outside the ring.*

one-sided Jacobians of the characteristic map do coincide on two sides of the boundaries of segments, so we can regard the Jacobian as being defined on U_1 without zero (see [22] for more details).

Sufficient condition for C^1-continuity. The following sufficient condition is a special case of the condition that was proved in [22]. Although all our constructions apply in the more general case, we state only a simplified version of the criterion sufficient for our purposes. This form captures the main idea of the sufficient condition. This condition generalizes Reif's condition [18].

Define for any two cyclic subspaces $\mathrm{ord}(J^i_j, J^k_l)$ to be $n^i_j + n^k_l$ if $J^i_j \neq J^k_l$; let $\mathrm{ord}(J^i_j, J^i_j) = 2n^i_j - 2$; note that for $n^i_j = 0$, this is a negative number, and it is less than $\mathrm{ord}(\cdot,\cdot)$ for any other pair. This number allows us to determine which components of the limit surface contribute to the limit normal (see [22, 20] for details). We say that a pair of cyclic subspaces J^a_b, J^c_d is *dominant* if for any other pair J^i_j, J^k_l we have either $|\lambda_a\lambda_c| > |\lambda_i\lambda_k|$, or $|\lambda_a\lambda_c| = |\lambda_i\lambda_k|$ and $\mathrm{ord}\,(J^a_b, J^c_d) > \mathrm{ord}\,\left(J^i_j, J^k_l\right)$; the blocks of the dominant pair may coincide.

THEOREM 2.1. *Let $\left\{b^i_{jr}\right\}$ be a basis in which a subdivision matrix S has Jordan normal form. Suppose that there is a dominant pair (J^a_b, J^c_d). If $\lambda_a\lambda_c$ is positive real, and the Jacobian of the characteristic map of J^a_b, J^c_d has constant sign everywhere on each segment of U_1, including the segment boundaries but excluding zero, then the subdivision scheme is tangent plane continuous on the k-regular complex. If the characteristic map is injective, the subdivision scheme is C^1-continuous.*

In the special case when all Jordan blocks are trivial, this condition reduces to an analogue of the Reif's condition.

To apply Theorem 2.1, we use self-similarity of the characteristic map: for any $t \in U_1$, the Jacobian $J[\Phi](t/2) = 4\lambda_a\lambda_b[\Phi](t)$. It is immediately clear that to prove regularity of the characteristic map it is sufficient to consider the Jacobian on a single annular portion of U_1 as shown in Figure 2.

Our goal is to develop an efficient general method that would allow us to apply Theorem 2.1 to arbitrary subdivision schemes. In the next two sections we develop a theoretical foundation for constructive application of this criterion: In section 3.1, we prove that regularity of the characteristic map can be verified using linear approximations to the map. This is sufficient to analyze tangent plane continuity. In section 3.2, we show that injectivity of the characteristic map can be verified by computing the index of a curve.

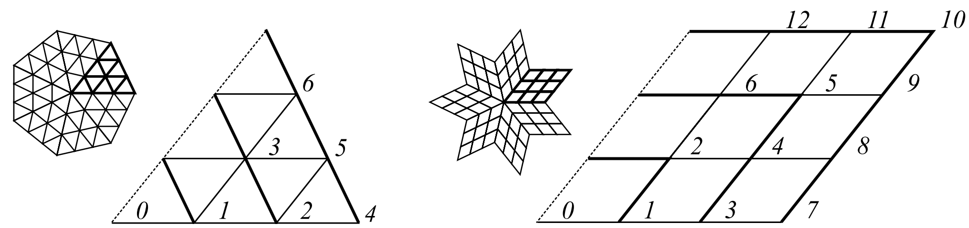

FIG. 3. *The numbers of the vertices in a sector of the control mesh for the characteristic map. Left: the numbering for triangular schemes; right: the numbering for quadrilateral schemes.*

2.4. Invariant schemes. Specific examples of schemes considered in this paper are invariant with respect to all isomorphisms of complexes.[3] For such schemes, the general algorithms we develop in subsequent sections can be simplified. Furthermore, the algorithm for verifying C^1-continuity is formulated for invariant subdivision schemes only.

In this section we introduce the notation for invariant schemes and describe transformations of subdivision matrices for such that reduces them to the block-diagonal form. We also state a necessary condition for C^1-continuity of invariant schemes, which can be used to prove that a scheme is *not* C^1-continuous. In section 5.1 we use it to prove that the original butterfly scheme is not C^1-continuous. The constructions of this section follow the ideas of Ball and Storry [1], also used in [24] and in [15].

If a scheme is invariant with respect to all isomorphisms of complexes, it is also invariant with respect to automorphisms of a k-regular complex. If ρ is an automorphism of a complex K, the coefficients of subdivision satisfy

$$a(v, w) = a(\rho(v), \rho(w)). \tag{2.4}$$

For k-regular complexes, the set of automorphisms consists of rotations around the extraordinary vertex, mirror reflections, and their combinations; we use only rotations.

Let L be the localization/control size for the subdivision scheme on a k-regular complex. In this case, the control set of U_1 is an L-neighborhood of the extraordinary vertex. One sector of this neighborhood (center excluded) contains $L(L+1)/2 = N$ vertices, the total number of vertices being $Nk + 1$. We will use notation $[s\, j]$ for the vertices; s is the number of the sector $s = 0 \dots k-1$, $j > 0$ is an arbitrarily chosen numbering of vertices within a sector. We use the numbering shown in Figure 3.

The central vertex is $[0\,0]$. We assume that the numbering is chosen consistently in each sector, that is, $R^m([s\, j]) = [s + m \bmod k\; j]$, where R^m corresponds to the rotation of the plane by $2m\pi/k$.

With this notation, (2.4) becomes $a([s'\, j'], [s\, j]) = a([(s'+m)\, j'], [(s+m)\, j])$ for any m, where the sums are modulo k.

The coefficients are functions of j,j' and $s - s'$ only; in the cases when $j = 0$ or $j' = 0$ (one of v, w is the extraordinary vertex), the coefficients do not depend on $s - s'$. We introduce notation $a([s\, j], [s'\, j']) = a_{j\,j'}(s - s')$, $b_j = a([0\,0], [s\, j])$, $c_j = a([s\, j], [0\,0])$, $a_{00} = a([0\,0], [0\,0])$. The subdivision matrix will have a convenient block form if we arrange the vertices "by symmetry class": $[0,0]$, $[0,1]$, $[1,1]$, $[2,1] \dots$ $[k-1,1]$, $[0,2] \dots [k-1,N]$. With this ordering of vertices, the subdivision matrix

[3] An example of a scheme which is *not* invariant with respect to some isomorphism is the piecewise-smooth scheme of Hoppe et al. [10]

has the form

$$S = \left(\begin{array}{c|ccc} a_{00} & \mathbf{b}_0^T & \cdots & \mathbf{b}_{N-1}^T \\ \hline \mathbf{c}_0 & A_{00} & \cdots & A_{0\,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_{N-1} & A_{N-1\,0} & \cdots & A_{N-1\,N-1} \end{array}\right), \tag{2.5}$$

where $A_{j\,j'}$ are $k \times k$ matrices with entries $a_{j\,j'}(s)$, $s = 0 \ldots k-1$. Clearly, these matrices are cyclic. $\mathbf{b}_j$ denotes the vector $[b_j, \ldots, b_j]^T$ of size k with equal entries; similarly, $\mathbf{c}_j$ is the vector $[c_j, \ldots, c_j]^T$.

A cyclic matrix can be reduced to a diagonal form using the DFT. Let $\mathcal{D} = \mathrm{diag}(1, \frac{1}{k}D_k \ldots \cdots \frac{1}{k}D_k)$, where D_k is the DFT matrix of size k. The number of DFT blocks in $\mathcal{D}$ is N.

Applying the transform to S, we obtain

$$\mathcal{D}\, S\, \mathcal{D}^{-1} = \left(\begin{array}{c|ccc} a_{00} & \mathbf{b}_0^T \overline{D_k} & \cdots & \mathbf{b}_{N-1}^T \overline{D_k} \\ \hline \frac{1}{k} D_k \mathbf{c}_0 & \frac{1}{k} D_k A_{00} \overline{D_k} & \cdots & \frac{1}{k} D_k A_{0\,N-1} \overline{D_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k} D_k \mathbf{c}_{N-1} & \frac{1}{k} D_k A_{N-1\,0} \overline{D_k} & \cdots & \frac{1}{k} D_k A_{N-1\,N-1} \overline{D_k} \end{array}\right).$$

The matrices $(1/k) D_k A \overline{D_k}$ are diagonal with entries on the diagonal $D_k \mathbf{a}_{j\,j'}$, where $\mathbf{a} = [a_{j\,j'}(0), \ldots, a_{j\,j'}(k-1)]$. Note that vectors $D_k \mathbf{b}_j$ and $D_k \mathbf{c}_n$ have zeros in all positions except the first one or two: $D_k \mathbf{b}_j = [k, b_j, 0, \ldots, 0]^T$, $D_k \mathbf{c}_j = [c_j, 0, \ldots, 0]^T$.

Finally, the subdivision matrix can be reduced to block diagonal form by applying a permutation. Let P be the permutation that rearranges the entries of a vector of length $kN+1$ as follows: $[0, 1, 2, 3, \ldots, Nk] \to [0, 1, k+1, \ldots, (N-1)k+1, 2, k+2, \ldots (N-1)k+2 \ldots Nk]$. Applying this permutation, we obtain

$$P\, \mathcal{D}\, S\, \mathcal{D}^{-1} P^{-1} = \mathrm{diag}\left(Z, B\left(e^{\frac{2\pi i}{k}}\right), \ldots, B\left(e^{\frac{2(k-1)\pi i}{k}}\right)\right). \tag{2.6}$$

The matrix has $k-1$ $N \times N$ blocks $B(\omega)$, where $\omega = e^{2\pi i/k}, \ldots, e^{2(k-1)\pi i/k}$. Each $B(e^{2\pi m i/k})$ has entries $[D\mathbf{a}_{j\,j'}]_m$, i.e., is composed of mth entries of DFT transforms of all vectors $\mathbf{b}_{j\,j'}$. For $m = 0$ we have to consider a larger $(N+1) \times (N+1)$ matrix Z with vectors $\mathbf{b} = [b_0, \ldots, b_{N-1}]^T$ and $(1/k)\mathbf{c} = (1/k)[c_0, \ldots, c_{N-1}]^T$ added on two sides. Note that $B(e^{2m\pi i/k}) = \overline{B(e^{2(k-m)\pi i/k})}$ and the eigenvalues of these blocks are conjugate. If an eigenvalue happens to be real and corresponds to the block $B(e^{2m\pi i/k})$ with $m \neq k/2$ it necessarily has an eigenspace of dimension at least 2. If x is its complex eigenvector obtained from an eigenvector of $B(e^{2m\pi i/k})$, a pair of real eigenvectors in this subspace can be taken to be $\Re x$ and $\Im x$. If an eigenvalue λ is complex, the two-dimensional real eigenspace corresponding to λ and $\overline{\lambda}$ is also spanned by $\Re x$ and $\Im x$.

