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CONSTRUCTIVE APPROXIMATION

# Smoothness of Subdivision Surfaces with Boundary

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**Abstract** Subdivision rules for meshes with boundary are essential for practical applications of subdivision surfaces. These rules have to result in piecewise $C^\ell$-continuous boundary limit curves and ensure $C^\ell$-continuity of the surface itself. Extending the theory of Zorin (Constr Approx 16(3):359–397, 2000), we present in this paper general necessary and sufficient conditions for $C^\ell$-continuity of subdivision schemes for surfaces with boundary, and specialize these to practically applicable sufficient conditions for $C^1$-continuity. We use these conditions to show that certain boundary rules for Loop and Catmull–Clark are in fact $C^1$ continuous.

**Keywords** Subdivision algorithms · Boundary rules · $C^1$-analysis · Characteristic maps · Loop · Catmull–Clark

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## 1 Introduction

Subdivision is a method to construct smooth surfaces out of polygonal meshes used in a variety of computer graphics and geometric modeling applications. Two features of subdivision algorithms are particularly important for applications. The first is the

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ability to handle a large variety of input meshes, including meshes with boundary. The second is the ease of modification of subdivision rules, which makes it possible to generate different surfaces (e.g., surfaces with sharp or soft creases) out of the same input mesh.

The importance of special boundary and crease rules has been recognized for some time [7,11,12,18]. However, most of the theoretical analysis of subdivision [15,17,23,24] focused on the case of surfaces without boundaries and schemes invariant with respect to rotations. The goal of this paper is to develop the necessary theoretical foundations for analysis of subdivision rules for meshes with boundary and to present analysis for boundary rules extending several well-known subdivision schemes, described in [1].

In this paper, we consider surfaces with piecewise-smooth boundary. This class readily extends to a broader class of piecewise-smooth surfaces with crease curves and corner points. We demonstrate how the standard constructions of subdivision theory (subdivision matrices, characteristic maps, etc.) generalize to surfaces with piecewise-smooth boundary. We demonstrate that convex and concave boundary corners inherently require separate subdivision rules for the surfaces to have well-defined normals in both cases.

We proceed to extend the techniques for analysis of $C^1$-continuity developed in [23] to the case of piecewise-smooth surfaces with boundary. This paper is based on the analysis in [24] and follows conventions and notation of that paper. While we briefly consider $C^\ell$-continuity, we focus on $C^1$-continuity conditions.

The result, allowing one to analyze $C^1$-continuity of most subdivision schemes for surfaces without boundaries, is the sufficient condition of Reif [17]. This condition reduces the analysis of stationary subdivision to the analysis of a single map, called the *characteristic map*, uniquely defined for each valence of vertices in the mesh. The analysis of $C^1$-continuity is performed in three steps for each valence:

1. compute the control net of the characteristic map;
2. prove that the characteristic map is regular;
3. prove that the characteristic map is injective.

We show that similar conditions hold for surfaces with boundary, and under commonly satisfied assumptions injectivity of the characteristic map for surfaces with boundary can be inferred from regularity.

Finally, we use the theory that we have developed to derive and analyze several specific boundary subdivision rules, initially proposed in [1].

*Previous work* The theory presented in this paper is based on the theory developed for closed surfaces in [17,23,24], which was recently extended to subdivision on manifolds in [19–21]. Subdivision schemes for closed surfaces were analyzed in [15, 23]. Most of the standard theory is also summarized in the book by Reif and Peters [16].

As far as we know, analysis of $C^1$-continuity of subdivision rules for surfaces with boundary was performed only in [18], where a particular choice of rules extending Loop subdivision was analyzed.

At the same time, a substantial number of papers proposed various boundary rules starting with the first papers on subdivision by Doo and Sabin, and Catmull and

Clark [2,4,7,11,13]. A method for generating soft creases was proposed in [3], and a complexity analysis was done in [14].

## 2 Surfaces with Piecewise-Smooth Boundary

### 2.1 Definitions

In this section, we define surfaces with piecewise-smooth boundary. Unlike the case of surfaces defined as manifolds of dimension 2, there is no commonly accepted definition that would be suitable for our purpose. Often definitions restrict the boundary to be a union of nonintersecting $C^\ell$-continuous curves (cf. [5,10]). This definition is, however, too narrow for geometric modeling applications, as surfaces with corners (e.g., surfaces obtained by smooth deformations of a rectangle) are quite common. To include corners, we must allow isolated singularities for the boundary curves. On the other hand, complex singularities such as cusps or $C^m$-continuous joints for $0 < m < \ell$ are too difficult to manage in the analysis and are not of interest within the framework of this paper. In practice, these complex cases are rare, so we assume that at boundary corners the tangents are different at the shared endpoint. We call such endpoints of boundary curve segments **nondegenerate corners.** *We consider surfaces with piecewise-smooth boundaries but without cusps or $C^m$-continuous joints for $0 < m < \ell$.*

Once higher-order contact cases are excluded, we define a surface with piecewise $C^\ell$-continuous boundary with nondegenerate corners more constructively using four local chart types: the disk $D$, the half disk $Q_2 = H \cap D$ with $H = \{(x, y)|y \geq 0\}$, a quarter of the disk $Q_1$, and three quarters of the disk $Q_3$. The domains $Q_i$ $i = 1, 3$ are defined as follows: $Q_1 = \{(x, y)|y \geq 0 \text{ and } x \geq 0\} \cap D$, $Q_3 = \{(x, y)|y \geq 0 \text{ or } x \geq 0\} \cap D$.

Recall that a map on a subset $A$ of $\mathbf{R}^d$ is differentiable if there is an extension of the map to an open neighborhood of the set, such that the extended map is differentiable in the standard sense. Under weak assumptions on the set $A$, which hold in all cases relevant to us, this is equivalent to existence and continuity of limits of the derivatives in the interior of $A$ at the boundary.

As subdivision surfaces can be viewed as meshes continuously *immersed* (but not necessarily embedded) in $\mathbf{R}^3$, we need to consider surfaces as images of two-dimensional domains, more precisely, topological spaces that are locally two-dimensional Euclidean with boundary; i.e., for any point of such space there is an open neighborhood homeomorphic to $\mathbf{R}^d$ or a half-space of $\mathbf{R}^d$ (boundary points).

**Definition 2.1** Consider a surface $(M, f)$ where $M$ is a topological locally Euclidean space with boundary, and $f$ is a continuous immersion $f : M \to \mathbf{R}^d$. The surface $(M, f)$ is called **$C^\ell$-continuous with piecewise $C^\ell$-continuous boundary with nondegenerate corners** if for any $x \in M$ there is a neighborhood $U_x$ of $x$ with bijective image $V_x \subset f(M)$ and a $C^\ell$-continuous and homeomorphic parametrization $P \to V_x$, with $P$ being one of the domains $Q_i$, $i = 1, \ldots 3$, or the disk $D$, mapping 0 to $f(x)$. We call $x$ an interior point if $P = D$; otherwise, we call it a boundary point. We distinguish two main types of boundary points: if $V_x$ is diffeomorphic to $Q_2$, the boundary point is called smooth; otherwise, it is called a corner. There are two types of corners:

- convex corners ($V_x$ is diffeomorphic to $Q_1$);
- concave corners ($V_x$ is diffeomorphic to $Q_3$).

Definition 2.1 is similar to the definition of manifolds with boundary, which have a differentiable structure in [8]. There, only the half space is used, while we consider also the quarter and the three-quarter space of $\mathbf{R}^2$.

Observe that the same type of smoothness structure could be defined on $f(M)$ viewed as a point set if $f$ were an embedding. However, in the setting we consider, we need to ensure that our definition is compatible with smoothness in the ambient metric on the one hand (which is ensured by requiring maps $V_x \to Q_i$ to be $C^\ell$) and, on the other hand, be able to handle self-intersections, for which we need to use $M$, to differentiate between points of $M$ with coinciding images under $f$. Surfaces satisfying Definition 2.1 can be used to model a large variety of features; for example, by joining the surfaces along boundary lines, we can obtain surfaces with creases. However, in addition to boundary cusps, a number of useful features such as cones cannot be modeled unless degenerate configurations of control points are used.

### 2.2 Tangent Plane Continuity and $C^1$-continuity

As we will see in Sect. 3, analysis of subdivision focuses on the behavior of surfaces which are known to be at least $C^1$-continuous in a neighborhood of a point, but nothing is known about the behavior at the point. In this case, it is convenient to first establish **tangent plane continuity**, for which we use the exterior product $\wedge$ to describe a 2-dimensional subspace in $\mathbf{R}^d$ (vector product for $d = 3$). The result of the wedge product of the vectors is an element in the exterior algebra. Only if $d = 3$ can we identify this element with the normal vector of the corresponding hyperplane. Normally it describes a 2-dimensional subspace, which can be identified with a vector in the exterior algebra. We denote by $[\cdot]_+$ the normalization of a vector.

**Definition 2.2** Suppose a surface $(M, f)$ in a neighborhood of a point $x \in M$ is parametrized by $h : U \to f(M) \subset \mathbf{R}^d$, where $U$ is an open subset of the unit disk $D$, $Q_1$, $Q_2$, or $Q_3$ containing 0, which is regular (h is $C^1$-continuous and the Jacobi matrix has maximal rank) everywhere except 0, and $h(0) = f(x)$. For $y \in U$, let $\pi(y) = [\partial_1 h \wedge \partial_2 h]_+$, where $\partial_1 h$ and $\partial_2 h$ are derivatives with respect to a choice of coordinates in the plane of the disk $D$ or one of the domains $Q_i$. The surface is **tangent plane continuous** at $x$ if the limit $\lim_{y \in U \to 0} \pi(y)$ exists.

We can then prove an equivalent proposition as Proposition 1.2 [24].

**Proposition 2.3** *Suppose a surface $(M, f)$ is $C^1$-continuous with $C^1$-continuous boundary everywhere excluding a boundary point $x \in M$. The surface is $C^1$-continuous at $x$ with piecewise $C^1$-continuous boundary with nondegenerate corners if and only if:*

1. *a parametrization as in Definition 2.2 exists, which is tangent plane continuous at $x$,*
2. *the projection of the surface into the tangent plane is injective, in a neighborhood.*

3. *the boundary either has a nondegenerate corner at $x$ (this means the limits of the tangent vectors are different) or is $C^1$-continuous at $x$.*

*Proof* The necessity is obvious, and in order to prove the sufficiency of the condition, we follow the proof of Proposition 1.2 from [24]. It shows that the projection $P$ of a subset $V$ of the surface to the tangent plane is injective and $C^1$ at all points; it applies without changes to surfaces with boundary. What is left to show is that the image of $V$ or a subset is $C^1$-diffeomorphic to one of the domains $Q_i$, $i = 1, 2, 3$. As the two boundary curves of $V$, which we call $\gamma_1$ and $\gamma_2$, are $C_1$-continuous, and their tangents are in the tangent plane to the surface at all points, their projections $P(\gamma_1)$ and $P(\gamma_2)$ into the tangent plane at $x$ are also $C^1$-continuous. At the point $x$, the tangents to the curves are in the tangent plane at $x$, and coincide with the tangents to the projections. Using the third condition of the proposition, we conclude that the images of the curves at $x$ either form a nondegenerate corner or a $C^1$-continuous curve. Topologically, $V$ and $P(V)$ are homeomorphic to a half disk. We have just shown that the image of the boundary diameter of the half disk is $C^1$-continuous or $C^1$-continuous with a nondegenerate corner at $x$. We can choose the neighborhood $V_x$ of $x$ so that the image $P(V_x)$ has a boundary consisting of two segments of the curves $P(\gamma_1)$ and $P(\gamma_2)$ and a part of a circle, intersecting each of these at a single point. Let $l_1$ and $l_2$ be the rays along tangent directions to $\gamma_1$ and $\gamma_2$ (possibly collinear). Then for sufficiently small radius of the neighborhood, we can assume that orthogonal projections of $\gamma_i$ to $l_i$ in the plane are injective. Now we can directly construct a $C_1$-diffeomorphism of the $P(V_x)$ to one of the domains $Q_i$. □

## 3 Subdivision Schemes on Complexes with Boundary

In this section, we summarize the main definitions and facts about subdivision on complexes that we use. More details for the case of surfaces without boundaries can be found in [22,24].