Keeping in mind the support size of the scheme, it is easy to show that each block B also has a particular structure, for a suitable choice of numbering of vertices in each sector:

$$B(\omega) = \begin{pmatrix} B_{00}(\omega) & 0 \\ B_{10}(\omega) & B_{11}(\omega) \end{pmatrix}.$$

In this way, the size of the matrices that have to be analyzed is further reduced; for example, for the butterfly scheme considered in section 5.1, $B(\omega)$ is 6×6 and $B_{00}(\omega)$ is 3×3.

Necessary condition for tangent plane continuity of invariant schemes. Before we proceed to the analysis of specific subdivision schemes, we formulate a necessary condition for C^1-continuity. We need this condition to show that the butterfly scheme is *not* C^1-continuous for most valences.

Each eigenvalue of the subdivision matrix is an eigenvalue of a block $B(e^{2m\pi i/k})$, $m = 1 \dots k-1$ or Z. Each eigenvector can be obtained by taking an eigenvector of one of the blocks, setting the rest of the entries to 0, and transforming it using $\mathcal{D}P$. This means that the eigenvectors have symmetries that can be used to establish necessary conditions on dominant eigenvalues of the subdivision matrix.

A condition of this type was proposed in [15, Theorem 3.1]. The theorem of Peters and Reif states that the dominant eigenvalues for a subdivision scheme with an injective characteristic map necessarily have to be the eigenvalues of the blocks $B(e^{2\pi i/k})$ and $B(e^{2(k-1)\pi i/k})$. Intuitively, it appears that this is true for any "reasonable" subdivision scheme. However, it is possible to construct examples of C^1-continuous schemes with eigenvalues corresponding to the characteristic map being in other blocks. Typically, such schemes would have a noninjective characteristic map. Injectivity of a characteristic map is not strictly necessary for C^1-continuity of the scheme, contrary to Theorem 2.2 of [15]. However, the cases when the scheme is C^1-continuous and the characteristic map is not injective are quite degenerate and are unlikely to be practically useful.

A weaker version of the conditions of Peters and Reif under some additional assumptions is proved in [20]. We offer a further simplified version of the condition, which is sufficient for our purposes.

As the subdivision matrix for an invariant scheme can be reduced to the block diagonal form, each cyclic subspace of the matrix is also a cyclic subspace of one of Z, $B(\omega)$, $\omega = e^{2\pi i/k}, \dots, e^{2(k-1)\pi i/k}$.

LEMMA 2.2. *Suppose that the subdivision matrix for a subdivision scheme has a pair of dominant cyclic subspaces J_b^a, J_d^c, which either coincide and both have order 1, or are distinct and have order 0. Suppose these subspaces correspond to the blocks $B(e^{2\pi mi/k})$ and $B(e^{2\pi(k-m)i/k})$, $m \neq 1$, and the Jacobian of the characteristic map of this pair of cyclic subspaces is not identically zero. Let λ be an eigenvalue of the block $B(e^{2\pi i/k})$ and let x be a corresponding complex eigenvector.*

Suppose that for the limit map $f : U_1 \to \mathbf{R}^2$ generated by the pair $\Re x$, $\Im x$ the following two conditions hold: $f^{-1}(0) = \{0\}$, and the winding number of the curve obtained by restricting f to the boundary of U_1 is 1. Then the scheme is not C^1-continuous.

Proof. See [20] for the proof. $\square$

3. Results on regularity and injectivity.

3.1. Regularity on regular complexes. We have observed that regularity of the characteristic map can be established, if it is known that the scheme is regular on an annular region (ring) shown in Figure 2. All control vertices for a ring are regular, and the subdivision rules that are used to compute the limit surface on the ring are the rules used for the regular complex. Clearly, the ring cannot be identified with a subset of a regular complex. However, such identification can be done for each of the k segments of the ring together with its control points. Therefore, if we can prove regularity of a limit map on the regular complex, we can apply the same algorithm to prove regularity of the characteristic map for each segment.

Our method for verifying regularity is based on the observation that we can define a subdivision scheme for the vector of differences. The limit function of this scheme is the vector of partial derivatives of the characteristic map. We estimate the error of the piecewise-linear approximations produced by this scheme. From linear approximations and errors we compute upper and lower bounds for the Jacobian of the characteristic map. If these bounds have the same sign, we can conclude that the map is regular.

Our derivations are similar to the derivations in Chapters 2 and 3 of Cavaretta, Dahmen, and Micchelli [2], and those found in Dyn, Levin, and Micchelli [7]. We have to consider convergence not only of the scheme, but also of the corresponding scheme for differences, which, in general, is a *matrix subdivision scheme*. For this reason, some of the theorems in [2] have to be generalized to the matrix case. Cohen, Dyn, and Levin have developed the basic theory of univariate matrix schemes in [3]. We use *multivariate matrix subdivision schemes*, that is, we need a synthesis of the theories presented in [2] and [3]. The theory of matrix subdivision differs from the theory of scalar subdivision in a nontrivial way, when the components of the limit functions generated by the scheme are interdependent [3]. However, this case is of little interest to us: if the components of the difference scheme are interdependent, the limit surfaces are degenerate. Hence we can assume independence of components. With this assumption, the results in [2] can be readily extended to the matrix case.

Definitions. For a regular complex, the vertices can be identified with the integer points in the plane. In general, we can consider functions on the integer lattice $\mathbf{Z}^2$ in $\mathbf{R}^2$. Most of the discussion applies to integer lattices $\mathbf{Z}^s$ of arbitrary dimension with minor changes. We perform the derivations for the case $s = 2$ to simplify the presentation. We use Greek letters to denote multi-indices corresponding to the points of the lattice: $\alpha = (\alpha_1, \alpha_2)$. A stationary matrix subdivision scheme on a regular complex with the vertex set $V = \{v_\alpha | \alpha \in \mathbf{Z}^2\}$ is defined by the equation

$$(Sp)\,(v_\alpha) = \sum_\beta A_{\alpha-2\beta}\, p(v_\beta),$$

where A_α are $n \times n$ matrices and p is in $\mathcal{P}(V, \mathbf{R}^n) = \left(\ell_2^\infty\right)^n = \left(\ell^\infty(\mathbf{Z}^2) \times \dots \ell^\infty(\mathbf{Z}^2)\right)$, the space of two-dimensional sequences of n-dimensional vectors with bounded norm. In this section, we use notation p_α for $p(v_\alpha)$. As we are interested in schemes with finite support, all results can be extended to arbitrary vectors p_α in a straightforward manner (see [2]). We are primarily interested in the cases $n = 1, 2, 4$, corresponding to scalar subdivision, difference schemes, and second difference schemes, respectively.

If a subdivision scheme converges on the regular complex, there is a matrix function $\Xi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, such that any limit function $f[p]$ generated from the initial values $p \in \left(\ell_2^\infty\right)^n$, can be written as $f[p](t) = \sum_\alpha \Xi(t-\alpha)p_\alpha$, where $t \in \mathbf{R}^2$. The matrix refinable function Ξ satisfies the refinement relation

$$\Xi(t) = \sum \Xi(2t - \alpha)A_\alpha.$$

The function Ξ can be obtained as a limit of subdivision applied to initial matrix data Δ with $\Delta_\alpha = 0$ if $\alpha \neq 0$, and Δ_0 is an $n \times n$ identity matrix.

We say that a matrix scheme is *nondegenerate* if the vectors $f[p](t)$, $t \in \mathbf{R}^2$, for all t for some $p \in \left(\ell_2^\infty\right)^n$ span the whole space $\mathbf{R}^n$. It is straightforward to show, following the derivation in [2], Proposition 2.1, that for a nondegenerate matrix scheme to be convergent the following condition is necessary.

THEOREM 3.1. *For a nondegenerate matrix scheme S to be convergent it is necessary that for any $e = (e_1, e_2)$, $e_i \in \{0, 1\}$,*

$$\sum_{\alpha} A_{e-2\alpha} = I, \tag{3.1}$$

where I is the identity matrix.

A matrix subdivision scheme is *stable* (or, more precisely, L^∞-stable) [3] if the matrix refinable function Ξ corresponding to the scheme satisfies the inequalities

$$c_1 \sup_{\alpha} \left\| p_\alpha \right\|_\infty \leq \sup_{t \in \mathbf{R}^2} \left\| \sum_{\alpha} \Xi(t-\alpha) p \right\|_\infty \leq c_2 \sup_{\alpha} \left\| p_\alpha \right\|_\infty \tag{3.2}$$

for some positive constants c_1, c_2 and any $p \in \left(\ell_2^\infty\right)^n$.

Convergence condition. To analyze convergence of matrix subdivision schemes, we use the *contraction functions*. Let $D(p)$ be a real-valued nonnegative function defined on $\left(\ell_2^\infty\right)^n$. A subdivision scheme S is *contractive* relative to D if there is an integer N and a positive constant $\gamma_N < 1$ for which we have

$$D(S^N p) \leq \gamma_N D(p) \quad \text{for any } p \in \left(\ell_2^\infty\right)^n.$$

A typical contraction function has the form $\left\| \nabla p \right\|_\infty$, where ∇p denotes the vector of directional differences.

The following theorem is a direct generalization of [2, Theorem 3.1], with the proof extended without any changes from the scalar case.

THEOREM 3.2. *Let S be a matrix subdivision scheme and D a contraction function. Suppose that for some scheme B, which we call a* comparison *scheme,*

$$\left\| Sp - Bp \right\|_\infty \leq cD(p) \quad \textit{for any } p \in \left(\ell_2^\infty\right)^n,$$

where c is a constant.

If the comparison scheme is stable and converges, then S also converges.

We use three types of comparison schemes in our analysis: schemes that produce piecewise constant, piecewise linear, and piecewise bilinear limit functions.

Error estimates. Using Theorem 3.2, we can derive error estimates for the piecewise linear, bilinear, or constant approximations of the limit function of subdivision. When applied to the difference schemes, these estimates allow us to estimate the Jacobians of the maps produced by subdivision. Let L^m be the limit function obtained by applying the comparison scheme B to the control values p^m. This is our approximation. The choice of B guarantees that the limit functions of B can be computed trivially. Then we have

$$\left\| L^{m+1} - L^m \right\|_\infty = \left\| (S - B) p^m \right\|_\infty < cD(p^m).$$

Suppose that $m = kN + q$, with $0 \le q < N$; then $D(p^m) < \gamma^k D(p^q)$ and

$$\begin{aligned}
\left\|L^\infty - L^m\right\|_\infty &\le \sum_{j=0}^{\infty} \left\|L^{m+j+1} - L^{m+j}\right\|_\infty \\
&= \sum_{j=0}^{\infty} \left\|(S-B)p^{j+m}\right\|_\infty \le c\sum_{j=0}^{\infty} D\left(p^{m+j}\right) = c\sum_{i=1}^{\infty}\sum_{q=1}^{N} D\left(p^{m+iN-q}\right) \\
&= c\sum_{i=1}^{\infty}\left(\sum_{q=1}^{N-1} D\left(p^{m+iN-q}\right) + D\left(p^{m+(i-1)N}\right)\right).
\end{aligned}$$

If $m \ge N$,

$$\left\|L^\infty - L^m\right\|_\infty \le \frac{c}{1-\gamma}\left(\gamma \sum_{q=1}^{N-1} D\left(p^{m-q}\right) + D\left(p^m\right)\right). \tag{3.3}$$

Estimating c and γ. To use the estimates (3.3), we need to compute the constants γ_N and c. These constants clearly depend on the choice of the contraction function and on the choice of the comparison scheme. We use the contraction functions of the form $\left\|\nabla p\right\|_\infty$, where $\nabla:\ \left(\ell_2^\infty\right)^n \to \left(\ell_2^\infty\right)^{2n}$ is a difference operator, which assigns to each vector p of length n the vector of differences in two independent directions of length $2n$. Specific choice of ∇ can be adapted to the scheme; the simplest choice is $\left[\left(\nabla p_{2m+i}\right)_\alpha\right]^m = \left[p_{\alpha+e_i}\right]^m - \left[p_\alpha\right]^m = \Delta_{e_i}\left[p_\alpha\right]$, where e_i, $i = 0, 1$, is one of the multi-indices $(0,1)$ and $(1,0)$. The superscript m of p denotes the component of p, $m = 1 \dots n$. Other possible choices include replacing vectors e_i by displacements in other directions. Specific choices are discussed in sections 5.1–5.3. Whenever a scheme S converges, there is a difference scheme S' satisfying the commutation formula

$$\nabla S p = S' \nabla p \tag{3.4}$$

for any p. If $\left\|S'^N\right\|_\infty < 1$ for some N, we can use $\left\|\nabla p\right\|_\infty$ as a contraction function, because $\left\|\nabla S^N p\right\|_\infty \le \left\|S'^N\right\|_\infty \left\|\nabla p\right\|_\infty$. While in most cases the simplest choice of ∇ is theoretically possible, in practice the constant N can be quite large. In certain cases, such as the butterfly scheme, different contraction functions yield better results.

For contraction functions of the type described above, computation of γ_N is reduced to computing the (matrix) difference scheme S' and its sup norm. Lemma 2.3 from [2], which directly generalizes to the matrix case, yields formulas for computing the matrix Laurent polynomial of the difference scheme.

Formulas for γ. We describe the formulas for computing the difference scheme in the bivariate case, for the simplest comparison function $D(p) = \left\|\nabla p\right\|_\infty$, $p \in \left(\ell_2^\infty\right)^n$, with ∇p described above. In a more explicit form,

$$D(p) = \sup_\alpha \max_m \left(\left[\Delta_{(1,0)} p_\alpha\right]^m, \left[\Delta_{(0,1)} p_\alpha\right]^m\right).$$

Let $A(z)$ be the Laurent polynomial with matrix coefficients of a matrix scheme S, with $z = (z_1, z_2)$. The commutation formula in the z-domain can be written as

$$\begin{aligned}
\begin{pmatrix} (z_1-1)A(z)P(z) \\ (z_2-1)A(z)P(z) \end{pmatrix} &= A'(z)\begin{pmatrix} (z_1^2-1)P(z) \\ (z_2^2-1)P(z) \end{pmatrix} \\
&= \begin{pmatrix} \left(T_{11}(z)(z_1^2-1) + T_{12}(z)(z_2^2-1)\right)P(z) \\ \left(T_{21}(z)(z_1^2-1) + T_{22}(z)(z_2^2-1)\right)P(z) \end{pmatrix}
\end{aligned} \tag{3.5}$$

for any p, where $P(z)$ denotes the Laurent polynomial corresponding to p. The matrix Laurent polynomial A' with the coefficients of size $2n \times 2n$ is the Laurent polynomial of the difference scheme S'. Each coefficient of A' is composed out of the coefficients of $n \times n$ blocks $T_{ij}(z)$. The blocks T_{ij} are not defined uniquely and can be computed in a variety of ways. A method suggested by Lemma 2.3 of [2] yields the following formulas for a decomposition $R(z) = (z_1^2 - 1)T_1(z) + (z_2^2 - 1)T_2(z)$:

$$\begin{aligned} T_1(z) &= \frac{(1-z_2)R(z_1,-1) + (1+z_2)R(z_1,1)}{2(z_1^2-1)}, \\ T_2(z) &= \frac{2R(z) - (1-z_2)R(z_2,-1) - (1+z_2)R(z_1,1)}{2(z_2^2-1)}. \end{aligned} \tag{3.6}$$

Whenever a scheme with the Laurent polynomial A satisfies Theorem 3.1, it follows that $A(-1,-1) = A(1,-1) = A(-1,1) = 0$. Then $T_1(z)$ and $T_2(z)$ are guaranteed to be matrix Laurent polynomials, rather than rational functions, for $R(z) = (z_i - 1)A(z)$, $i = 1, 2$.

These formulas can be used to compute the blocks for each line in (3.5). Note that the formulas are asymmetric, and care must be taken to choose the order of variables z_1 and z_2 to obtain better estimates. The rule of thumb is that the norm of the off-diagonal blocks should be small; then for schemes with factorizable Laurent polynomials the $2n \times 2n$ difference scheme can be decomposed into two $n \times n$ schemes.

The same method can be used to compute difference schemes for other choices of ∇ that we use. Once S' is known, we compute γ_N as $\left\|S'^N\right\|_\infty$, which is equal to $\left\|A(z)A(z^2)\dots A(z^{2^N})\right\|_\infty$ using the formula for the sup-norm of a bivariate polynomial with $n \times n$ matrix coefficients $A_{ij} = [a^{lm}]_{ij}$:

$$\left\|C(z)\right\|_\infty = \max_{e_1,e_2\in\{0,1\}, m=1\dots n} \sum_{l=1\dots n} \sum_{ij} \left|c^{lm}_{2i+e_1\, 2j+e_2}\right|. \tag{3.7}$$

Computing c. To compute c, it is sufficient to observe that if the comparison scheme B is nondegenerate and convergent, then $B(z) - A(z)$ can be always decomposed in a way similar to $R(z)$ in (3.6). This means that we can represent $B - S$ as $\tilde{S}\nabla$, for some $2n \times 2n$ matrix scheme $\tilde{S}$, which leads to the estimate of c as $\left\|\tilde{S}\right\|_\infty$.

Summary of the estimates. We have obtained the following estimates for the pairs of constants (γ_N,c), and (γ_N^D,c^D), characterizing the errors of the approximations of the limit functions and its derivatives respectively via (3.3). Let S is a convergent scalar subdivision scheme, satisfying commutation formula $\nabla S p = S'\nabla p$, where $p \in \ell_2^\infty$, and S' is a 2×2 matrix scheme. Let B be a scalar comparison scheme, and $(B - S)p = \tilde{S}\nabla p$. Then we can take $\gamma_N = \left\|S'^N\right\|_\infty$, and $c = \left\|\tilde{S}\right\|_\infty$. If the difference scheme $2S'$ converges and satisfies the commutation formula $\nabla 2S'p = S''\nabla p$, where S'' is a 4×4 matrix subdivision scheme, $\gamma_N^D = \left\|S''\right\|_\infty$. If the difference scheme $2B'$ corresponding to a comparison scheme B is also a comparison scheme, and $(2S' - 2B')p = \tilde{S}'\nabla p$, then $c^D = \left\|\tilde{S}'\right\|_\infty$.

Limitations of the method. While for any C^1-continuous subdivision scheme and for all operators ∇ that we use the difference schemes S' and S'' are defined, it is not guaranteed that there is N such that $\left\|S'^N\right\|_\infty < 1$ or $\left\|S''^N\right\|_\infty < 1$. This is the main limitation on the applicability of our method. Even for a convergent scheme it might be possible that $N \left\|S'^N\right\|_\infty > 1$ for all N. However, note that a sharper estimate can be made for γ: we are interested in the action of the difference scheme not on arbitrary elements of $\left(\ell_2^\infty\right)^{2n}$, but only on the elements that have the form

∇p for some $p \in \left(\ell_2^\infty\right)^n$. Therefore, we can use $\left\| S'^N|_\nabla \right\|_\infty$ instead of $\left\| S'^N \right\|_\infty$, where $|_\nabla$ denotes restriction to the subspace of differences. If we use this norm, then the following theorem holds.

THEOREM 3.3. *A matrix subdivision scheme S is nondegenerate, convergent, and stable if and only if the there is a difference scheme S' satisfying the commutation formula* 3.4 *and such that* $\lim_{N\to\infty} \left\| S'^N|_\nabla \right\|_\infty = 0$.

The proof of the theorem is identical to the proof of [2, Theorem 2.3]. The additional condition (stability of the scheme) is quite weak and in all cases of interest is likely to be satisfied.

Computing the norm $\left\| \cdot|_\nabla \right\|_\infty$ is possible for schemes with finite support, but is substantially more complicated than computing the sup norm. For schemes with factorizable Laurent polynomials for suitable choices of ∇, $\left\| \cdot|_\nabla \right\|_\infty$ is often equal to $\left\| \cdot \right\|_\infty$ (for example, this is true for tensor product schemes). However, for certain schemes with nonfactorizable polynomials computing $\left\| \cdot|_\nabla \right\|_\infty$ may be the only option.

3.2. Injectivity of the characteristic map. In general, it is difficult to establish injectivity of a map defined as a limit of a subdivision process. Even if the Jacobian of a map is nonzero everywhere, only local injectivity is guaranteed. However, the special structure of the characteristic maps allows one to reduce the injectivity test to computing the index of a curve, a relatively simple and fast operation; for example, the index can be computed counting the number of intersections of the curve with a line.

A step in this direction was made by Peters and Reif [15]. However, their method still required closed-form expression for the derivative of the characteristic map along a line and was formulated only for schemes invariant with respect rotations of k-regular complexes (see section 2.4).

The characteristic map can be continuously extended using scaling relations to the whole plane. The proof of this fact is straightforward but tedious; it can be found in [20]. Moreover, if the scheme is C^1-continuous on regular complexes, then the extension is differentiable on each of the k sectors of the k-regular complex and has continuous one-sided derivatives on the boundaries of segments, excluding zero. In the following theorem we assume that the characteristic map is defined on $\mathbf{R}^2$. Recall that although the map has only one-sided derivatives defined on the boundaries of sectors, the Jacobian is well-defined everywhere (section 2.1).

THEOREM 3.4. *Suppose a characteristic map $\Phi = (f_a, f_c)$ satisfies the following conditions:*

1. *the preimage $\Phi^{-1}(0)$ contains only one element,* 0;
2. *the characteristic map has a Jacobian of constant sign everywhere on $\mathbf{R}^2$ except zero.*

Then the extension of the characteristic map is a surjection and a covering away from 0*. In particular, if the winding number with respect to the origin of the image $\Phi(\gamma)$ of a simple curve is* 1*, the characteristic map is injective and the scheme is C^1-continuous.*

Proof. Three cases are possible: the characteristic map is defined by a pair of real eigenvectors, by two generalized eigenvectors from the same Jordan block corresponding to a real eigenvector, or by the real and imaginary parts of an eigenvector corresponding to a complex eigenvalue.

A pair of real eigenvectors. In the first case the components of the characteristic map satisfy the scaling relations of the simplest form $f_a(y/2) = \lambda_a f_a(y)$, $f_c(y/2) = \lambda_c f_c(y)$.

First, we establish the following important fact: if a characteristic map satisfies the first two conditions of the theorem, then the map is continuous at infinity.

Consider two circles of radii r and $2r$ centered at 0 in the domain of Φ. The image $\Phi(R)$ of the ring R bounded by the two circles is compact and does not contain 0. Thus, there is a constant $M > 0$ such that for any point p in the ring $\|\Phi(p)\| \geq M$.