### 3.1 Definitions

*Polygonal complexes* Subdivision surfaces are naturally defined as functions on two-dimensional polygonal complexes. A two dimensional polygonal complex $K$ is a set of vertices, edges, and planar simple polygons (faces) in $\mathbf{R}^N$ such that for any face its edges are in $K$, and for any edge its vertices are in $K$ and the intersection of two elements is also an element in the complex. We assume that there are no isolated vertices or edges (a pure or homogeneous complex). $|K|$ denotes the union of faces of the complex regarded as a subset of $\mathbf{R}^N$ with induced metric. $|K|$ is the locally Euclidean domain with boundary $M$ in Definition 2.1. We say that two complexes $K_1$ and $K_2$ are *isomorphic* if there is a homeomorphism between $|K_1|$ and $|K_2|$ that maps vertices to vertices, edges to edges, and faces to faces.

A 1-neighborhood $N_1(v, K)$ of a vertex $v$ in a complex $K$ is the complex formed by all faces that have $v$ as a vertex. The $m$-neighborhood of a vertex $v$, $N_m(v, K)$ is defined recursively as a union of all 1-neighborhoods of vertices in the $(m-1)$-neighborhood of $v$. We are primarily interested in schemes that work on either quadrilateral or triangle meshes. A $k$-regular complex with boundary $\mathcal{R}_k$ is the regular tiling of the half plane, consisting of identical triangles (quads), with all internal vertices of valence 6(4) and all vertices on the boundary of valence 4(3), excluding the vertex at the origin which has valence $k+1$. Here, valence denotes the number of edges sharing a vertex.

*Tagged complexes* The vertices, edges, or faces of a complex can be assigned one element of a finite set of tags. These tags can be used to choose a type of subdivision rule applied at a vertex. In this paper, we use tags in a very limited way: specifically, a boundary vertex can be tagged as a *convex* or *concave* corner, or a smooth boundary vertex. However, as is discussed below, the tags can be used to create creases in the interior of meshes and for other purposes. Subdivision on tagged complexes merits a separate detailed consideration in a future paper.

Isomorphisms of tagged complexes with identical tag sets can be defined as isomorphisms of complexes which preserve tags; i.e., if a vertex has a tag $\tau$, its image also has a tag $\tau$.

*Subdivision of complexes* We can construct a new complex $D(K)$ from a complex $K$ by subdivision. For a triangular scheme, $D(K)$ is constructed by adding a new vertex for each edge of the complex and replacing each old triangle with four new triangles. For a quadrilateral scheme, $D(K)$ is constructed by adding a vertex for each edge and face and replacing each quadrilateral face with 4 quadrilateral faces. Note that $k$-regular complexes with boundary are self-similar; that is, $D(\mathcal{R}_k)$ and $\mathcal{R}_k$ are isomorphic.

We use notation $K^j$ for $j$ times subdivided complex $D^j(K)$ and $V^j$ for the set of vertices of $K^j$. Note that the sets of vertices are nested: $V^0 \subset V^1 \subset \cdots$.

If a complex is tagged, it is also necessary to define rules for assigning tags to the new edges, vertices, and faces. For our vertex tags, we use a trivial rule: all newly inserted boundary vertices are tagged as smooth boundary vertices.

*Subdivision schemes* Next, we attach values to the vertices of the complex; in other words, we consider the space of functions $V \to B$, where $B$ is a vector space over $\mathbf{R}$, typically $\mathbf{R}^l$ or $\mathbf{C}^l$ for some $l$. We denote this space by $\mathcal{P}(V, B)$, or $\mathcal{P}(K, B)$ if the set of vertices comes from the complex $K$.

A *subdivision scheme* for any function $p \in \mathcal{P}(K, B)$ computes a function $p^1 \in \mathcal{P}(K^1, B)$. We consider only subdivision schemes that are linear; i.e.,

$$p^1(v) = \sum_{w \in V} a_{vw} p^0(w)$$

for all $v \in V^1$. The coefficients $a_{vw}$ may depend on $K$.

We restrict our attention to subdivision schemes which are finitely supported, locally defined, and invariant with respect to a set of isomorphisms of tagged complexes and affinely invariant. A subdivision scheme is *finitely supported* if there is an integer $M$ such that $a_{vw} \neq 0$ only if $w \in N_M(v, K^1)$ for any complex $K$. We call the minimal possible $M$ the *support size* of the scheme. We assume our schemes to be *locally* *defined* and *invariant with respect to isomorphisms of tagged complexes*. Together these two requirements can be defined as follows: there is a constant $L$ such that if two neighborhoods $N_L(v_1, K_1)$ and $N_L(v_2, K_2)$ of complexes $K_1$ and $K_2$ are isomorphic with tag-preserving isomorphism $\rho : N_L(v_1, K_1) \to N_L(v_2, K_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$ such that $\rho(v_1) = v_2$, then $a_{v_1 w} = a_{v_2 \rho(w)}$. In most cases, the *localization size* $L = M$.

The final requirement that we impose on subdivision schemes is *affine invariance*: if $T$ is an affine transformation $B \to B$, then for any $v$, $Tp^{j+1}(v) = \sum_{w \in V} a_{vw} Tp^j(w)$. This is equivalent to requiring that all coefficients $a_{vw}$ for a fixed $w$ sum up to 1.

For each vertex $v \in \cup_{j=0}^{\infty} V^j$, there is a sequence of values $p^i(v), p^{i+1}(v), \ldots$ where $i$ is the minimal number such that $V^i$ contains $v$.

**Definition 3.1** A subdivision scheme is called *convergent* on a complex $K$, if for any function $p \in \mathcal{P}(K, B)$ there is a continuous function $f$ defined on $|K|$ with values in $B$ such that

$$\lim_{j \to \infty} \sup_{v \in V^j} \left\| p^j(v) - f(v) \right\|_2 \to 0.$$

The function $f$ is called the limit function of subdivision.

Notation: $f[p] : |K| \to B$ is the limit function generated by subdivision from the initial values given by the function $p \in \mathcal{P}(K)$. It is easy to show that if a limit function exists, it is unique. The surface $(|K|, f)$ is called a subdivision surface if $B = \mathbf{R}^3$ and the complex $K$ has only simple links. Recall that a *link* of a vertex is the set of edges of $N_1(v, K)$ that do not contain $v$. Simple links are links that are connected simple polygonal lines, open or closed.

*Subdivision matrices* We consider subdivision on $\mathcal{R}_k$ (a $k$-regular complex with boundary as defined above). The points $p^{j+1}(v) \in \mathbf{R}^3$ for $v \in N_Q(0, \mathcal{R}_k^{j+1})$ (the $Q$-neighborhood of the $j+1$ times subdivided $k$-regular complex with boundary) can be computed using only control points $p^j(w)$ for $w \in N_Q(0, \mathcal{R}_k^j)$. The integer $Q$ is the minimal one such that this holds. Let $N+1$ be the number of vertices in $N_Q(0, \mathcal{R}_k)$. Let $\bar{p}^j \in \mathbf{R}^{(N+1)\times 3}$ be the vector of control points $p^j(v)$ for $v \in N_Q(0, \mathcal{R}_k^j)$.

As the subdivision operators are linear, $\bar{p}^{j+1}$ can be computed from $\bar{p}^j$ using a $(N+1) \times (N+1)$ matrix $S^j$: $\bar{p}^{j+1} = S^j \bar{p}^j$.

If for some $m$ and for all $j > m$, $S^j = S^m = S$, we say that the subdivision scheme is *stationary on the k-regular complex*, or simply stationary, and call $S$ the *subdivision matrix* of the scheme. We will assume that the subdivision scheme is stationary from now on. It will be crucial to understand the limit function of the subdivision scheme on $U_1 = |N_1(0, \mathcal{R}_k)|$. This will be done by eigenvalue analysis and the introduction of the universal map.

*Eigenbasis functions* Let $\lambda_0 = 1 > \lambda_1 \geq \cdots \geq \lambda_{\tilde{N}}$, where $\tilde{N} \leq N+1$, be the distinct eigenvalues of the subdivision matrix in nonincreasing order; the condition $\lambda_0 > |\lambda_1|$ is necessary for convergence, and $\lambda_0 = 1$ is given by affine invariance. Let us consider the real Jordan normal form of the subdivision matrix $S$ given by $S = PJP^{-1}$, where $J$ is block diagonal such that

$$P^{-1}SP = J = \begin{pmatrix} J^{(1)} & & \\ & \ddots & \\ & & J^{(p)} \end{pmatrix}, \text{ where } J^{(i)} = \begin{pmatrix} J_1^{(i)} & & \\ & \ddots & \\ & & J_{q^{(i)}}^{(i)} \end{pmatrix}.$$

The $J_j^{(i)}$ are the Jordan blocks which look like

$$J_j^{(i)} = \begin{pmatrix} \lambda_i & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot \; 1 \\ & & & \lambda_i \end{pmatrix}, \text{ for real } \lambda_i \quad j = 1, \ldots, q^{(i)},$$

$$J_j^{(i)} = \begin{pmatrix} \begin{pmatrix} Re(\lambda_i) & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & Re(\lambda_i) \end{pmatrix} & \begin{pmatrix} -Im(\lambda_i) & & \\ & \ddots & \\ & & -Im(\lambda_i) \end{pmatrix} \\ \begin{pmatrix} Im(\lambda_i) & & \\ & \ddots & \\ & & Im(\lambda_i) \end{pmatrix} & \begin{pmatrix} Re(\lambda_i) & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & Re(\lambda_i) \end{pmatrix} \end{pmatrix}$$

$$\text{for complex-conjugate pair } \lambda_i, \bar{\lambda_i}, \quad j = 1, \ldots, q^{(i)}.$$

Let $n_j^{(i)}$ be the size of the Jordan block $J_j^{(i)}$ minus one (real eigenvalue) or half the size of the Jordan block $J_j^{(i)}$ minus one (complex eigenvalue). Let $c_{jr}^{(i)} \in \mathbf{R}^{N+1}$, $r = 0, \ldots, n_j^{(i)}$ or $r = 0, \ldots, 2n_j^{(i)} + 1$, be the generalized eigenvectors, which are the corresponding column of the matrix $P$. They span the subspace called $\mathbf{J}_j^{(i)}$. The vectors $c_{jr}^{(i)}$ satisfy for *real* eigenvalues:

$$Sc_{jr}^{(i)} = \lambda_i c_{jr}^{(i)} + c_{j\,r-1}^{(i)} \quad \text{if } r > 0, \quad Sc_{j0}^{(i)} = \lambda_i c_{j0}^{(i)}, \tag{3.1}$$

and for *complex-conjugate pairs* of eigenvalues:

$$\begin{aligned} Sc_{jr}^{(i)} &= Re(\lambda_i)c_{jr}^{(i)} + c_{j\,r-1}^{(i)} - Im(\lambda_i)c_{j\,r-n_j^{(i)}-1}^{(i)} \quad \text{if } r > n_j^{(i)} + 1, \\ Sc_{j\,n_j^{(i)}+1}^{(i)} &= Re(\lambda_i)c_{j\,n+1}^{(i)} - Im(\lambda_i)c_{j0}^{(i)}, \\ Sc_{jr}^{(i)} &= Re(\lambda_i)c_{jr}^{(i)} + c_{j\,r-1}^{(i)} + Im(\lambda_i)c_{j\,r+n_j^{(i)}+1}^{(i)} \quad \text{if } n_j^{(i)} + 1 > r > 0, \\ Sc_{j0}^{(i)} &= Re(\lambda_i)c_{j0}^{(i)} + Im(\lambda_i)c_{j\,n_j^{(i)}+1}^{(i)}. \end{aligned}$$

The *eigenbasis functions* are the limit functions defined by $f^{(i)}_{jr} = f[c^{(i)}_{jr}] : U_1 \to \mathbf{R}$.