Consider any point q in the domain of Φ. There is a number $k \in \mathbf{Z}$ such that $2^k q$ is contained in the ring R. Thus, by scaling relations, $\|\Phi(q)\| > \min(|\lambda_a|, |\lambda_c|)^k M$. Clearly, as $\|q\| \to \infty$, $k \to \infty$, and for any C there is C' such that if $\|q\| > C'$, $\|\Phi(q)\| > C$.

Consider the stereographic map P from the plane into the sphere without one point. The map Φ corresponds to a map on the sphere: $\Phi_S = P\Phi P^{-1} : S^2\backslash\{N\} \to S^2$, where N is the center of projection. From the continuity of Φ at infinity it follows that if we extend the mapping by setting $\Phi_S(N) = N$, we get a continuous mapping. As we have assumed that the Jacobian of the characteristic map has constant sign where it is defined, the mapping is also a local homeomorphism away from 0. The sphere is compact, thus its image is compact, hence closed, i.e., contains its boundary. But under local homeomorphism the points on the boundary of the image can be images only of the points of the boundary of the domain. Therefore, the only points that can be contained in the boundary of the image are 0 and N. We conclude that the image has no boundary, i.e., the mapping is surjective.

Finally, for any q set $\Phi_S^{-1}(q)$ is finite: if it were not finite, it would have a limit point (S^2 is compact). As $\Phi_S^{-1}(q)$ is a discrete set for any local homeomorphism, the only limit points that it may have are 0 and N. But $\Phi(0) = 0$ and $\Phi(N) = N$, so this is impossible. We conclude that for any point q the set $\Phi_S^{-1}(q)$ is finite. As any point $y \in \Phi_S^{-1}(q)$, $q \neq 0, N$ has a neighborhood $U(y)$ such that $\Phi_S|_{U(y)}$ is a homeomorphism, then the intersection of all neighborhoods $V = \Phi_S(U(y))$ has inverse image consisting of disjoint homeomorphic images of V. This proves that Φ_S is a covering away from 0.

Two generalized eigenvectors. The case of the characteristic map generated by imaginary and real parts of a complex eigenvector corresponding to a complex eigenvalue is similar to the case of two real eigenvectors; we proceed directly to the proof for the case of two generalized eigenvectors from a single Jordan block $\Phi = (f_0, f_1)$, satisfying $f_0(\frac{y}{2}) = \lambda f_0(y)$ and $f_1(\frac{y}{2}) = \lambda f_1(y) + f_0(y)$.

From these equations we immediately obtain

$$\Phi(2^s y) = \frac{1}{\lambda^s} \begin{pmatrix} 1 & 0 \\ 1 & -s/\lambda \end{pmatrix} \Phi(y) = \frac{1}{\lambda^s} T\Phi(y). \tag{3.8}$$

Consider the image of a circle γ of radius r centered at 0. Let $\mathrm{Int}(\gamma)$ be the interior domain of the simple curve γ. As $\Phi^{-1}(0)$ by assumption is $\{0\}$, then 0 is an interior point of the image of $\mathrm{Int}(\gamma)$ and there is an open disk centered at 0 of some radius r', which is contained in $\Phi(\mathrm{Int}(\gamma))$. For any s the image of the disk bounded by $2^s\gamma$ is determined by (3.8). It can be obtained from the image of the disk bounded by γ by affine transform $\frac{1}{\lambda^s}T$ from (3.8). If a disk D_r of radius r is contained in $\Phi(\mathrm{Int}(\gamma))$, then the interior of the ellipse $\frac{1}{\lambda^s}TD_r$ is contained in $\Phi(\mathrm{Int}(2^s\gamma))$. We can estimate the length of the minor axis of this ellipse: it can be represented parametrically as $\left(\frac{r}{\lambda^s}\cos t, \frac{r}{\lambda^s}(\sin t - \frac{s}{\lambda}\cos t)\right)$. The square of the distance from 0 to a point on the ellipse is

$$\frac{r^2}{\lambda^{2s}}\left(\cos^2 t + \left(\sin t - \frac{s}{\lambda}\cos t\right)^2\right) = \frac{r^2}{\lambda^{2s}}\left(1 + \frac{s^2}{2\lambda}(\cos 2t + 1) - \frac{s}{\lambda}\sin 2t\right).$$

This quantity can be estimated from below by $\left(r^2/\lambda^{2s}\right)\left(1+s^2/\lambda - s/\lambda\right)$. As $\lambda < 1$, the length of the minor axis increases with s for sufficiently large s. We conclude that as $s \to \infty$, the image of the exterior of $2^s\gamma$ is arbitrarily far from zero, and Φ is continuous at infinity. Then the rest of the argument of the previous case applies.

Finally, our covering is injective if and only if the winding number of the image of a simple curve around zero is 1. This fact can be seen by looking at the fundamental groups of the domain and the image. The assumptions guarantee that both have fundamental group $\mathbf{Z}$. As for a covering the fundamental group of the covering space is a subgroup of the fundamental group of the base space, with a monomorphism induced by the covering map. A simple curve around zero is the generating element of the fundamental group of the domain. Thus, the mapping of fundamental groups is an isomorphism which is necessary and sufficient for the covering mapping to be an injection, if and only if the simple curve maps to a curve homotopic to a simple curve, i.e., one with winding number 1. $\square$

Computing the winding number. In general, we do not have a closed-form expression for any curves on the limit surface. One way to compute the winding number of a curve is to choose a sufficiently close linear approximation and compute the winding number of the approximation. The following proposition can be easily proved (see [20] for details).

PROPOSITION 3.5. *Let $\gamma(t)$ be a curve in the domain of Φ, and let L_m be a piecewise linear approximation to Φ. Suppose for some ϵ $\sup_t \|\Phi(\gamma(t)) - L_m(\gamma(t))\| \leq \epsilon$ and $\inf_t \|\Phi(\gamma(t))\| \geq 2\epsilon$. Then the winding number of $L_m(\gamma(t))$ with respect to zero is equal to the winding number of $\Phi(\gamma(t))$.*

As subdivision computes linear approximations to the surface, and the approximation estimates are known, we can use this proposition to compute the winding number.

4. Algorithms.

In this section we describe the algorithms for verification of C^1-continuity of subdivision near extraordinary points based on the theorems presented in sections 3.1 and 3.2. Two algorithms are used to analyze C^1-continuity of a scheme near an extraordinary point of a fixed valence: the first one verifies regularity, the second verifies injectivity. We give a brief description of the algorithms; more details can be found in [20]. The source code is available from the author.

We assume that the eigenvectors and eigenvalues of the subdivision matrix defining the characteristic map are known with guaranteed precision: if x is a component of an eigenvector or an eigenvalue, it is represented by a pair of exactly representable numbers $[x_d, x_u]$ such that $x_d \leq x \leq x_u$.

All calculations are performed in interval arithmetics, which makes it possible to obtain guaranteed bounds on the computed quantities, despite using finite-precision arithmetics. In this section, all underlined variables are intervals, with arithmetic operations defined following [14].

4.1. Verification of C^1-continuity for a fixed valence.

C^1-continuity is verified by checking regularity and injectivity of the characteristic map on a ring, as described in section 2.

First, we verify *regularity*, computing successive linear approximations to the characteristic map and using error estimates of section 3.1 to estimate the range for the Jacobian. If the computed bounds are on the same side of zero, this guarantees that the Jacobian has constant sign on the domain. To guarantee that the condition $\Phi^{-1}(0)$ of Theorem 3.4 and the assumption of Proposition 3.5 are satisfied, it is necessary to verify that the image of the characteristic map on the ring is sufficiently

```
TestRegular( G^m, γ^D, c^D )

  Compute ε_ij, i, j = 1, 2, using (3.3)
  J_min := [+∞, +∞], J_max := [−∞, −∞]
  foreach vertex v_ij ∈ vertices(G^m)
    d^1 := G^m(i+1, j) − G^m(i, j)
    d^2 := G^m(i, j+1) − G^m(i, j)
    compute 16 numbers J_l, l = 1...16,
    choosing signs in (d^1_1 ± ε_11)(d^2_2 ± ε_22) − (d^1_2 ± ε_12)(d^2_1 ± ε_21)
    J_min := min(J_min, J_l, l = 1...16)
    J_max := max(J_max, J_l, l = 1...16)
  endforeach
  if 0 ∉ J_min and 0 ∉ J_max
     and J_min and J_max have the same sign
   then return true
  return undefined
```

FIG. 4. *The algorithm for testing regularity.*

far from zero. The algorithm is straightforward, and we omit the detailed description. Finally, we compute *the winding number* of the curve obtained by restricting the linear approximation to the characteristic map to the boundary of the domain. If the winding number is 1, this completes the proof that the characteristic map is injective and regular.

In the descriptions of algorithms, $\underline{G^m}$ denotes the control mesh of the characteristic map after m subdivision steps. The components of the control points are stored in interval representation. The numbers and ϵ_{ij}, $i, j = 1, 2$, are the estimates of the error of the approximation by divided differences to the derivatives of the characteristic map, computed using the right-hand side of (3.3) from $\underline{G^m}$, γ^D, and c^D, the convergence constants defined by formulas in section 3.1.

Regularity. Once we know the error in the approximation of the derivatives by the divided differences, we can estimate the Jacobian. Observe that the Jacobian $J[\partial_1 f_1, \partial_2 f_2, \partial_2 f_1, \partial_1 f_2] = \partial_1 f_1 \partial_2 f_2 - \partial_2 f_1 \partial_1 f_2$ is a bilinear function of the derivatives. If the intervals for the derivatives are known, the Jacobian can be regarded as a bilinear function on a four-dimensional cube, and it attains its minimal/maximal value at a vertex of the cube. We present a slightly simplified version of the algorithm, which does not detect the situation when the Jacobian is guaranteed to change sign, and the map is verifiably nonregular. A complete version can be found in [20]. In the pseudocode shown in Figure 4, $d^s = [d_1^s, d_2^s]$, $s = 1, 2$ are discrete approximations of the two directional derivatives of the characteristic map; $\underline{J_{min}}$ and $\underline{J_{min}}$ are intervals containing the estimates of the maximal and minimal value of the Jacobian of the characteristic map. We use standard interval arithmetic definitions of algebraic operations on interval as well as *min* and *max* [14].

Computing the winding number. While the simplest approach to this problem is to count the number of intersections with a straight line, numerically this is not the best choice when the curve is piecewise-linear. Instead, we choose a different approach:

```
ComputeProj(G^i)
  projLength := 0
  for every segment s of the curve
      n^s  := head(s)/max( head(s)_1, head(s)_2)
      n^f  := tail(s)/max( tail(s)_1, tail(s)_2)
      if |Sides(n^s)| > 2  or |Sides(n^f)| > 2
       then return fail
      intervSides  := Sides(n^s) ∪ Sides(n^f)
      if |intervSides| > 2  then  return undefined
      if 1 ∈ intervSides  then projLength += n^f_2 − n^s_2
      if 2 ∈ intervSides  then projLength += n^s_1 − n^f_1
      if 3 ∈ intervSides  then projLength += n^s_2 − n^f_2
      if 4 ∈ intervSides  then projLength += n^f_1 − n^s_1
  endfor
  return projLength
```

FIG. 5. *The algorithm for computing the projected length of a curve.*

we observe that the winding number for a piecewise-linear curve can be computed as 1/4 of the sum of signed lengths of projections of segments onto a unit square centered at zero. As the coordinates of the vertices are represented by intervals, the actual calculation becomes somewhat more complicated. For each interval endpoint of the segment we determine the sides of the square on which the endpoint may be projected. We require the calculation to be sufficiently precise (i.e., the size of the intervals for the points to be sufficiently small) for the total number of sides intersecting the projection of the interval to be no more than two.