Consider $\bar{p}^0 \in \mathbf{R}^{(N+1)\times 3}$. It can be represented as a linear combination of the $c^{(i)}_{jr}$:

$$\bar{p}^0 = \sum_{i,j,r} c^{(i)}_{jr} \alpha^{(i)}_{jr},$$

with $\alpha^{(i)}_{jr} \in \mathbf{R}^{1\times 3}$. Any subdivision surface $f[p] : U_1 \to \mathbf{R}^3$ can be represented as

$$f[p](y) = \sum_{i,j,r} \alpha^i_{jr} f^{(i)}_{jr}(y). \tag{3.2}$$

One can show using the definition of limit functions of subdivision and (3.1) that the eigenbasis functions satisfy the following set of *scaling relations* for real $\lambda_i$:

$$f^{(i)}_{jr}(y/2) = \lambda_i f^{(i)}_{jr}(y) + f^{(i)}_{j\,r-1}(y) \quad \text{if } r > 0, \quad f^{(i)}_{j0}(y/2) = \lambda_i f^{(i)}_{j0}(y).$$

We can assume that the coordinate system in $\mathbf{R}^3$ is always chosen in such a way that the single component of $f[p]$ corresponding to eigenvalue 1 is zero. This allows us to reduce the number of terms in (3.2) to $N$.

### 3.2 Reduction to Universal Surfaces

In [24], it was shown that for surfaces without boundary, the analysis of smoothness of subdivision can be reduced to the analysis of *universal surfaces*. In this section, we introduce universal surfaces for neighborhoods of boundary vertices.

*Universal map* The universal map is defined as $\psi(y) = \sum_{i,j,r} f^{(i)}_{jr}(y) h^i_{jr} : U_1 \to \mathbf{R}^N$, where $h^{(i)}_{jr}$ is an orthonormal basis of $\mathbf{R}^N$. Let $\alpha^1, \alpha^2, \alpha^3 \in \mathbf{R}^N$ be the vectors such that

$$\Big( (h^i_{jr}, \alpha^1),\ (h^i_{jr}, \alpha^2),\ (h^i_{jr}, \alpha^3) \Big) = \alpha^{(i)}_{jr} \in \mathbf{R}^3.$$

Then (3.2) can be rewritten as

$$f[p](y) = \Big( (\psi(y), \alpha^1),\ (\psi(y), \alpha^2),\ (\psi(y), \alpha^3) \Big),$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbf{R}^N$. This equation indicates that all surfaces generated by a subdivision scheme on $U_1$ can be viewed as projections of a single surface in $\mathbf{R}^N$. We call $\psi$ the *universal map,* and the surface specified by $\psi$ the *universal surface.* In [24], it was demonstrated that the analysis of tangent plane continuity and $C^\ell$-continuity of subdivision can be reduced to analysis of the universal surface. Not surprisingly, we will see that this also holds for subdivision schemes with boundary.

In the chosen basis, the matrix $S$ is in the real Jordan normal form. Note that by the definition of $S$ for any $\mathcal{X} \in \mathbf{R}^N$,

$$(\mathcal{X}, \psi(y/2)) = (S\mathcal{X}, \psi(y)) \Rightarrow \psi(y/2) = S^T \psi(y).$$

The universal map $\psi$ is only piecewise $C^\ell$ under the assumption that the subdivision scheme is $C^\ell$-smooth on regular complexes. Differentiability is computed with respect to the standard metric on $U_1 \subset \mathbf{R}^2$. However, one can easily construct a map $\kappa$ (see [24]) such that $\phi = \psi \circ \kappa^{-1}$ is $C^1$-continuous away from the center.

We will impose the following condition on the subdivision schemes.

**Condition A** For any $y \in U_1$,

$$\partial_1\phi(y) \wedge \partial_2\phi(y) \neq 0 \quad \text{for all } y \in U_1,\ y \neq 0.$$

This condition holds for all known practical schemes.

*Reduction theorem* We say that a subdivision scheme has a certain property if this property holds for almost all subdivision surfaces. The following theorem holds under the assumption that the subdivision scheme is $C^1$ with $C^1$-smooth boundary on regular complexes with boundary.

**Theorem 3.2** *For a subdivision scheme satisfying Condition A to be tangent plane continuous on a k-regular complex with boundary, it is necessary and sufficient that the universal surface be tangent plane continuous; for the subdivision scheme to be $C^\ell$-continuous with piecewise $C^\ell$-continuous boundary with nondegenerate corners (as in Definition 2.1), it is necessary and sufficient that the universal surface is $C^\ell$-continuous with piecewise $C^\ell$-continuous boundary with nondegenerate corners. Almost all surfaces generated by a given subdivision scheme on a k-regular complex with boundary are locally diffeomorphic to the universal surface.*

*Proof* Sufficiency is clear as any surface is a linear projection of the universal surface. Following [24], we see that if the universal surface is not tangent plane continuous, then a set of subdivision surfaces of nonzero measure is not tangent plane continuous. Also, if the universal surface has noninjective projection into the tangent plane, the same is true for a set of subdivision surfaces of nonzero measure. Furthermore, if the projection of the universal surface into the tangent plane is not $C^\ell$, the same is true for a set of subdivision surfaces of nonzero measure. It remains to prove that if the boundary of the universal surface is not $C^\ell$-continuous, or it is not $C^\ell$-continuous with nondegenerate corner, the same is true for a set of subdivision surfaces of nonzero measure. By assumption, the boundary of the surface is $C^1$-continuous away from zero. Let the two pieces of the boundary be $\gamma_i : (0, 1] \to \mathbf{R}^p$, $i = 1, 2$, with $\gamma_1(1) = \gamma_2(1)$. We can assume both pieces to be $C^1$-continuous away from one. Suppose $\gamma_1$ does not have a tangent at one; then there are at least two directions $\tau_1$ and $\tau_2$ which are limits of sequences of tangent directions to $\gamma_1(t)$ as $t$ approaches one. There is a set of three-dimensional subspaces $\pi$ of measure nonzero in the space of all three-dimensional subspaces, for which the projections of both vectors $\tau_1$ and $\tau_2$ to the subspace are not zero. If we project the universal surface to any of these subspaces, the boundary curve of the resulting surface will not be tangent continuous. For curves, tangent continuity is equivalent to $C^1$-continuity. For $C^\ell$-continuity, the proof for curves is identical to the proof for surfaces. We conclude that the curves $\gamma_1$ and $\gamma_2$ should be $C^\ell$-continuous. Similarly, if the curves are joined with continuity less than $\ell$, then almost all curves obtained by projection into $\mathbf{R}^3$ will have the same property. Finally, if the tangents to the curves coincide, the same is true for almost all projections of the curves, which means that almost all projections do not have a nondegenerate corner. □

The following important corollary immediately follows from Theorem 3.2:

**Corollary 3.3** *Almost all surfaces generated by a given $C^\ell$-continuous subdivision scheme on a k-regular complex are diffeomorphic.*

This corollary implies in particular that the same subdivision rule cannot generate convex and concave corners simultaneously in a stable way, and separate rules are required for these cases. We note that the most commonly used boundary corner rules for Loop and Catmull–Clark surfaces do not distinguish between these two cases, so cannot produce both convex and concave corners.

## 4 Criteria for Tangent Plane and $C^1$-Continuity

We will again follow [24] to establish the $C^1$-continuity criteria. We focus on a sufficient condition for $C^1$-continuity ([24] Theorem 3.6 and Theorem 4.1), which is most relevant for applications. More general necessary and sufficient conditions can be extended in a similar way. The sufficient conditions will be conditions on the eigenstructure of the subdivision matrix. We assume the scheme to be $C^1$-smooth in the interior and on regular boundary points. To state the sufficient condition, we need to define *characteristic maps*, which are commonly used to analyze $C^1$-continuity of subdivision surfaces. We use a definition somewhat different from the original definition of Reif [17]. We will define a characteristic map for a pair of subspaces. We will later introduce the dominant characteristic map which is the characteristic map defined on the dominant subspaces.

### 4.1 Conditions on Characteristic Maps

In the following, we will refer to the subspaces $\mathbf{J}_j^{(i)}$, which were defined at the end of Sect. 3.1.

**Definition 4.1** The **characteristic map** $\Phi : U_1 \to \mathbf{R}^2$ is defined for a pair of subspaces $\mathbf{J}_b^{(a)}, \mathbf{J}_d^{(c)}$ of the subdivision matrix as:

1. $(f_{b0}^{(a)}, f_{d0}^{(c)})$ if $\mathbf{J}_b^{(a)} \neq \mathbf{J}_d^{(c)}$, $\lambda_a, \lambda_c$ are real,
2. $(f_{b0}^{(a)}, f_{b\,n+1}^{(a)})$ if $\mathbf{J}_b^{(a)} = \mathbf{J}_d^{(c)}$, for a complex-conjugate pair $\lambda_a, \bar{\lambda}_a$,
3. $(f_{b0}^{(a)}, f_{b1}^{(a)})$ if $\mathbf{J}_b^{(a)} = \mathbf{J}_d^{(c)}$, $\lambda_a$ is real.

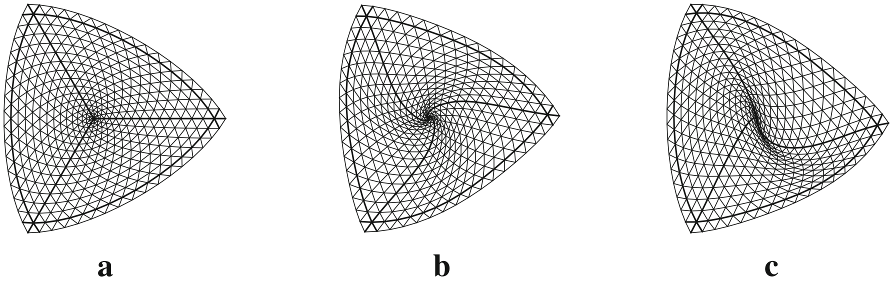


**Fig. 1** Characteristic maps: control points after 4 subdivision steps: **a** Two real eigenvalues. **b** A pair of complex-conjugate eigenvalues. **c** Single eigenvalue with Jordan block of size 2

We are not interested in other cases of pairs of subspaces since only maps of enumerated types determine tangent plane behavior of a subdivision surface. Three types of characteristic maps are shown in Fig. 1 for the closed-surface case.

The domain of a characteristic map is the neighborhood $U_1$, consisting of $k$ faces of the regular complex with boundary; we call these faces *segments*. We assume that the subdivision scheme generates $C^1$-continuous limit functions on regular complexes, and the characteristic map is $C^1$-continuous inside each segment and has continuous one-sided derivatives on the boundary. One can show (see [24]) that the Jacobian is actually continuous across those boundaries.

The characteristic map satisfies the scaling relation $\Phi(t/2) = T\Phi(t)$, where $T$ is one of the matrices

$$T_{\text{scale}} = \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_c \end{pmatrix}, \quad T_{\text{rot}} = |\lambda_a| \begin{pmatrix} cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad T_{\text{skew}} = \begin{pmatrix} \lambda_a & 1 \\ 0 & \lambda_a \end{pmatrix},$$

where $\phi$ is the argument of a complex $\lambda_a$.

*Sufficient condition for* $C^1$*-continuity* The following sufficient condition is a special case of the condition that was proved in [24]. Although all our constructions apply in the more general case, we state only a simplified version of the criterion sufficient for most applications. This condition generalizes Reif's condition [17].

Define for any two subspaces $\operatorname{ord}\left(\mathbf{J}_j^{(i)}, \mathbf{J}_l^{(k)}\right)$ to be $n_j^{(i)} + n_l^{(k)}$, if $\mathbf{J}_j^{(i)} \neq \mathbf{J}_l^{(k)}$; let $\operatorname{ord}\left(\mathbf{J}_j^{(i)}, \mathbf{J}_j^{(i)}\right) = 2n_j^{(i)} - 2$; note that for $n_j^{(i)} = 0$, this is a negative number. This number allows us to determine which components of the limit surface contribute to the limit normal (see [22,24] for details). We say that a pair of subspaces $\mathbf{J}_b^{(a)}, \mathbf{J}_d^{(c)}$ is *dominant* if for any other pair $\mathbf{J}_j^{(i)}, \mathbf{J}_l^{(k)}$ we have either $|\lambda_a\lambda_c| > |\lambda_i\lambda_k|$ or $|\lambda_a\lambda_c| = |\lambda_i\lambda_k|$ and $\operatorname{ord}\left(\mathbf{J}_b^{(a)}, \mathbf{J}_d^{(c)}\right) > \operatorname{ord}\left(\mathbf{J}_j^{(i)}, \mathbf{J}_l^{(k)}\right)$. Note that the blocks of the dominant pair may coincide. If a dominant pair exists, we call the corresponding characteristic map a **dominant characteristic map**.

**Theorem 4.2** *Suppose that there is a dominant pair* $\mathbf{J}_b^{(a)}$*,* $\mathbf{J}_d^{(c)}$*. If* $\lambda_a\lambda_c$ *positive real, and the Jacobian of the dominant characteristic map has constant sign everywhere* *on $U_1$ except zero, then the subdivision scheme is tangent plane continuous on the $k$-regular complex.*

*If this characteristic map is injective, the subdivision scheme is $C^1$-continuous.*

This theorem can be proved following the proof of the criterion of [24] also for surfaces with boundary, without any changes. In the special case when all Jordan blocks are trivial, this condition reduces to an analog of Reif's condition. However, Theorem 4.2 does not characterize the behavior of the boundary curve.

*Criterion for piecewise $C^1$-continuity of the boundary.* Assuming that the scheme at a boundary vertex satisfies the conditions of Theorem 4.2, we establish additional conditions which guarantee that the scheme for almost all control meshes generates $C^1$-continuous surfaces with piecewise $C^1$-continuous boundary with nondegenerate corners. We call $I_1$ and $I_2$ the two parts of the boundary line of $U_1$ achieved by excluding the center vertex. We use notation $\partial_1$ for the derivative in the direction of this boundary line. $\partial_2$ will be the orthogonal direction. We will call the two components of the dominant characteristic map $f_1$ and $f_2$ in the following theorem.

**Theorem 4.3** *Suppose a subdivision scheme satisfies the conditions of Theorem 4.2 for boundary vertices of valence $k$. Then the scheme is piecewise $C^1$-continuous with nondegenerate corners for boundary vertices of valence $k$ if the following conditions are satisfied:*

1. *$\lambda_a$ and $\lambda_c$ are positive real.*
2. *Suppose $\lambda_a > \lambda_c$ (diagonal scaling matrix, asymmetric scaling). Then the scheme is boundary $C^1$-continuous if and only if $\partial_1 f_1 \neq 0$ and has the same sign on $I_1$ and $I_2$, or $\partial_1 f_1 \equiv 0$ on $I_1$ and $I_2$.*
   *The scheme is a nondegenerate corner scheme if and only if $\partial_1 f_1 \neq 0$ on $I_1$ and $\partial_1 f_1 \equiv 0$ on $I_2$. The same is true if $I_1$ and $I_2$ are exchanged.*
3. *Suppose $\mathbf{J}_b^{(a)} = \mathbf{J}_d^{(c)}$ (scaling matrix is a Jordan block of size 2) and $\partial_1 f_1$ does not vanish on $I_1$ and $I_2$. The scheme is boundary $C^1$-continuous if $\partial_1 f_2$ has the same sign everywhere on $I_1$ and $I_2$, and if $\partial_1 f_2(t_1) = 0$ for a $t_1 \in I_1 \cup I_2$, then $\partial_1 f_1(t_1)$ needs to have this sign as well. Nondegenerate corners cannot be generated by a scheme of this type.*
4. *Suppose $\lambda_a = \lambda_c$ (diagonal scaling matrix, symmetric scaling). The boundary is $C^1$-continuous if and only if there is a nontrivial linear combination $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ identically vanishing on $I_1$ and $I_2$, and any other independent linear combination has the same sign on $I_1$ and $I_2$. The scheme is a corner scheme if and only if there is a linear combination $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ identically vanishing on $I_1$ and a different linear combination $\beta_1 \partial_1 f_1 + \beta_2 \partial_1 f_2$ identically vanishing on $I_2$, with $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ linearly independent.*

*Proof* For each of the boundary segments defined on $I_1$ and $I_2$, we need to show that the limit of the tangent exists at the common endpoint. If these limits coincide, then the boundary curve of the universal surface is $C^1$-continuous; if the limits have different directions, then the universal surface has a nondegenerate corner.

First, we observe that by assumption the dominant characteristic map has nonzero Jacobian on the boundary. This means that one of the components has nonzero derivative along the boundary $\partial_1 f_1(t) \neq 0$, or $\partial_1 f_2(t) \neq 0$ at any point $t \in I_1 \cup I_2$. Consider the tangent to the boundary of the surface defined by the dominant characteristic map. It is a two-dimensional vector $v(t) = (\partial_1 f_1(t), \partial_1 f_2(t))$, where $t$ is a point of $I_1$ or $I_2$. The tangent satisfies the scaling relation of the form $v(t/2) = 2Tv(t)$, where $T$ is the scaling matrix for the dominant characteristic map. The direction of the tangent has a limit if and only if $T$ is either $T_{\text{scale}}$ or $T_{\text{skew}}$ and its eigenvalues are positive (Lemma 3.1, [24]). As the projection of the universal surface is arbitrarily well approximated by the dominant characteristic map, or coincides with it for simple Jordan structures of the subdivision matrix, we conclude that for the universal surface boundary to have well-defined tangents at zero, the eigenvalues of the dominant characteristic map have to be positive and real. However, this condition is not sufficient for the existence of tangents.

*Diagonal scaling matrix, asymmetric case* First we consider the case of the dominant subspace pair $J_b^a$, $J_d^c$ with $a \neq c$ (different eigenvalues). In this case, the sequences $\partial_1 f_1(t/2^m)$ and $\partial_1 f_2(t/2^m)$, for $\partial_1 f_1(t), \partial_1 f_2(t) \neq 0$, change at a different rate. This can be easily seen from the scaling relation. Moreover, the ratio $\|\partial_1 f_2(t/2^m)\|/\|\partial_1 f_1(t/2^m)\|$ approaches zero as $m \to \infty$.

Suppose at some points $t_1$, $t_2$ of $I$, $\partial_1 f_1(t_1) \neq 0$, and $\partial_1 f_1(t_2) = 0$. Then $\partial_1 f_2(t_2) \neq 0$, and the tangents at points $t_2/2^m$ all point in the direction $\pm e_2$, where $e_2$ is the unit vector along the coordinate axis corresponding to $f_2$. $\|\partial_1 f_2(t_1/2^m)\|/\|\partial_1 f_1(t_1/2^m)\| \to 0$ as $m \to \infty$; thus, at points $t_1/2^m$, the direction of the tangent approaches $\pm e_1$. We conclude that there is no limit, unless $\partial_1 f_1$ is either nowhere or everywhere zero $I_1$. Same applies to $I_2$. Conversely, if $\partial_1 f_1$ is nowhere zero, then the limit tangent direction at the center is $\pm e_1$. If it is zero everywhere, then by assumption about the dominant characteristic map, $\partial_1 f_2$ is nowhere zero, and the limit tangent direction is $\pm e_2$. The choice of sign in each case depends on the sign of $\partial_1 f_1$ or $\partial_1 f_2$.

If $\partial_1 f_1$ is not zero and has the same sign on both $I_1$ and $I_2$, then the tangent is continuous and the boundary curve is $C^1$-continuous. If $\partial_1 f_1 \equiv 0$ on $I_1$ and $I_2$, the images of $I_1$ and $I_2$ under the dominant characteristic map are straight lines on the $e_2$ axis, and therefore the boundary curve is $C^1$-continuous. If it is zero on $I_1$ and nonzero on $I_2$, then the tangents are not parallel, and the surface defined by the dominant characteristic map has a corner, and the same applies for $I_1$ and $I_2$ interchanged, which proves the second part.

*Scaling matrix is a Jordan block of size 2* The second condition of the theorem applies if the dominant characteristic map components correspond to a subspace of size 2, i.e., satisfy $f_1(t/2) = \lambda_a f_1(t) + f_2(t)$. Thus, $\partial_1 f_1 \equiv 0$ implies $\partial_1 f_2 \equiv 0$ on $I_1$ or $I_2$. Otherwise $v(t/2^m)$ converges to $\pm e_1$ for any $t$ on $I_1$ as well as $I_2$. If $\partial_1 f_2(t) \neq 0$, its sign determines the sign of the limit tangent.