In the algorithm shown in Figure 5, `head` and `tail` return the endpoints of a segment of the curve, subscripts 1 and 2 denote the coordinates, and the function `sides(`x`)` returns the set of sides (identified by numbers 1, ... ,4) to which a point x with interval coordinates is projected.

The algorithms described in this section are quite efficient—even in the case of the butterfly scheme, which required six subdivision levels to verify C^1-continuity of a single valence, the execution time per valence was about seven seconds on a 300MHz Pentium II.

4.2. Verification of C^1-continuity for all valences. The algorithms of the previous section allow us to prove that a scheme is C^1-continuous for any given valence. We have made only weak assumptions about invariance of the schemes (rotational invariance is not required), and we have not assumed any relations between the subdivision rules used near extraordinary vertices of different valences. Although one can verify C^1-continuity for a number of valences that is sufficiently large for all practical purposes, as was done, for example, in [19] and [15], this approach is not satisfying theoretically.

We propose an algorithm that verifies C^1-continuity for all sufficiently high valences. Our approach to analysis of subdivision for large valences applies to schemes invariant with respect to rotations of k-regular grids around the extraordinary vertex.

In this case, the segments of the characteristic map are identical, and the analysis has to be performed for a single segment. As the valence grows, the control points of a segment approach a degenerate configuration for which all control points are on a single line. However, by rescaling the control points in one direction by $1/\sin(2\pi/k)$, where k is the valence, we typically remove the singularity. Because subdivision is affine-invariant, verifying regularity and injectivity of the rescaled characteristic map is equivalent to verifying injectivity and regularity of the original map. For all common subdivision rules, when the coefficients are defined as functions of $\cos(2\pi/k)$ and $\sin(2\pi/k)$, as k approaches infinity, the control points of the rescaled segment approach a nondegenerate limit configuration. More precisely, assume that the eigenvalues λ_a and λ_b and eigenvectors e_a and e_b defining the characteristic map are values of continuous functions of a variable c, evaluated at discrete points $c = \cos(2\pi/k)$, $k = 3 \dots$. Further, assume that these functions can be computed in the interval form: given an interval of values of c, we can compute an interval of values of $\lambda_{a,b}$.

Recall that all algorithms that we have described operate with interval representations. Let $\underline{e_a}$ and $\underline{e_b}$ be the vectors with interval components obtained by evaluating $e_a(c)$ and $e_b(c)$ on the interval $\underline{c} = [1-\epsilon, 1]$. If we have verified that the limit map defined by $\underline{e_a}$ and $\underline{e_b}$ is injective for these interval eigenvectors, we have verified that it is injective for any valence for which $\cos(2\pi/k) > 1-\epsilon$. Thus, we obtain a proof of C^1-continuity for all valences greater than some k_0 at no additional cost — all we have to do is to choose the value of $\underline{c}$ to be $[1-\epsilon, 1]$.

Finally, we observe that it is not always possible to represent an eigenvalue of the subdivision matrix as an explicit function of c. For example, for Kobbelt's scheme the characteristic polynomial has degree 6 and is not factorizable. However, the eigenvectors can always be represented as explicit functions of the eigenvalues and coefficients of the subdivision scheme. Therefore, the problem is reduced to computing the eigenvalue as a function of $\underline{c}$ with guaranteed intervals. While there are always cases when this is difficult if at all possible, it appears to be an achievable goal — see section 5.2 for an example.

5. Analysis of specific schemes. In this section we use our algorithms to analyze several subdivision schemes: butterfly, modified butterfly, Kobbelt's scheme, Catmull–Clark, and Loop schemes. C^1-continuity of the first three schemes was not previously established. C^1-continuity of the last two schemes was analyzed for large ranges of valences; we present analysis for all valences.

5.1. Analysis of the butterfly and modified butterfly schemes. The Butterfly subdivision scheme was proposed by Dyn, Levin, and Gregory in [6]. In [7], it was proved that the scheme produces C^1-continuous limit functions for regular meshes. Here we present an analysis of the scheme near extraordinary vertices. It turns out that for valences $k = 3$ and $k \geq 8$ the scheme is not C^1-continuous. We also show that the modified butterfly scheme [24] is C^1-continuous for all valences.

Definition of the schemes. The butterfly scheme is an interpolating scheme: once a vertex is added to the complex, the control point corresponding to the vertex does not change. In [6], the coefficients of the scheme are parameterized by a parameter w. The scheme has maximal approximation order for $w = 1/16$; we analyze the scheme for this value of w. The mask of the subdivision rule for newly inserted vertices is shown in Figure 6 on the left. The attractive feature of the scheme is its simplicity: the rules are the same for all vertices. However, as we will prove, the scheme does not produce C^1-continuous surfaces.

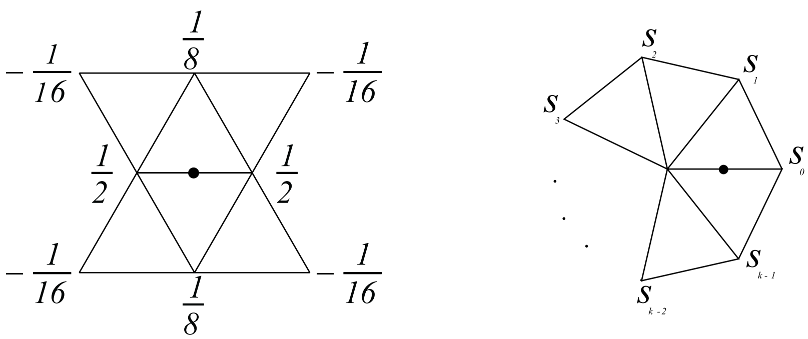

FIG. 6. *The masks of the butterfly and modified butterfly schemes.*

In [24] we have proposed a modification of the butterfly scheme, which does not have this problem. The rule for the immediate neighbors of an extraordinary vertex is modified in such a way that the spectrum of the subdivision matrix is similar to the spectrum of the subdivision matrix for valence 6, i.e., has eigenvalues 1, 1/4 in block $B(0)$, 1/2 in blocks $B(e^{2\pi i/k})$ and $B(e^{2(k-1)\pi i/k})$, and 1/2 in blocks $B(e^{4\pi i/k})$ and $B(e^{2(k-2)\pi i/k})$. The rest of the eigenvalues should be less than 1/8. In order to achieve this, we use a mask with coefficients $s_0, \ldots, s_{k-1}$, as shown in Figure 6 on the right. Note that this mask is asymmetric. For vertices on levels finer than 0, this is not a problem: we are modifying coefficients of the scheme only for neighbors of extraordinary vertices, and only one of the two neighbors can be extraordinary after one subdivision step. On the top level both neighbors can be extraordinary. The choice that we make on the top level does not affect C^1-continuity. We make an *ad hoc* choice to take the average of the results produced by each of the two possible choices. For $K \geq 5$ the coefficients are $s_j = (1/k)\,(1/4 + \cos(2\pi/k) + 1/2\cos(4\pi/k))$, $j = 0, \ldots, k-1$. For $k = 3$ we use $s_0 = 5/12$, $s_{1,2} = -1/12$, and for $k = 4$, $s_0 = 3/8$, $s_2 = -1/8$, $s_{1,3} = 0$. The properties of the scheme are discussed in greater detail in [24, 20].

Subdivision matrices. For the butterfly scheme, the size of the blocks $B(\omega)$ (section 2.4) is 6×6. There is no need to consider the block number zero separately, as it can be split into a trivial 1×1 block and a 6×6 block.

All blocks have eigenvalues 0 and $-1/16$, the eigenvalue $-1/16$ having multiplicity 2 for each block.

The other eigenvalues are eigenvalues of the upper left 3×3 subblock $B_{00}(\omega)$. Let $c = \Re\omega = \cos(2m\pi/k)$. Then the subblocks $B_{00}(\omega)$ have the form

$$B_{00}(\omega) = \begin{pmatrix} \frac{1}{2} + \frac{1}{4}c - \frac{1}{8}(2c^2 - 1) & -\frac{1}{16}\,\overline{\omega} - \frac{1}{16} & 0 \\ \frac{1}{2} + \frac{1}{2}\omega - \frac{1}{16}\overline{\omega} - \frac{1}{16}\,\omega^2 & \frac{1}{8} & -\frac{1}{16} - \frac{1}{16}\,\omega \\ 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$P(\lambda, d) = \lambda^3 + \left(-1/4 - 3/2\,d + d^2\right)\lambda^2 + \left(\frac{1}{64} + \frac{23}{64}\,d - 3/16\,d^2\right)\lambda - \frac{1}{64}\,d,$$

where $d = c^2$.

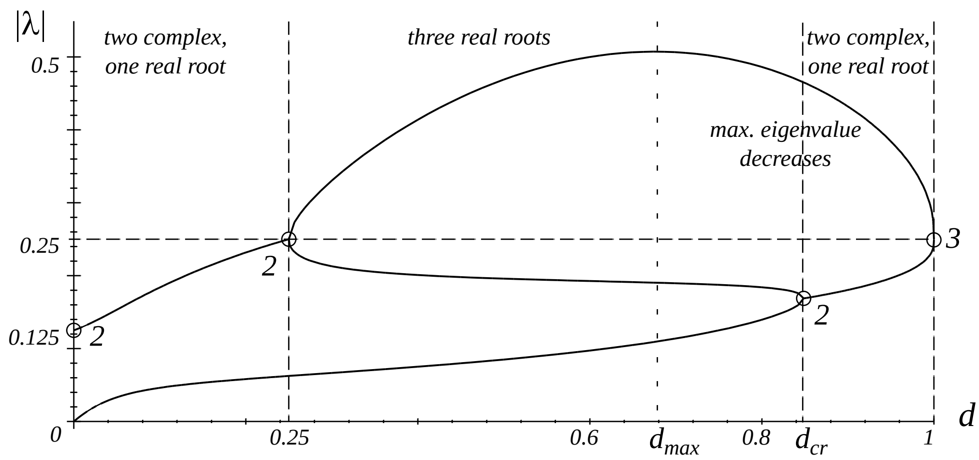

FIG. 7. *The magnitudes of the eigenvalues of the block* $B_{00}(\omega)$ *as functions of* $d = \cos^2(\omega)$ *for the butterfly scheme. The vertical lines indicate the values of* d *for which the matrix has a nontrivial Jordan structure. The circles indicate eigenvalues with multiplicity greater than* 1*, the numbers next to the circles are multiplicities.*

For the modified butterfly scheme, the first row of $B_{00}(\omega)$ is replaced by $[\lambda(\omega), 0, 0]$, where $\lambda(\omega)$ is the prescribed largest eigenvalue for the block $B(\omega)$. The roots of the characteristic polynomial in this case are $\lambda(\omega)$, 0, and 1/8.