*Diagonal scaling matrix, symmetric case* In the symmetric case where $a = c$, the sequences $\partial_1 f_1(t/2^m)$ and $\partial_1 f_2(t/2^m)$ change at the same rate, and any linear combination $\alpha_1 f_1 + \alpha_2 f_2$ is also an eigenbasis function. Suppose $f_1$ and $f_2$ come from different subspaces of the same eigenvalue which have the same size. Suppose $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ does not vanish identically on $I_1$ for any nontrivial choice of $\alpha_1$

and $\alpha_2$. Pick $t_1$ and $t_2$ such that the vectors $v(t_i) = [\partial_1 f_1(t_i), \partial_1 f_2(t_i)]$ are linearly independent. Then the sequences $v(t_1/2^m)$ and $v(t_2/2^m)$ converge to different limit directions. Therefore, for the limit tangents at zero to exist, there should be a nontrivial linear combination of $\partial_1 f_1$ and $\partial_1 f_2$ which vanishes on $I_1$. If $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ is such a combination, it is easy to see that the limit tangent direction is, up to the sign, the direction of the vector $[-\alpha_2, \alpha_1]$. For the boundary to be $C^1$-continuous, the direction should be the same on two sides. Finally, the tangents on two sides exist and do not coincide if the vectors $(\alpha_1, \alpha_2)$ for $I_1$ and $I_2$ are linearly independent. □

An interesting corollary of this theorem is that in the symmetric case, the images of $I_1$ and $I_2$ under the dominant characteristic map are straight line segments. In this case, we have that $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2 \equiv 0$, which means that $\alpha_1 f_1 + \alpha_2 f_2$ is constant and the image of $(f_1, f_2)$ is a straight line segment. Note that this is not necessary if the eigenvalues $\lambda_a$ and $\lambda_c$ are different.

### 4.2 Analysis of Characteristic Maps

To verify conditions of Theorem 4.2, we need to establish that the dominant characteristic map is regular and injective and verify that it has the expected behavior on the boundary. Typically, analysis of the boundary behavior is relatively easy, as in most cases the boundary curve is independent from the interior. In this section, we focus on regularity and injectivity of the dominant characteristic map.

*Regularity of the characteristic map* Just as in the case of interior points, we use self-similarity of the dominant characteristic map to verify the regularity condition of Theorem 4.2: for any $t \in U_1$, the Jacobian satisfies $J[\Phi](t/2) = 4\lambda_a \lambda_c J[\Phi](t)$. It is immediately clear that to prove regularity of the dominant characteristic map, it is sufficient to consider the Jacobian on a single annular portion of $U_1$. As all vertices of such a ring are either regular or boundary regular, we can estimate the Jacobian of the dominant characteristic map using tools developed for analysis of subdivision on regular grids. This time we have to consider subdivision schemes not just on regular meshes but on regular meshes with boundary.

*Injectivity of the characteristic map* Even if the Jacobian of a map is nonzero everywhere, only local injectivity is guaranteed. For interior vertices, self-similarity of the dominant characteristic maps allows one to reduce the injectivity test to computing the index of a closed curve around zero [23]. A closed curve with winding number $\pm 1$ gives injectivity in a small neighborhood of zero. This test cannot be applied for boundary points, as there are no closed curves around zero.

For boundary points, a different test (Theorem 4.4) suffices, which is even easier to apply in all cases that we have considered. However, unlike the curve index test, it does not immediately yield a computational algorithm that applies to an arbitrary scheme.

The dominant characteristic map can be extended using scaling relations to a complete $k$-regular complex with boundary. In the following theorem, we assume that the dominant characteristic map is defined on the whole complex $|\mathcal{R}_k|$.

**Theorem 4.4** *Suppose a characteristic map $\Phi$ satisfies the following conditions:*

1. *the preimage $\Phi^{-1}(0)$ contains only one element, 0;*
2. *the characteristic map has a Jacobian of constant sign at all points of the domain besides 0;*
3. *the image of the boundary of the characteristic map has no self-intersections;*
4. *the image of the characteristic map is not the whole plane.*

*Then this characteristic map is injective.*

*Proof* We can show that the characteristic map is continuous at infinity, and if P is the stereographic projection of the sphere to the plane such that the south pole gets mapped to 0, $\tilde{\Phi} = P^{-1}\Phi P$ is a continuous mapping of a subset $D = P^{-1}(|\mathcal{R}_k|)$ of the sphere into the sphere, with the south pole mapped to the south pole. $\tilde{\Phi}$ is a local homeomorphism away from the south pole.

We observe that the points of the boundary of the image $\tilde{\Phi}(D)$ can be images only of the boundary of $D$ due to the properties of local homeomorphisms. Therefore, $\partial\,\tilde{\Phi}(D) \subset \tilde{\Phi}(\partial D)$.

The image of the boundary $\tilde{\Phi}(\partial D)$ has no self intersections. It is easy to see that the boundary of the domain $\partial D$ is a simple closed Jordan curve, and so is its image $\tilde{\Phi}(\partial D)$. Suppose $\partial\,\tilde{\Phi}(D) \neq \tilde{\Phi}(\partial D)$. Then there is a point $y \in \tilde{\Phi}(\partial D)$ which is an interior point of $\tilde{\Phi}(D)$. As $\tilde{\Phi}(\partial D)$ separates the sphere into two linearly connected domains, we can connect each point in either domain to point $y$ with a continuous curve which does not intersect $\partial\,\tilde{\Phi}(D)$. Thus, any two points on the sphere can be connected by a continuous curve which does not intersect $\partial\,\tilde{\Phi}(D)$. We conclude that the image $\tilde{\Phi}(D)$ is the whole sphere. Therefore, either $\partial\,\tilde{\Phi}(D) = \tilde{\Phi}(\partial D)$, or the image is the whole sphere. The latter option contradicts the last condition of the theorem. We conclude that $\tilde{\Phi}(D)$ is simply-connected since its boundary is a simple closed Jordan curve. In order to prove injectivity, we will show that the mapping on the domain excluding the south pole is a covering and coverings from connected domains to simply connected domains are injective. If we exclude the south pole of the sphere, the mapping is a local homeomorphism. Consider an interior point $y$ of the image, and the set $\tilde{\Phi}^{-1}(y)$. Suppose it is infinite. Then it has a limit point, which cannot be an interior point of $D$ (otherwise, $\tilde{\Phi}$ is not a local homeomorphism at that point). Similarly, it cannot be a boundary point, unless it is the south pole. It cannot be the south pole $x_s$ for which $P(x_s) = 0$, because then $\tilde{\Phi}(x_s) = y$, which means that $P(y) = 0$, which contradicts the assumption $\Phi^{-1}(0) = \{0\}$. We conclude that $\tilde{\Phi}^{-1}(y)$ is finite for each point y of the interior of the image. A similar argument holds for boundary points away from the poles. $\tilde{\Phi}$ is a local homeomorphism and maps the boundary exactly to the boundary. Let $y$ be a point of the image away from poles, and let $x_1, x_2, \ldots x_n$ be points of $\tilde{\Phi}^{-1}(y)$. We know there exist pairwise disjoint neighborhoods $U_i$ of the $x_i$ such that $\tilde{\Phi}|_{U_i} : U_i \to f(U_i)$ is a homeomorphism to an open subset of $\tilde{\Phi}(D)$ containing $y$ for all $i$. The set $\bigcap_i \tilde{\Phi}(U_i)$ is an evenly covered neighborhood of $x$ if $\tilde{\Phi}^{-1}(\bigcap_i \tilde{\Phi}(U_i))$ is wholly contained in $\bigcup_i U_i$, which is not always the case. Instead, we consider $V = \bigcap_i \tilde{\Phi}(U_i)\backslash\tilde{\Phi}(U_i^C)$, where $U_i^C$ are the complements of $U_i$. Then $V$ is the required evenly covered open neighborhood of $y$. Indeed, $V$ is open and $y \in V$ because $\bigcup_i U_i$ contains all preimages of $y$. Finally, by construction, $\tilde{\Phi}(V)$ is the disjoint union of the $\tilde{\Phi}^{-1}(V) \cap U_i$, each of which is homeomorphically mapped onto $V$ by $\tilde{\Phi}$.

We conclude that the characteristic map is injective. □

# 5 Verification of $C^1$-continuity

In the following, we will analyze certain boundary rules of the two most important subdivision schemes, namely Loop [9] and Catmull–Clark [2]. We will show that they lead to subdivision surfaces that are $C^1$ boundary continuous with nondegenerate corners as defined in Definition 2.1. In order to do that, we need to show that the characteristic map defined by the dominant eigenpair satisfies the conditions given in Theorem 4.3 and 4.4. In both cases, we know that we can extend the subdivision over regular boundary vertices by mirroring. This would create a $C^1$-smooth subdivision surface and the boundary is a $C^1$-smooth curve. We therefore only have to look at tagged corner vertices or nonregular boundary points. As we discuss later, we have to choose the right boundary subdivision rules for the regular smooth boundary vertices, of course.

## 5.1 Loop Scheme

The Loop scheme with boundary is given by the rules in Fig. 2. For each valence of a boundary vertex, there are 4 free parameters that we will pick in the following discussion: $\alpha_k, \beta_k, \gamma_k, \delta_k$. These rules are such that the support of each mask is the same as for Loop or the cubic B-spline on the boundary and are equivalent to those for regular meshes with boundary. We also keep the symmetry for rules in boundary regions. Some parts of our analysis are similar to the analysis performed by Schweitzer [18].

We assume that all coefficients in the masks are positive. This choice is sufficiently general and allows a complete parameter-dependent eigenanalysis. For the specific schemes that we consider, the boundaries do not depend on the control points in the interior. We will furthermore assume that $\alpha_3 = 1/8$ and $\beta_3 = 1/2$ when the point is

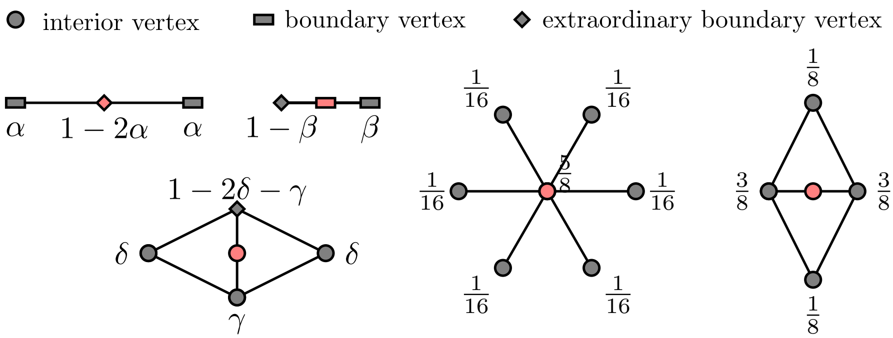


**Fig. 2** Modified Loop subdivision rules for meshes with boundary [1]

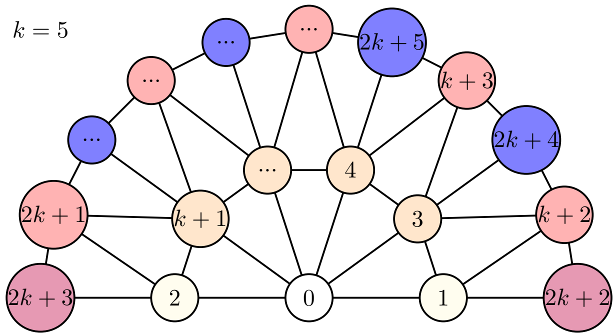


**Fig. 3** Ordering for the subdivision matrix

not marked a corner, which are the regular rules for the cubic B-spline. This guarantees a boundary curve that is $C^1$. Therefore we only need to consider boundary vertices which are either marked as corners or have a valence different from 3.