Convergence rates. For both schemes, which coincide on the regular complexes, we use the contraction function $\|\nabla p\|_\infty$, with the difference operator $\nabla : \left(\ell_2^\infty\right) \to \left(\ell_2^\infty\right)^2$, $\nabla = [\Delta_{(1,0)}, \Delta_{(0,1)}]$ (cf. [7]). For the difference scheme acting on the vectors $[\Delta_{(1,0)}p, \Delta_{(0,1)}p]$ we use the difference operator $\nabla' = [\Delta_{(0,1)}, \Delta_{(1,1)}, \Delta_{(1,0)}, \Delta_{(1,1)}] : \left(\ell_2^\infty\right)^2 \to \left(\ell_2^\infty\right)^4$. The convergence constants for the butterfly scheme are $c = 1/2$, $\gamma_1 = 7/8$, $\gamma_2 = 31/64$, $\gamma_3 = 261/1024$. The constants for the difference scheme are $c^D = 1/2$, $\gamma_1^D = 1$, $\gamma_2^D = 7/8$, $\gamma_3^D = 11/16$.

We have chosen to use γ for $N = 3$ levels of subdivision, as after three levels of subdivision the convergence factor *per level* is close enough to what we would get if we were to use more levels: $\gamma_3^{1/3}$ is close to $\gamma_4^{1/4}$.

These estimates indicate that the convergence for derivatives is quite poor: γ stays close to 0.9 per level. However, this is a worst-case estimate and in practice the scheme converges much faster. The reason for this is that for schemes with negative coefficients, the "worst case" happens when the initial values have changing signs, i.e., consist primarily out of high frequency components, which is uncommon for surfaces.

Analysis of the butterfly scheme. The following proposition summarizes the information about the roots of the characteristic polynomial that we need to analyze the scheme. Whenever an approximate value of a root is given, it is implied that the precision is given by the last digit. Roots as functions of c are shown in Figure 7. The proof is straightforward, but tedious. An outline is presented in Appendix A.1. A detailed proof with calculations can be found in [21] in the form of a Maple worksheet.

PROPOSITION 5.1.

1. *For* $d \in (1/4, d_{cr})$, $d_{cr} \approx 0.84868$ *there are three real roots. The largest root*

is real, and is greater than $1/4$*; the other roots are less than* $1/4$*.*

2. *For* $d \in (0, 1/4) \cup (d_{cr}, 1)$ *there are two complex and one real roots. The magnitude of the complex roots is always less than* $1/4$*. For* $d \in (d_{cr}, 1)$ *the real root is greater than* $1/4$*; for* $d \in (0, 1/4)$*, it is less than* $1/4$*.*

3. *For* $d = 0$*, the characteristic polynomial has a double root* $1/8$ *and a single root* 0*.*

For $d = 1/4$*, there is a double root* $1/4$ *and a single root* $1/16$*.*

For $d = d_{cr}$*, there is a single root* $\lambda_1 \approx 0.46503$ *and a double root* $\lambda_2 \approx 0.16887$*.*

For $d = 1$*, there is a triple root* $1/4$*.*

For $d \in (d_{max}, 1)$*,* $d_{max} \approx 0.67600$*, the largest root decreases as a function of* d*.*

Using the information about the eigenvalues given by Proposition 5.1, we conclude that for $k \geq 10$, the maximum eigenvalue of the second block $B(4\pi/k)$ is greater than the largest eigenvalue of the first block $B(2\pi/k)$. Evaluating eigenvalues for $k = 3 \ldots 9$ directly, we can see that this is also true for $k = 8$ and $k = 9$. For $k = 3$, $(d = 1/4)$, the blocks $B(2\pi/3)$ and $B(4\pi/3)$ have double eigenvalues $1/4$.

Once the eigenvalues are known, the eigenvectors corresponding to the pair of dominant eigenvalues can be found from the complex eigenvector of $B(\omega)$ given by

$$e_0(\omega) = \left[\lambda, 1, \frac{(\omega+1)((2c-9)\lambda+1)}{2-16\lambda}\right].$$

To compute the pair of eigenvectors defining the characteristic map, we first extend $e(\omega)$ to an eigenvector of a full 6×6 block using $v(\omega) = (\lambda I - B_{11})\, B_{10} e_0$. Then the complete complex eigenvector $\mathbf{e}$ for valence k is

$$[0, e(0), e(2\pi/k), e(4\pi/k), \ldots, e(2(k-1)\pi/k)].$$

From this vector we obtain two real vectors $\Re\mathbf{e}$ and $\Im\mathbf{e}$ defining the characteristic map.

The algorithms that we use to check regularity of the characteristic map with minor changes can be used to verify the assumptions of Lemma 2.2. For valences $k = 4, 5, 7$ we use these algorithms to show injectivity and regularity of the characteristic map.

Our findings are summarized in the following proposition.

PROPOSITION 5.2. *The butterfly scheme is* C^1*-continuous for valences* $k = 4, 5, 6, 7$*; it is not* C^1*-continuous for any other valence and is not tangent plane continuous for* $k = 3$*.*

While the scheme is not formally C^1-continuous, the actual appearance of the surfaces generated by the scheme is not obviously nonsmooth: for valences other than three, the scheme produces tangent plane continuous surfaces, and the "twist" that makes the surfaces non-C^1-continuous is a relatively subtle effect. For more details, see [20].

Modified butterfly scheme. As the eigenvalues of the subdivision matrix in this case are prescribed, no eigenvalue analysis is necessary. The eigenvectors can be determined in the same way as it was done for the butterfly scheme.

The control mesh for the ring consists of six rings of vertices around the central vertex as shown in Figure 8.

The convergence rates for the modified butterfly scheme are exactly the same as for the butterfly scheme, as these schemes coincide on the regular complexes. We used our algorithms for verification of regularity and injectivity of the characteristic map to prove that the scheme is C^1-continuous for any fixed valence.

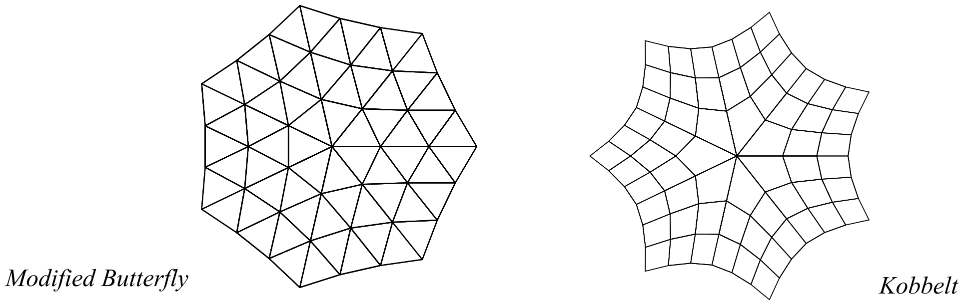

FIG. 8. *Control nets of the characteristic maps for the modified butterfly scheme and the Kobbelt scheme.*

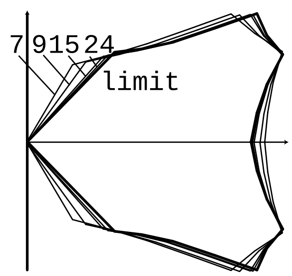

FIG. 9. *The convergence of the normalized control meshes of one segment of the characteristic maps for the modified butterfly scheme as valence increases. Only the boundaries of the meshes are shown.*

As was discussed above, it is possible to prove convergence for all valences if suitably chosen affine transforms of the control nets for one segment of the characteristic map converge to a limit as $k \to \infty$ and the normalized segment in the limit is regular and injective. This is the case for the modified butterfly scheme; the affine transform that we use is simply scaling along the y-axis by $\sin(2\pi/k)$.

Normalized control nets for several valences and the limit mesh are shown in Figure 9.

The algorithm of section 4 steps through the valences, verifying C^1-continuity for each valence which has sufficiently different control net (Figure 10). In the case of the modified butterfly scheme we were able to use only a relatively small step size 2.6×10^{-6}, with all tests passing only after 6 steps of subdivision. The maximal valence k_0 that had to be tested (1481) is determined by the condition $|\cos(2\pi/k) - 1| < \delta$, such that the tests succeed for $\mathbf{e}\,(\lambda([1-\delta, 1]))$ (recall that all quantities are represented as intervals). For each tested valence, we increase the interval size for control points, in order to be able to analyze many valences simultaneously for large valences. We have used the interval size 0.7×10^{-5} for valence greater than 6. The total number of valences that had to be analyzed separately was 406.

5.2. Analysis of Kobbelt's scheme. Kobbelt's subdivision scheme [12], is an interpolatory scheme defined on quad meshes; in the regular case, the scheme reduces to the tensor product of four-point schemes [5]. There are two challenges in the analysis of this scheme: First, as for the butterfly scheme, the limit surface cannot be expressed in explicit form. In addition, the eigenvalues of the subdivision matrix cannot be found explicitly.

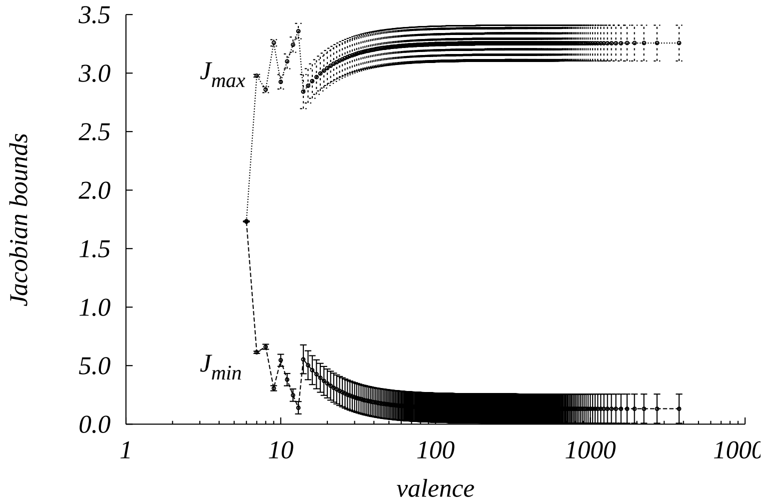

FIG. 10. *The upper and lower bounds for the Jacobians; the error bars show the interval for each bound; the step of the algorithm was chosen to be* 2.6×10^{-6} *so that the lower bound of the interval for* J_{min} *is close to zero. The initial anomaly in the lower bound is due to the fact that eight subdivision steps, rather than six as for all other valences, were required to verify regularity for valence* 3.

Let $\alpha = (8+w)/16$ and $\beta = -w/16$ be the coefficients of the four point scheme, where w is a parameter. Let $p^j_{i,l}$ be the control point corresponding to the vertex with number l in sector i at subdivision level j. Then the control points for level $j+1$ are computed from the values on level j in two steps. First, the *edge points* are computed; all vertices are computed in the usual manner using the four-point rule, excluding the vertices $p^{j+1}_{i,1}$ immediately adjacent to the extraordinary vertex. These vertices are computed using the formulas

$$\begin{aligned} p^{j+1}_{i,1} &= \alpha c + \alpha p^j_{i,1} + \beta u^j_i + \beta p^j_{i,3}, \\ u^j_i &= \frac{4}{k}\sum_{i=0}^{k-1} p^j_{i,1} - (p^j_{i-1,1} + p^j_{i,1} + p^j_{i+1,1}) \\ &\quad - \frac{\beta}{\alpha}(p^j_{i-2,2} + p^j_{i-1,2} + p^j_{i,2} + p^j_{i+1,2}) + \frac{4\beta}{\alpha k}\sum_{i=0}^{k-1} p^j_{i,2}, \end{aligned}$$

where u^j_i are intermediate "virtual points."

Next, the *face points* are computed. All face points are computed in the same way: four-point coefficients are applied to four consecutive edge points on level $j+1$, as shown in Figure 11. It is important to note that there are two ways to choose four consecutive edge points; the coefficients for the scheme are chosen in such way that both choices produce the same result.

We performed the analysis of the scheme for $w = 1$, which is the value for which the four point scheme has maximal smoothness. Let k be the valence, $\omega = e^{2im\pi/k}$,

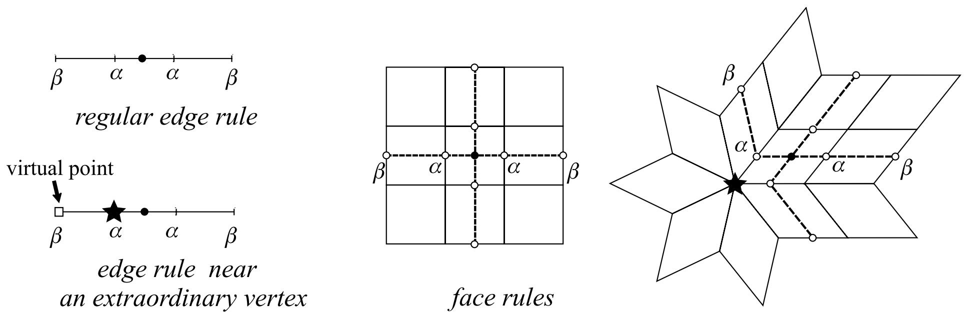

FIG. 11. *Rules for Kobbelt's scheme. The stars indicate extraordinary vertices,* $\alpha = (8+w)/16$, $\beta = -w/16$. *In the masks for face control points (right) empty circles are edge vertices inserted on the* same *subdivision step. The dashed lines show the two possible sequences of four edge points that are used to compute a face point.*

$m = 1 \dots k$, and

$$\begin{aligned} c_{00} &= \alpha + 4\,\beta\,\delta_{m,0} - \beta\,(1+2\,c), & c_{01} &= 4\,\beta^2\delta_{m,0}/\alpha - \beta^2\left(\bar{\omega}^2 + 2\,c + 1\right)/\alpha, \\ c_{10} &= 4\,\beta\,\alpha\,\delta_{m,0} + \left(\alpha^2 - \alpha\,\beta\right)(1+\omega), & c_{11} &= 4\,\beta^2\delta_{m,0} - \beta^2\,(1+2\,c) + 2\,\alpha\,\beta\,c + \alpha^2. \end{aligned}$$

After the standard operation of applying DFT to the subdivision matrix, we obtain the following 12×12 matrix $B(\omega)$:

$$\left[\begin{array}{cccccc|cccccc} c_{00} & c_{01} & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{10} & c_{11} & (1+\omega)\alpha\,\beta & \beta^2\omega+\alpha\,\beta & \beta^2 & \beta^2\bar{\omega}+\alpha\,\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha+\beta\,\bar{\omega} & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha\,\omega & \alpha+\beta\,\omega & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \alpha & 0 & \alpha & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ \beta^2\bar{\omega}+\alpha^2+\alpha\,\beta\,\omega & \alpha\,\beta\,\bar{\omega}+\alpha^2 & \beta^2\omega+\alpha^2 & \alpha^2 & \alpha\,\beta & \alpha\,\beta\,(1+\bar{\omega}) & \alpha\,\beta & \alpha\,\beta & \beta^2 & 0 & 0 & \beta^2\bar{\omega} \\ \beta\,\omega & \alpha & 0 & \alpha & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ (1+\omega)\alpha\,\beta & \alpha^2 & (1+\omega)\alpha\,\beta & \alpha^2 & \alpha^2 & \alpha^2 & \beta^2(1+\omega) & \alpha\,\beta & \alpha\,\beta & \beta^2 & \alpha\,\beta & \alpha\,\beta \\ \beta & \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & \beta \\ \alpha\,\beta+\alpha^2\omega+\beta^2\omega^2 & \alpha^2+\alpha\,\beta\,\omega & \beta^2+\alpha^2\omega & (1+\omega)\alpha\,\beta & \alpha\,\beta & \alpha^2 & \alpha\,\beta\,\omega & \beta^2\omega & 0 & 0 & \beta^2 & \alpha\,\beta \end{array}\right].$$

As discussed in section 2.4, for any subdivision scheme each block can be separated into subblocks, with eigenvalues of lower-right 6×6 subblock not depending on the valence. The eigenvalues of this subblock are equal to $-1/16$, $-1/32$, $-1/64$ (double), $-1/128$, and $1/256$.

The larger eigenvalues are always eigenvalues of the upper-left 6×6 subblock. The roots of the characteristic polynomial of that subblock cannot be found explicitly. However, for fixed m and k, we can easily find the roots numerically, with guaranteed lower and upper bounds on the roots. The characteristic polynomial has the form

$$\begin{aligned} P(c,\lambda) = \lambda^6 &+ \left(-\frac{3}{64}\,c - \frac{15}{16}\right)\lambda^5 \\ &+ \left(\frac{297}{1024} - \frac{9}{1024}\,c\right)\lambda^4 + \left(-\frac{335}{8192} + \frac{21}{4096}\,c - \frac{7}{16384}\,c^2\right)\lambda^3 \\ &+ \left(\frac{183}{65536} - \frac{9}{16384}\,c\right)\lambda^2 + \left(-\frac{45}{524288} + \frac{9}{524288}\,c\right)\lambda + \frac{1}{1048576}. \end{aligned}$$

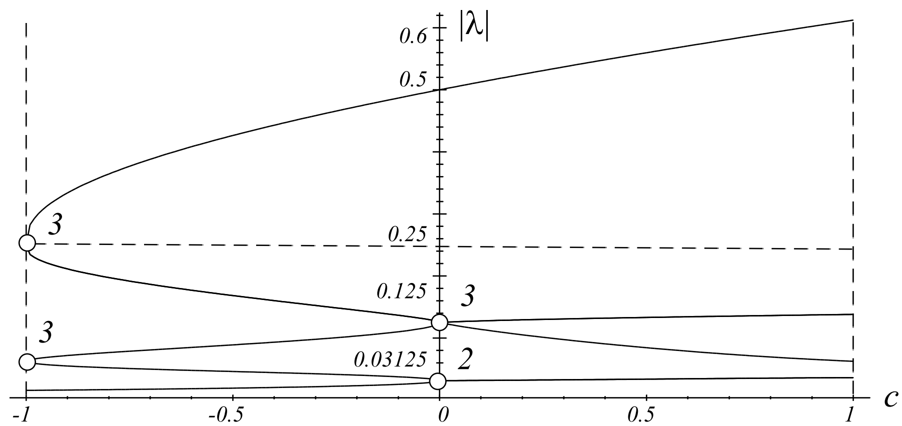

FIG. 12. *The magnitudes of the eigenvalues of the subdivision matrix for Kobbelt's scheme as functions of c. Only the eigenvalues of the upper-right* 6×6 *subblock are shown. Note that the magnitudes of the complex conjugate eigenvalues coincide, and there are fewer than six distinct curves.*

Numerically computed roots of this polynomial are plotted as functions of c in Figure 12.

Analysis of the eigenvalues. From the plot it is clear that the largest eigenvalue increases as a function of c; therefore, it appears that the largest eigenvalue of the subdivision matrix for any valence corresponds to $m = 1$. Moreover, our calculations indicate that the largest eigenvalue is always real. Using interval methods, we prove the following proposition.

PROPOSITION 5.3. *For any valence k, and any $m = 1 \ldots k-1$, the largest eigenvalue is real and unique, and for any block $B(2m\pi/k)$, $m \neq k-1, 1$, the largest eigenvalue is less than the largest eigenvalue of the blocks $B(2\pi/k)$ and $B(2\pi(k-1)/k)$. The unique largest eigenvalue is the only eigenvalue in the interval* $[0.5, 0.613]$ *for $k > 4$.*

The detailed proof with all calculations, including the Maple code, can be found in [21].

Here we present an outline of the proof. The proof is performed in several steps:

1. We show that for $c < 0$, all roots of the characteristic polynomial $P(c, \lambda)$ are likely to be less than 0.51
2. We show that for any $c \in [0 \ldots 1]$, there is a unique real root μ in the interval $[0.5, 0.613]$, and the function $\mu(c)$ is C^1-continuous and increases.
3. We "deflate" the characteristic polynomial (that is, divide by the monomial $\lambda - \mu$) and verify that all roots of the deflated polynomial are inside the circle of radius 0.5 for $c \in [0, 1]$.

Marden–Jury test. On steps 1 and 3 we have to show that the roots of a polynomial are inside a circle of radius r in the complex plane. This task is similar to the task of establishing stability of a filter with the transfer function $1/a(z)$, where $a(z) = \sum_{i=0}^{M} a_i z^i$ is a polynomial. The filter is stable, if all roots of the polynomial $a(z)$ are inside the unit circle. A variety of tests exist for this condition; for our purposes, the algebraic Marden–Jury test is convenient [11]. With appropriate rescaling of the variable it can be used to prove that all roots of a polynomial are inside the circle of any given radius r. As the test requires only a simple algebraic calculation on the coefficients of the polynomial, it can easily be performed for symbolic and interval coefficients.

To perform the test for a real polynomial $\sum_{m=0}^{M} \lambda^m$, a table is constructed. The first line is given by $r_i^0 = a_i$, $i = 0 \dots M$. The rest of the table is defined recursively:

$$r_j^{i+1} = \begin{vmatrix} r_0^i & r_{M-i-j}^i \\ r_{M-i}^i & r_j^i \end{vmatrix}, \quad j = 0 \dots M-i.$$

Each row contains one element less than the previous row. Once the table is computed, the necessary and sufficient condition for stability is $r_0^1 < 0$, $r_0^i > 0$ for $i = 1 \dots M$.

A more detailed discussion of the proof of Proposition 5.3 can be found in Appendix A.2.

Analysis of C^1-continuity. Using Proposition 5.3, we can easily compute the largest eigenvalue with guaranteed bounds for any k: this will be the unique real root in the interval $[0.5, 0.613]$. We compute roots up to a maximal valence k_0. Once the eigenvalues are known, the eigenvectors defining the characteristic map are computed, and the tests described in section 4 are applied to establish C^1-continuity of the scheme for any fixed valence.

For this scheme we use the standard difference operator ∇ defined in section 3.1; because this is a tensor product scheme, the matrix difference scheme is diagonal and can be decomposed into scalar schemes. The convergence constants are $c = 13/32$, $\gamma_1 = 25/32$, $\gamma_2 = 105/256$, $\gamma_3 = 425/2048$. The convergence constants for the difference scheme are $c^D = 31/64$, $\gamma_1^D = 5/4$, $\gamma_2^D = 15/16$, $\gamma_3^D = 5/8$.

To complete analysis of the scheme we need to describe the behavior of $\mu(c)$ at infinity. Specifically, to use our algorithm for verification of smoothness for all valences, for any interval value $c = [1-\epsilon, 1]$ we need to estimate the corresponding interval value $\mu(c)$. As $\mu(c)$ changes slowly, linear approximation is sufficient for our purposes; the upper bound for the derivative $\mu'_c = 1/c'_\mu$ can easily be computed. This allows us to compute the interval eigenvectors at infinity and verify C^1-continuity for all valences greater than k_0.