*Subdivision matrix and eigenstructure* We assume that $k > 1$ first. The subdivision matrix (in the ordering of Fig. 3 for a boundary vertex with $k$ adjacent triangles) has the following form (more details in [6]):

$$S = \left(\begin{array}{c|cc|c|c|cc|c}
1-2\alpha & \alpha & \alpha & & & & & \\
\hline
1-\beta & \beta & & & & & & \\
1-\beta & & \beta & & & & & \\
\hline
a_1 & A_{10} & & A_{11} & & & & \\
\hline
a_2 & A_{20} & & A_{21} & \frac{1}{8} I_k & & & \\
\hline
1/8 & 3/4 & & & & 1/8 & & \\
1/8 & & 3/4 & & & & 1/8 & \\
\hline
a_3 & & & A_{31} & A_{32} & & & \frac{1}{16} I_{k-1}
\end{array}\right).$$

The vectors $a_1$ and $a_3$ have length $k-1$, the vector $a_2$ has length $k$, $I_k$ and $I_{k-1}$ are unit matrices of sizes $k$ and $k-1$. Note that the eigenvalues of the matrix are $1/8$, $1/16$, the eigenvalues of the upper-left $3 \times 3$ block $A_{00}$, and the eigenvalues of the matrix $A_{11}$. The matrix $A_{11}$ is tridiagonal, of size $k-1 \times k-1$. The eigenvalues of $A_{00}$ are $1, \beta, \beta - 2\alpha$, where the eigenvector to 1 is the vector $\mathbf{e} = [1, \ldots, 1]$. Following [18], we observe that $k-1 \times k-1$ tridiagonal symmetric matrices have the following eigenvectors, independent of the matrix:

$$v^j = [\sin j\theta_k, \sin 2j\theta_k, \ldots \sin (k-1) j\theta_k], \quad j = 1 \ldots k-1, \tag{5.1}$$

**Table 1** Eigenvector of $\beta$

| | one eigenvector ($i$th entry) | two eigenvectors ($i$th entries) |
|---|---|---|
| $r > 2,\ k$ odd | $(-1)^i \cosh{(i-k/2)\,\theta}$, $r = 2\cosh\theta$ | $(-1)^i \sinh i\theta,\ (-1)^i \sinh{(i-k)\,\theta}$, $r = 2\cosh\theta$ |
| $r > 2,\ k$ even | $(-1)^i \sinh{(i-k/2)\,\theta}$, $r = 2\cosh\theta$ | $(-1)^i \sinh i\theta,\ (-1)^i \sinh{(i-k)\,\theta}$, $r = 2\cosh\theta$ |
| $r = 2,\ k$ odd | $(-1)^i$ | $(-1)^i i,\ (-1)^i (i-k)$ |
| $r = 2,\ k$ even | $(-1)^i\ (n-k/2)$, | $(-1)^i i,\ (-1)^i (i-k)$ |
| $-2 < r < 2$ | $\sin{(i-k/2)\,\theta}$, $r = -2\cos\theta$ | $\sin i\theta,\ \sin{(i-k)\,\theta}$, $r = -2\cos\theta$ |
| $r = -2$ | $i - k/2$ | $i,\ i-k$ |
| $r < -2$ | $\sinh{(i-k/2)\,\theta}$, $r = -2\cosh\theta$ | $\sinh i\theta,\ \sinh{(i-k)\,\theta}$, $r = -2\cosh\theta$ |

where $\theta_k = \pi/k$. Multiplying the matrix $A_{11}$ to the vectors, we see that the eigenvalues are $\lambda_j = 2\delta\cos j\theta_k + \gamma_k$. We have determined that the eigenvalues of the subdivision matrix are 1, $\beta_k$,$\beta_k - 2\alpha_k$, 1/8, 1/16, and $\lambda_j = 2\delta\cos j\theta_k + \gamma_k$, $j = 1\dots k-1$. The eigenvectors corresponding to the eigenvalues $\lambda_j$ do not depend on the matrix and are given by (5.1). The eigenvectors corresponding to the eigenvalue $\beta$ depend on the ratio $r = (\gamma - \beta)/\delta$; for $\alpha \neq 0$, there is a single eigenvector. For $\alpha = 0$, there is a pair of eigenvectors for the case when $\beta$ is not an eigenvalue of $A_{11}$. If $\beta$ is an eigenvalue of $A_{11}$, it has a nontrivial Jordan block of size 2. Depending on the range of $r$, Table 1 shows the eigenvector for $v^\beta$ or the 2 eigenvectors if $\beta$ is a double eigenvalue.

If k=1, the matrix in this case has eigenvalues $\beta$, $\beta - 2\alpha$, and a triple eigenvalue 1/8. The eigenvectors can be trivially computed.

*Coefficients for smooth boundary vertices* One possible choice was given by Hoppe et al. [7] and examined in detail in [18]. In our notation, this choice corresponds to $\beta_k = 5/8$, $\alpha_k = 1/8$, $\gamma_k = 3/8$, $\delta_k = 1/8$ for extraordinary vertices $k \neq 3$, and $\beta_3 = 1/2$. (In the following, we will drop the index $k$.) Remarkably, the ratio $r$ is $-2$. The disadvantage of this choice is that the shape of the boundary curve depends on the valence of the vertices on the boundary; hence it becomes impossible to join two meshes continuously along a boundary if extraordinary vertices on two sides do not match. If we require the boundary curve to be a cubic spline, $\beta$ has to be 1/2 and $\alpha$ has to be 1/8. Further, we pick $\delta = 1/2$.

We consider the cases $k > 2$, $k = 2$, and $k = 1$ separately.

*Case $k > 2$*. Once $\alpha$, $\beta$, and $\delta$ are fixed, the eigenvalues of the subdivision matrix become 1, $\beta = 1/2$, $\beta - 2\alpha = 1/4$, 1/8, 1/16, and $\lambda_j = (1/4)\cos j\theta_k + \gamma$.

There are two choices of $\gamma$ that we find particularly interesting: $\gamma = 1/4$ and $\gamma = 1/2 - 1/4\cos\theta_k$. The first choice, $\gamma = 1/4$, is the maximal value of $\gamma$ independent of $k$ for which it is in the correct range, meaning $1/2 \geq |\lambda_1| > 1/4$ and $|\lambda_1| > |\lambda_i|$, for all $k > 2$. Note that in this case, $r = -2$ again. The second choice leads to equal subdominant eigenvalues $\beta = \lambda_1 = 1/2$. In this case, $r = -2\cos\theta_k$. The expressions

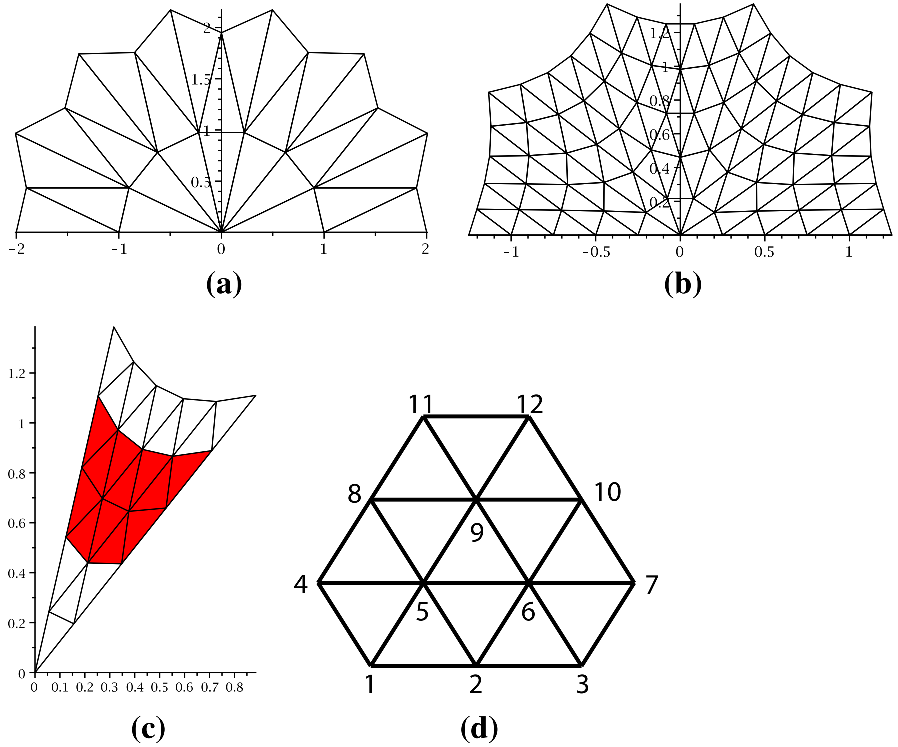

**Fig. 4** **a** $K = N_2(0, R_7)$; **b** $N_2^2(0, R_5)$; **c** annular part of characteristic map in one sector; **d** control net of a triangle patch

for the subdominant eigenvectors are $v_j^1 = \sin j\theta_k$ and $v_j^\beta = \cos j\theta_k$; i.e., they form a half of a regular $2k$-gon.

The choice of $\gamma = 1/2 - 1/4 \cos\theta_k$, although being slightly more complex, appears to be more natural. It has the additional advantage of coinciding with the regular value $\gamma = 3/8$ for $k = 3$.

*Case* $k = 2$. In this case, the eigenvalues are 1, 1/2, 1/4, 1/8, 1/16, and $\lambda_1 = \gamma$. Thus, we need to pick $1 > \gamma > 1/4$ to get the same eigenvectors as in the case $k > 2$. We observe that the choice of $\gamma = 1/4$ also results in a $C^1$ surface, although the behavior of the scheme becomes less desirable.

*Case* $k = 1$. The subdominant eigenvalues are 1/2 and 1/4.

**Proposition 5.1** *Let* $\beta = 1/2, \alpha = 1/8, \delta = 1/8,$ *and* $\gamma = 1/2 - 1/4\cos\theta_k$. *Let* $\Phi$ *be the characteristic map which is defined by the eigenvectors of* $\beta = 1/2$ *and* $\lambda_1 = 1/4\cos(\theta_k) + \gamma = 1/2$ . *Then the conditions of Theorem 4.4 are satisfied.*

*Proof* We consider the 2-neighborhood of the k-regular complex with boundary $K = N_2(0, R_k)$ shown in Fig. 4a. The control points $p \in \mathcal{P}(K, \mathbf{R}^2)$ are given by the eigenvector data of the eigenvectors corresponding to 1/2:

We subdivide twice as seen in Fig. 4b. The $\mathbf{R}^2$ triangular patch of the red triangles shown for one sector in Fig. 4c are given by:

$$\begin{aligned}
&f[p](u, v, w) = B \cdot Q \cdot P, \text{ where} \\
&B = \Big(u^4, 4u^3v, 4u^3w, 6u^2v^2, 12u^2vw, 6u^2w^2, \\
&\quad 4uv^3, 12uv^2w, 12uvw^2, 4uw^3, v^4, 4v^3w, 6v^2w^2, 4vw^3, w^4\Big)
\end{aligned}$$

and $Q$ is a $15 \times 12$ matrix given in [18] and $P \in \mathbf{R}^{12\times 2}$ such that $P_i \in \mathbf{R}^2$ is the eigenvector data on the point $i$ numbered as shown in Fig. 4d.