The control mesh for the characteristic map of Kobbelt's scheme for valence 7 is shown in Figure 8. Figure 13 shows the dependence of the upper and lower estimates of computed Jacobians on the valence. Valences up to 812 had to be examined; because eigenvectors for large valences were sufficiently close, it was possible to perform analysis for a number of valences simultaneously; thus, only 193 valences had to be tested.

We conclude that *Kobbelt's scheme is C^1-continuous for all valences.*

5.3. Other schemes. C^1-continuity of the Loop scheme was verified in [19] for valences up to 150. For the Catmull–Clark scheme, C^1-continuity was analyzed for valences up to 10,000 in [15]. As the subdivision matrices have relatively simple form, and the eigenvalues and eigenvectors can be explicitly computed, our algorithms can be applied without any extra effort to obtain a proof of C^1-continuity for all valences. Spline-based schemes have a remarkable property: the convergence rates γ_1 and γ_1^D for the scheme and the difference scheme are both $1/2$; this is due to the fact that both the scheme and the derivative scheme have only positive coefficients. In addition, replacing the coefficients of the scheme by intervals, we establish not only C^1-continuity of the schemes, but also stability for small perturbations of the nonzero coefficients, as long as the perturbed coefficients lead to a convergent scheme. Due to fast convergence and high stability, intervals of large size can be used in the analysis,

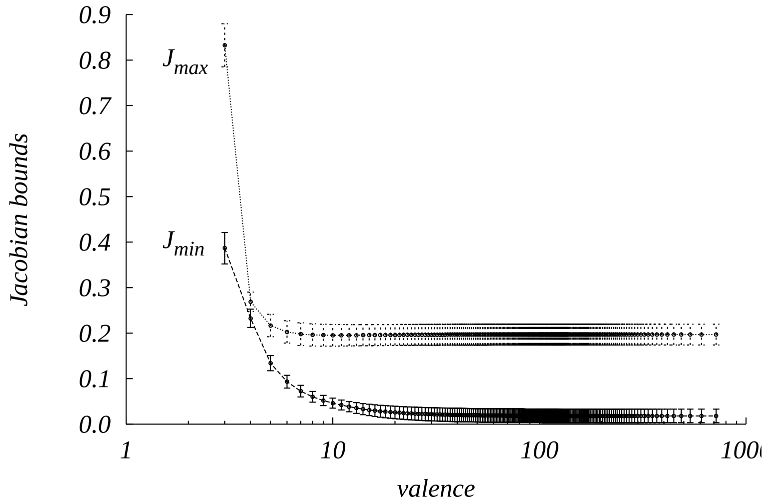

FIG. 13. *The upper and lower bounds for the Jacobian of the characteristic maps as functions of the valence for Kobbelt's scheme. The error bars indicate the size of the interval; the step of the algorithm was chosen to be* 3×10^{-5}*; the maximal examined valence was* 818*; the total number of valences for which the test was performed was* 193*.*

and only few valences (up to 58 for Loop, up to 89 for Catmull–Clark) have to be analyzed. The number of subdivision iterations required to verify regularity is also quite small: 3 iterations are sufficient in both cases.

Our techniques can be also applied in virtually unchanged form to the dual, or corner cutting, subdivision schemes. Two schemes of this type are known to us: the Doo–Sabin subdivision scheme [4] and the Midedge subdivision scheme [9, 16]. For these schemes, C^1-continuity was already proved for all valences [15, 9, 16]. Using our method, it is possible to perform perturbation analysis of the type that we have described above. We will discuss issues related to stability of smoothness properties of subdivision in greater detail in a future paper.

6. Conclusions. We have presented a general method for the verification of C^1-continuity of stationary subdivision. This method allows us to analyze schemes which are not derived from spline subdivision and perform most of the analysis automatically.

Our method opens the way for a general characterization of invariant C^1-continuous schemes with small support: such schemes are defined by a small number of parameters, and our interval algorithms can be used to prove C^1-continuity for continuous ranges of these parameters.

Applications of our algorithms are not restricted to the invariant schemes for closed meshes: in fact, we have successfully used them to establish C^1-continuity on the boundary of several variations of common subdivision schemes. These results are discussed in a future paper [23].

One important, although typically not the most difficult, aspect of the problem is not addressed by our method. As it could be seen from our analysis of the butterfly and Kobbelt subdivision schemes, the eigenstructure of a particular scheme or family of schemes still has to be analyzed separately in each case. To apply our method, we need to find the Jordan normal form of the subdivision matrix; in general, it is not always possible to do this numerically or in closed form. However, we have demonstrated, using the Kobbelt scheme as an example, that it is possible to use a combination of symbolic and numerical methods to obtain all necessary information. We believe that a satisfactory solution of this problem requires methods similar to those developed in [8]. While it might not be possible to determine the Jordan normal form exactly, one can find all possible Jordan normal forms of matrices that are obtained from the original subdivision matrix by a small perturbation. This approach is likely to yield an algorithm that would be capable of performing the analysis C^1-continuity of a scheme given only the coefficients of the scheme as the input.

While our method can be used to analyze parametric families of subdivision schemes, due to its seminumeric nature, it cannot be used for such tasks as finding precise ranges of the values of the parameters for which the scheme remains C^1-continuous. While this is of little relevance for practical applications, it can be regarded as a theoretical drawback.

While in principle our method can be used to verify C^1-continuity of any stable scheme, in practice it is limited by the computational resources. If the convergence of the difference scheme is too slow, the number of subdivision steps required to apply the method may become prohibitive. Comparison of the performance of the method for the loop scheme and the butterfly scheme is revealing: As the convergence rate of the butterfly scheme is substantially slower, 6 subdivision steps are required and 1481 different valences have to be examined to verify C^1-continuity. In contrast, for the loop scheme the convergence rate is high, and only 3 subdivision steps and 58 valences are necessary. The method is likely to perform very well for any scheme with positive coefficients, but take substantially more time for schemes with negative coefficients, which typically have slower convergence rates.

Appendix. Technical proofs. In this appendix we outline the proofs of two propositions used in the analysis of the butterfly and Kobbelt schemes. These proofs use symbolic and numeric computations. The complete Maple code with explanation is available separately.

A.1. Proposition 5.1. The roots of the characteristic polynomial of the butterfly scheme can be found explicitly; depending on the value of $d = c^2$, there can be either one real and two complex roots or three real roots. For four special values of d the matrix has nontrivial Jordan blocks; the special values of d are the roots of the discriminant of the characteristic polynomial, which is a polynomial in d. These roots are 0, 1/4, 1, and $d_{cr} \approx 0.84868$. The types of roots depend on the sign of the discriminant of the characteristic polynomial; the discriminant is positive on $(0, 1/4)$ and $(d_{cr}, 1)$, negative on $(1/4, d_{cr})$. On each interval, well-known formulas can be used to find the roots of the characteristic polynomial as functions of d explicitly. To determine the largest root for any value of d, that is, the largest magnitude of an eigenvalue of a subblock $B_{00}(\omega)$ for a given ω, we consider the cases of three real roots and one real root separately.

Suppose $\lambda_i(d)$, $i = 1, 2, 3$, are the three real roots for $d \in [1/4, d_{cr}]$. As zero is a root only for $d = 0$, the three real roots do not change signs for $d \in [0, 1]$. It is sufficient to compute the value of roots at any point to show that all three roots are nonegative. Therefore, the curves $|\lambda_1(d)|$, $|\lambda_2(d)|$, $|\lambda_3(d)|$ can intersect only if the roots coincide, which means that the discriminant is zero. Thus, the curves cannot intersect on the interval $(1/4, d_{cr})$. The largest root can be determined simply by evaluating the roots with guaranteed precision at any point of the interval.

If there is only one real root, we can easily show that for $d < 1/4$, $P(1/4, d) < 0$ and for $d > 1/4$, $P(1/4, d) > 0$. We conclude that for $d > d_{cr}$, the single real root is greater than 1/4, and for $d < 1/4$ it is less than 1/4. The magnitude of the complex roots also satisfies a cubic equation. Using the same method, we can show that these roots have magnitudes less than 1/4 on $(1, 4)$ and $(3/4, 1)$.

Finally, we determine the range of d for which the largest root decreases as a function of d. Note that $\lambda_{max}(d)$ is a unique solution of the equation $P(\lambda, d)$ in the domain $(1/4, 1) \times (1/4, \infty)$. We can determine the zeros of the derivative $\lambda'_{max}(d)$ from the system of the equations $P(\lambda, d) = 0$, $P'_d(\lambda, d) = 0$, solving for d. Excluding λ using standard Gröbner basis techniques, and solving with guaranteed precision for d, we obtain the value d_{max}. As the derivative is not zero on $(d_{max}, 1)$, we determine the sign simply evaluating it at a point, and we conclude that it is negative.

A.2. Proposition 5.3. Here we describe how steps 1–3 of the proof of Proposition 5.3 are performed.

Step 1. We have to verify that all roots of the polynomial for $c \in [0, 1]$ have magnitude less than 1. We split the interval into sufficiently small subintervals, so that we can evaluate Marden–Jury test in interval arithmetics for each subinterval with definite results.

The following observation is crucial for steps 2 and 3. Although the characteristic polynomial has degree 6 in λ, it is only quadratic in c, and has two solutions, $c_1(\lambda)$ and $c_2(\lambda)$.

Step 2. Using interval evaluation of the derivative, one of the two solutions, say, $c_1(\lambda)$, can be shown to be increasing for $\lambda \in [0.5, 0.613]$. As $c_1(0.5) = 0$, and $c_1(0.613) > 1$, we conclude that for $c \in [0, 1]$, there is always a real solution in the range $[0.5, 0.613]$. If we evaluate the second root $c_2(\lambda)$ for the same interval of λ, we can observe that the values of c_2 are outside the range $[-1, 1]$. Therefore, for $c \in [0, 1]$ there is a unique real solution λ in the range $[0.5, 0.613]$. Because $c_1(\lambda)$ is C^1-continuous and its derivative is positive, the inverse function $\mu(c)$ has the same properties (we use μ to distinguish between the real eigenvalue that we have identified from the indeterminate λ of the characteristic polynomial).

Step 3. To show that all other roots of the characteristic polynomial for $c \in [0, 1]$ are smaller than $\mu(c)$, we perform deflation symbolically, using μ as a parameter: we divide $P(c, \lambda)$ by $(\lambda - \mu)$ symbolically and substitute $c = c_1(\mu)$. The coefficients of the resulting polynomial are functions of μ. Again, we separate the range of μ into subintervals, small enough to be able to obtain a definite result from the Marden–Jury test. This proves that for all values of μ in $[0.5, 0.613]$, and, therefore, for all $c \in [0, 1]$, $\mu(c)$ is the largest root of $P(\lambda, c)$.

The proposition is derived from the three statements in section 5.2 in the following way.

As for $k > 4$, $\cos \frac{2\pi}{k} > 0.51$, the largest eigenvalue cannot possibly correspond to a block m, for which $\cos \frac{2m\pi}{k} \le 0$. From step 3, it follows that the largest root has to be the real root $\mu(c)$ for some c. As for any $m > 1, m < k - 1$, $\cos \frac{2m\pi}{k} < \cos \frac{2\pi}{k}$, and we have shown in step 1 that $\mu(c)$ increases, and for any c $\mu(c)$ is the largest root (step 3), we conclude that the largest eigenvalue always corresponds to $m = 1$, is real, and is the unique eigenvalue in the range $[0.5, 0.613]$.

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