We have to consider the 12 triangles in each sector independently. Furthermore, boundary sectors also need to be considered separately. The limit map behaves as if we would extend the mesh by its mirror image over the boundary. We have to separately check the following cases: (1) $k = 1$, (2) $k = 2$, (3) $k = 3, i = 1$ or $3$, (4) $k = 3, i = 2$, (5) $k > 3, i = 3, \ldots, k-2$, (6) $k > 3, i = 2$ or $k-1$ (7) $k > 3, i = 1$ or $k$.

For each of the triangles, we compute the polynomial $f[p](u, v, w)$ in Bernstein–Bezier coordinates. To prove that a polynomial in Bernstein–Bezier coordinates is positive on the given triangle we need to check that all the coefficients are positive.

1. In order to prove that there is no other element than 0 in the preimage $\Phi^{-1}(0)$, we check that $f_1^2 + f_2^2 > 0$ in each triangle of each sector. Then by the scaling property we know that
$$f_1(t/2)^2 + f_2(t/2)^2 = \lambda = 1/4\left(f_1(t)^2 + f_2(t)^2\right) > 0.$$
Since $\|\Phi(t)\| > 0$ for all $t \neq 0$, we proved the first statement.
2. We compute the Jacobian
$$\begin{aligned}
J[\Phi] = \partial_x f_1 \partial_y f_2 - \partial_x f_2 \partial_y f_1 &= (\partial_u f_1 - \partial_w f_1)(\partial_v f_2 - \partial_w f_2) \\
&\quad - (\partial_u f_2 - \partial_w f_2)(\partial_v f_1 - \partial_w f_1)
\end{aligned}$$
in each triangle and see that the coefficients of J (a polynomial in Bezier coordinates) are all of the same sign independent of $k$ and $i$. Therefore the polynomial has the same sign everywhere. By the scaling property, we can extend it from the ring to the sector. The scaling property for the Jacobian is
$$J[\Phi](t/2) = 4\beta\lambda_1 J[\Phi](t) = J[\Phi](t).$$
3. We take the 2 triangles in the third ring that form the boundary to the second ring and find the expression of the polynomial that describes the boundary curve. We want to show that the angle grows monotonically, and since the angle is given by $\arctan(f_1/f_2)$, it is enough to show that $f_1/f_2$ grows monotonically. We compute $f_1' f_2 - f_2' f_1$, the numerator of the derivative of $f_1/f_2$, and observe that all coefficients have the same sign. Since the denominator is a square, it is also positive. We conclude that $f_1/f_2$ is monotonic, and therefore the angle is monotonic, and in

each sector the curves cannot intersect. Neither there can be intersections between sectors as the curves limit lies strictly within their sectors.
4. As box spline surfaces lie strictly within the convex hull of their control net, the image of the characteristic map has to lie in the upper half-plane.

All explicit checks were done in Maple. □

We can now conclude by Theorem 4.4 that the characteristic map is injective. It is also regular, as the Jacobian of the characteristic map has constant sign everywhere. This means that in order for the scheme to be $C^1$-smooth with smooth boundary, we have to check the 4th condition of Theorem 4.3, since the subdominant eigenvalues are equal and span a 2-dimensional eigenspace. Since the boundary curve is a B-spline interpolating points on the x-axes, we get that $\partial_1 f_1 > 0$ and $\partial_1 f_2 = 0$, giving us the condition for a scheme that is $C^1$-smooth with smooth boundary.

*Coefficients for corner vertices* Separate rules have to be defined for corners. We choose the scheme interpolating the corner vertices, which results in $\alpha_k = 0$. As a consequence, the block $A_{00}$ has a double eigenvalue $\beta$. For a corner, the tangent plane is defined by the two tangents at the non-$C^1$-continuous point of the boundary. We leave all other parameters free, which means the rules of Hoppe et al. [7] are included. If $\beta$ has multiplicity 3 with Jordan blocks of size 2 and 1, which happens when it is an eigenvalue of $A_{11}$, the scheme is not likely to be tangent plane continuous; we assume that this is not the case. Otherwise the eigenvectors of interest can be found explicitly for various values of $r = (\gamma - \beta)/\delta$ as in Table 1, second column.

It is easy to see that positive values of $r$ are of little interest to us, because the components of the vectors alternate signs in these cases and are likely to produce nonregular characteristic maps. For $r \leq -2$, we are guaranteed to get a convex configuration of control points for the characteristic map, because the characteristic map interpolates the boundary curve. In the case $r \in (-2, 0)$, the eigenvectors corresponding to the eigenvalue $\beta$ can be taken to be $\sin i\theta$, $\sin\,(i-k)\,\theta$, where $\theta$ is such that $r = -2\cos\theta$. *This means that the corner is convex if $\theta < \theta_k$, and concave otherwise.* More explicitly, the convexity condition is $r = -2\cos\theta < -2\cos\theta_k$ or $\gamma < \beta - 2\delta\cos\theta_k$. Note that the same condition is required for the double eigenvalue $\beta$ to be subdominant. We conclude that for $r < 0$, the subdivision scheme can generate only convex smooth corners. One can show that this is true even if we do not assume that $\alpha = 0$. In the case $k = 1$, one can also immediately see that the corner produced by subdivision is convex.

*Concave corner vertices* We assume that $k > 1$. It is impossible to have stationary subdivision rules for a triangular mesh producing a concave corner for $k = 1$. As we have observed, concave corners cannot be produced simply by changing some of the coefficients using the same stencil. One can also show that no scheme with positive coefficients can produce interpolating smooth concave corners. It is possible to construct rules to produce $C^1$-continuous surfaces with concave corners, but negative coefficients and larger support have to be used.

Our approach to deriving the rules is based on the idea of reduction of the magnitudes of all eigenvalues, excluding 1 and $\beta = 1/2$. It turns out that this approach leads to particularly simple rules for subdivision.

For the scheme to produce smooth surfaces at a corner vertex, the eigenvectors $x^\beta$, $x'^\beta$ of the eigenvalue $\beta = 1/2$ should be subdominant. If we choose these eigenvectors to be $x^\beta = [0, 0, 1, v_1^\beta / \sin k\theta, \ldots]$, $x'^\beta = [0, 1, 0, v'^\beta_1 / \sin k\theta \ldots]$, corresponding left eigenvectors are very simple: $l = [-1, 0, 1, 0, \ldots]$, $l' = [-1, 1, 0, 0, \ldots 0]$. The left eigenvector $l^0$ for the eigenvalue 1 is $[1, 0, \ldots 0]$. Consider the following modification of the vector of control points:

$$\tilde{p} = (1-s)p + s\left((l^0, p)x^0 + (l, p)x^\beta + (l', p)x'^\beta\right),$$

where $x_0$ is the eigenvector $[1, \ldots 1]$ of the eigenvalue 1. Substituting expressions for the left eigenvectors, we get

$$\tilde{p} = (1-s)p + s\left(p^0 x^0 + \left(p_0^1 - p^0\right)x^\beta + \left(p_k^1 - p^0\right)x'^\beta\right). \tag{5.2}$$

The first entry of $p$ is called $p^0$, and the second and third are called $p_k^1$ and $p_0^1$. The effect of this transformation is to scale all components of $p$ in the eigenbasis of the subdivision matrix by $(1-s)$ except those corresponding to the eigenvalues 1 and $\beta$. If repeated at each subdivision step, it is equivalent to scaling all eigenvalues except 1 and $\beta$ by $(1-s)$.

To simplify the rules, we observe that it is unnecessary to scale multiple eigenvalues 1/16 and 1/8 of the lower-right blocks of the subdivision matrix. If we apply the rules (5.2), not to the whole vector of control points $p$ but to a truncated part, we can write

$$\tilde{p} = Tp = diag(M, I)p,$$

where $M$ is such that equation (5.2) is satisfied for the first vertices. Multiplying this matrix by the subdivision matrix on the left, eigenvalues are by construction 1, 1/2, $(1-s)\,(2\delta\cos j\theta_k + \gamma)$, $j = 1 \ldots k-1$, and 1/8 and 1/16.

By choosing the value of $s$ so that $(1-s)\,(2\delta\cos\theta_k + \gamma) < 1/2$, we can ensure that $\beta = 1/2$ is the subdominant eigenvalue. The parameter $s$ can be viewed as a tension parameter for the corner, which determines how flat the surface is near the corner.

We consider the case of convex and concave corners together:

**Proposition 5.2** *Let $\beta = 1/2, \alpha = 0, \delta = 1/8$ and $\gamma = 1/2 - 1/4\cos(\theta)$, where $0 < \theta < \pi$ for convex corners and $\pi < \theta < 2\pi$ for concave corners. Then $\Phi$, the characteristic map, is defined by the eigenvectors corresponding to $\beta = 1/2$. Then conditions of Theorem 4.4 are satisfied.*

*Proof* The proof is done exactly the same way as in the noncorner case. The characteristic map we need to check has a parameter $\theta$. We can, however, still verify that for any $\theta$ in the given range, we obtain positive coefficients. In the case of the concave corner, the convex hull of the control points no longer lies in the upper half-plane.

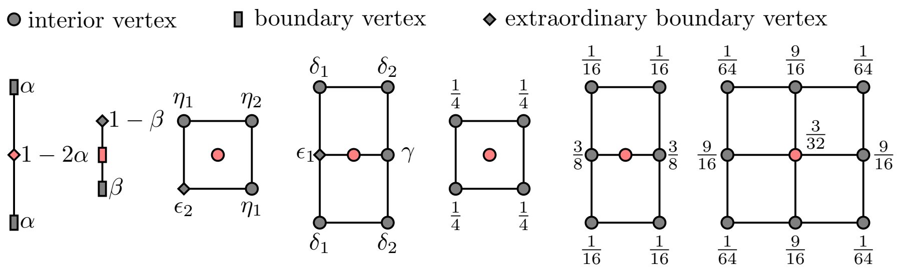


**Fig. 5** Modified Catmull–Clark subdivision rules for meshes with boundary [1]

However, we can look at the sectors individually and see that the limit function does not span the whole complex plane. □

With this proposition, we have established that the characteristic map is injective and regular. Now we need to check condition 4 in Theorem 4.3. Since the boundary of the control mesh away from 0 is a straight line for $k > 1$, the limit curve which is a B-spline is also a straight line. This means it satisfies the condition.

### 5.2 Catmull–Clark Scheme

The analysis of the eigenstructure of the boundary subdivision matrices becomes more complex in the case of the Catmull–Clark scheme. Using the Catmull–Clark scheme as an example, we describe a technique that can be used to analyze schemes with larger support.

The subdivision stencils are shown in Fig. 5. We have 6 parameters $\alpha$, $\beta$, $\gamma$, $\delta_1$, $\delta_2$, $\eta_1$, $\eta_2$, all dependent on the valence $k$, but we omit the subscript. We show some of the eigenstructure analysis, but also no scheme from this class can generate surfaces with smooth concave corners.

*Subdivision matrix* The subdivision matrix for a given valence $k$ has a somewhat more complex structure for the Catmull–Clark scheme (see also [6]). In the block form, the matrix can be written as

$$\left(\begin{array}{c|c|c|c} A_{00} & & & \\ \hline A_{10} & \frac{1}{8} I_2 & & \\ \hline A_{20} & A_{21} & A_{22} & \\ \hline A_{30} & A_{31} & A_{32} & \frac{1}{64} I_k \end{array}\right),$$

where the diagonal blocks are

$$
A_{00} = \left(\begin{array}{c|cc|cccc|cccc}
1-2\alpha & \alpha & \alpha & & & & & & & & \\
1-\beta & \beta & & & & & & & & & \\
1-\beta & & \beta & & & & & & & & \\
\hline
\epsilon_1 & \delta_1 & & \gamma & \delta_1 & & & \delta_2 & \delta_2 & & \\
\epsilon_1 & & & \delta_1 & \gamma & \delta_1 & & & \delta_2 & \delta_2 & \\
 & & & & & \ddots & & & & \ddots & \\
\epsilon_1 & & \delta_1 & & & \delta_1 & \gamma & & & \delta_2 & \delta_2 \\
\hline
\epsilon_2 & \eta_2 & & \eta_2 & & & & \eta_1 & & & \\
\epsilon_2 & & & \eta_2 & \eta_2 & & & & \eta_1 & & \\
\vdots & & & & \ddots & & & & & \ddots & \\
\epsilon_2 & & \eta_2 & & & & \eta_2 & & & & \eta_1
\end{array}\right),
$$

$$
A_{22} = \left(\begin{array}{cccc|cccc|cccc}
\frac{3}{32} & & & & 0 & \frac{1}{64} & & & \frac{1}{64} & & & \\
 & \frac{3}{32} & & & & & \frac{1}{64} & & & \frac{1}{64} & & \\
 & & \ddots & & & & \ddots & & & & \ddots & \\
 & & & \frac{3}{32} & & & & \frac{1}{64} & & & \frac{1}{64} & 0 \\
\hline
0 & & & & \frac{1}{16} & & & & & & & \\
\frac{1}{16} & & & & & \frac{1}{16} & & & & & & \\
 & \frac{1}{16} & & & & & \ddots & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & \frac{1}{16} & & & & \frac{1}{16} & & & & \\
\hline
\frac{1}{16} & & & & & & & & \frac{1}{16} & & & \\
 & \frac{1}{16} & & & & & & & & \frac{1}{16} & & \\
 & & \ddots & & & & & & & & \ddots & \\
 & & & \frac{1}{16} & & & & & & & & \frac{1}{16} \\
 & & & 0 & & & & & & & &
\end{array}\right).
$$

Note that all eigenvalues of $A_{22}$ are guaranteed to be less than $1/8$ (the sum of the magnitudes of the entries on any line does not exceed 1/8). Thus, only the eigenvalues of $A_{00}$ are of interest to us. Next, we observe that the matrix $A_{00}$ itself has two blocks on the diagonal; the first $3 \times 3$ block is identical to the block that we have considered for the Loop scheme; it has eigenvalues $1$, $\beta$ and $\beta - 2\alpha$. The remaining block denoted by $\bar{A}_{00}$ is the one we need to consider.

*Transformation of the subdivision matrix* Assume $k > 1$. The eigenvalues and eigenvectors of $\bar{A}_{00}$ can be found directly from the recurrences derived from the subdivision rules. We take a somewhat different approach, similar to the DFT analysis used for interior extraordinary vertices. This approach has greater generality and can potentially be applied to analyze subdivision schemes with larger supports. To find the eigenvalues of $\bar{A}_{00}$, we introduce a new set of control points. We replace the $k$ control points of type 2, $p_i^2$, $i = 0 \ldots k-1$, (convention as in Fig. 6) with $k+1$ control points $\tilde{p}_i^2$ satisfying

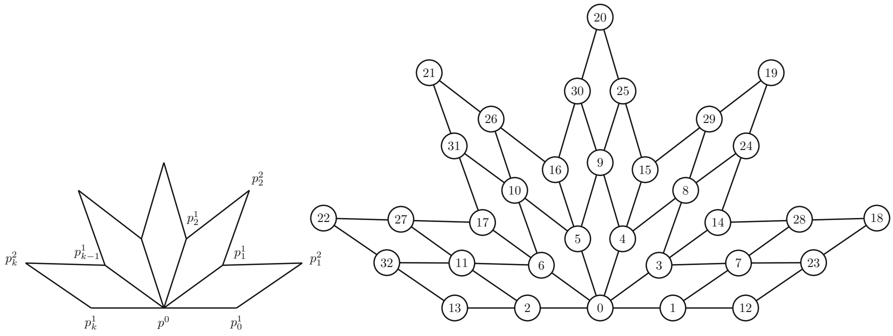


**Fig. 6** *Left* naming convention, *right* ordering

$$p_i^2 = \frac{1}{2}\left(\tilde{p}_i^2 + \tilde{p}_{i+1}^2\right) \tag{5.3}$$

for $i = 0 \ldots k-1$. We define the $k+1$ control points of type 1: $\tilde{p}_i^1 = p_i^1$. Note that we increase the number of control points. These equations clearly do not define the new control points uniquely, but an arbitrary choice of solution is adequate for our purposes. In matrix form, the relation between the original vector of control points of types 1 and 2 and the transformed vector $\tilde{p}$ can be written as $p = T\tilde{p}$, where $T$ is a $2k+1 \times 2k+2$ matrix.

In addition, we define the subdivision rules for the new control points. We choose the rules for $\tilde{p}$ in such a way that the relations (5.3) also hold after the subdivision rules are applied to $p$ and $\tilde{p}$. Let $\tilde{S}$ be the subdivision matrix for $\tilde{p}$. Then our choice of rules means that

$$ST\tilde{p} = T\tilde{S}\tilde{p}.$$

If $\lambda$ is an eigenvalue of $\tilde{S}$, then $\tilde{S}\tilde{p}^\lambda = \lambda\tilde{p}^\lambda$, where $\tilde{p}^\lambda$ is the corresponding eigenvector, and

$$ST\tilde{p}^\lambda = T\tilde{S}\tilde{p}^\lambda = \lambda T\tilde{p}^\lambda.$$

Therefore, $\lambda$ is also an eigenvalue of $S$, unless $T\tilde{p}^\lambda = 0$. Note that the null-space of $T$ has dimension 1 and contains the vector $p_i^1 = 0$, $\tilde{p}_i^2 = (-1)^i$. Hence a complete set of eigenvalues and eigenvectors of $S$ can be obtained from eigenvalues and eigenvectors of $\tilde{S}$ once we exclude the eigenvalue corresponding to this vector, if it happens to be an eigenvector.

We choose the subdivision rule for $\tilde{p}_i^2$ as follows:

$$\left[\tilde{S}\tilde{p}\right]_i^2 = \epsilon_2 p^0 + 2\eta_2 p_i^1 + \eta_1 \tilde{p}_i^2.$$

In terms of new control points, the rule for control points of type 1 becomes

$$[Sp]_i^1 = \epsilon_1 p^0 + \delta_1 \left(p_{i-1}^1 + p_{i+1}^1\right) + \gamma p_i^1 + \frac{\delta_2}{2}\left(\tilde{p}_{i-1}^2 + 2\tilde{p}_i^2 + \tilde{p}_{i+1}^2\right).$$

The matrix $\bar{A}_{00}$ is transformed into

$$\tilde{A}_{00} = \left(\begin{array}{cccc|cccccc} \gamma & \delta_1 & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & & & \\ \delta_1 & \gamma & \delta_1 & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & & \\ & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \\ & & \delta_1 & \gamma & & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} \\ \hline & & & & \eta_1 & & & & & \\ 2\eta_2 & & & & & \eta_1 & & & & \\ & 2\eta_2 & & & & & \eta_1 & & & \\ & & \cdot & & & & & \eta_1 & & \\ & & & 2\eta_2 & & & & & \eta_1 & \\ & & & & & & & & & \eta_1 \end{array}\right).$$

Note that $\tilde{p}_0^2$ and $\tilde{p}_k^1$ depend on $p_0^1$ and $p_k^1$, which are outside this matrix. Rearranging the entries, we get the matrix

$$\left(\begin{array}{cc|cccc|cccc} \eta_1 & & & & & & & & & \\ & \eta_1 & & & & & & & & \\ \hline \frac{\delta_2}{2} & & \gamma & \delta_1 & & & \delta_2 & \frac{\delta_2}{2} & & \\ & & \delta_1 & \gamma & \delta_1 & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & \\ & & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ & \delta_2 & & & \delta_1 & \gamma & & & \frac{\delta_2}{2} & \delta_2 \\ \hline & & 2\eta_2 & & & & \eta_1 & & & \\ & & & 2\eta_2 & & & & \eta_1 & & \\ & & & & \cdot & & & & \eta_1 & \\ & & & & & 2\eta_2 & & & & \eta_1 \end{array}\right).$$

This matrix has a double eigenvalue $\eta_1$. The rest of the eigenvalues are eigenvalues of the matrix $A$ consisting only of the 4 tridiagonal sub-blocks. We have already observed that three diagonal matrices have eigenvectors independent from the entries of the matrix. Denote by $H$ the matrix with entries $\sin ij\theta_k$, with $\theta_k = \pi/k$ as before, $i, j = 1 \ldots k-1$. This matrix has a similar role in the analysis of subdivision matrices of boundary vertices as the DFT matrix has in the analysis of subdivision matrices of interior vertices. The transform $\mathcal{H}$ is defined as diag $(H, H)$. The inverse of this matrix is $\mathcal{H}^{-1} = \text{diag}\,((2/k)H, (2/k)H)$. Then

$$\mathcal{H}A\mathcal{H}^{-1} = \frac{2}{k}\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \frac{2}{k}\begin{pmatrix} HB_{00}H & HB_{01}H \\ HB_{10}H & HB_{11}H \end{pmatrix},$$

and each block $HB_{ij}H$ is a diagonal matrix. Finally, we apply a permutation matrix such that the matrix $A$ is reduced to the block diagonal form

$$P\mathcal{H}A\mathcal{H}^{-1}P^{-1} = \begin{pmatrix} B(1) & & & \\ & B(2) & & \\ & & . & \\ & & & B(k-1) \end{pmatrix},$$

where the blocks $B(i)$, $i = 1 \ldots k-1$, are $2 \times 2$ matrices

$$B(i) = \begin{pmatrix} \gamma + 2\delta_1 \cos \dfrac{i\pi}{k} & \delta_2 \left(1 + \cos \dfrac{i\pi}{k}\right) \\ 2\eta_2 & \eta_1 \end{pmatrix}.$$

This allows us to compute the eigenvectors for $k \neq 1$. For $k = 1$, we can compute the eigenvalues and eigenvectors directly. We can now perform the whole analysis just as in the Loop case. In the case of a smooth boundary vertex choosing $\alpha = 1/8, \beta = 1/2, \gamma = 3/8 - 1/4 \cos \theta_k, \eta_1 = \eta_2 = 1/4, \delta_1 = \delta_2 = 1/16$, we get a scheme that is $C^1$-smooth with $C^1$-smooth boundary. For the corner vertices, we pick $\alpha = 0, \beta = 1/2, \gamma = 3/8 - 1/4 \cos \theta$ and the rest as above, and we get a scheme that is $C^1$-smooth with piecewise $C^1$-smooth boundary with nondegenerate corners. For the concave corner we have to do the same trick as in the Loop case.

## 6 Conclusions

We have presented constructive sufficient conditions for tangent plane continuity and $C^1$-continuity for surfaces with boundary. We have demonstrated for the two most commonly used schemes that the modified rules of [1] for Loop and Catmull–Clark subdivision schemes satisfy sufficient conditions for $C^1$-continuity. The techniques we used to analyze the Catmull–Clark subdivision matrix structure can be used for other schemes on surfaces with boundary.

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