Finding Limits of Multiscale SPDEs

David Kelly Martin Hairer

Mathematics Institute
University of Warwick
Coventry UK CV4 7AL

dtbkelly@gmail.com

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Multiscale Systems, Warwick

A Classic Result

Let X_ε satisfy the SDE

$$dX_\varepsilon(t) = \frac{1}{\varepsilon} b(X_\varepsilon/\varepsilon)dt + \sigma(X_\varepsilon/\varepsilon)dB(t)$$

with b, σ being periodic $\mathcal{C}^2$ functions and X_ε taking values on S^1.

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If σ is strictly positive and $\int b/\sigma^2 dx = 0$ then

$$X_\varepsilon \Rightarrow c\bar{X}$$

where $\bar{X}$ is a BM and $c > 0$ is some constant determined by b and σ.

Homogenization of PDEs

If we denote the generator of X_ε as

$$L_\varepsilon = \frac{1}{\varepsilon} b(x/\varepsilon)\partial_x + \frac{1}{2}\sigma^2(x/\varepsilon)\partial_x^2$$

where $x \in S^1$. We can use the classic result to **homogenize PDEs**

$$\begin{aligned} \partial_t u_\varepsilon &= L_\varepsilon u_\varepsilon \qquad \rightarrow \qquad & \partial_t u &= c\partial_x^2 u \\ u_\varepsilon(0) &= g & u(0) &= g \end{aligned}$$

since $u_\varepsilon(t) = \mathbb{E}[g(X_\varepsilon(t))]$.

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since $u_\varepsilon(t) = \mathbb{E}[g(X_\varepsilon(t))]$. We can also add a forcing term

$$\begin{aligned} \partial_t u_\varepsilon &= L_\varepsilon u_\varepsilon + f \\ u_\varepsilon(0) &= g \end{aligned} \qquad \rightarrow \qquad \begin{aligned} \partial_t u &= c\partial_x^2 u + f \\ u(0) &= g \end{aligned}$$

since $u_\varepsilon(t) = \mathbb{E}[g(X_\varepsilon(t))] + \int_0^t \mathbb{E}[f(X_\varepsilon(t-s), s)]ds$. This works provided $f = f(x,t)$ is nice enough.

Homogenization of SPDEs

We try to find a limit to

$$du_\varepsilon(t) = L_\varepsilon u_\varepsilon(t)dt + Q^{1/2}dW(t)$$
$$u_\varepsilon(0) = 0$$

where $\frac{dW}{dt}$ is space-time white noise and $Q^{1/2}$ is a positive, bounded linear operator with $Q^{1/2}e^{ikx} = \lambda_k e^{ikx}$ with $\lambda_k \geq 0$. Hence, we can also write

$$du_\varepsilon(t) = L_\varepsilon u_\varepsilon(t)dt + \sum_{k\in\mathbb{Z}} \lambda_k e^{ikx} dW_k(t)$$

where W_k are complex BMs with $W_k = W^*_{-k}$ and otherwise independent.

First Result

Theorem (Compact Q)

If $\lambda_k \to 0$ and u satisfies

$$du(t) = c\partial_x^2 u(t)dt + Q^{1/2}dW(t) \; ,$$

then

$$\mathbb{E} \sup_{t \in [0,T]} \|u_\varepsilon(t) - u(t)\|_{H^{-s}}^2 \to 0 \quad \text{as } \varepsilon \to 0,$$

for any $s > 3/4$.

- If we know $\lambda_k \ll k^{-\alpha}$ for some $\alpha > 0$ then we can improve to $s > (3/4 - 3\alpha/2) \wedge 0$.

Second Result

Theorem (Bounded Q)

If $\lambda_k \to \bar{\lambda}$ and $\hat{u}$ satisfies

$$d\hat{u}(t) = c\partial_x^2 \hat{u}(t)dt + \sum_k \left(|\lambda_k|^2 + |\bar{\lambda}|^2(\|\rho\|^2 - 1)\right)^{1/2} e^{ikx} d\hat{W}_k(t)$$

for some new sequence of BMs $\hat{W}_k$ and ρ satisfies $L^*\rho = 0$ with $\langle \rho, 1\rangle = 1$. Then there exists a sequence of processes $\hat{u}_\varepsilon \stackrel{dist}{=} u_\varepsilon$ such that

$$\mathbb{E} \sup_{t\in[0,T]} \|\hat{u}_\varepsilon(t) - \hat{u}(t)\|_{H^{-s}}^2 \to 0 \quad \text{as } \varepsilon \to 0,$$

for any $s > 1$.

- eg. For STWN, $d\hat{u} = c\partial_x^2 \hat{u} dt + \|\rho\| d\hat{W}$

Sketch of Proof

We use the following **interpolation** strategy

$$\mathbb{E}\|u_\varepsilon(t) - u(t)\|_{H^{-s}}^2 = \sum_{|m|<\varepsilon^{-\beta}} \mathbb{E}|\langle u_\varepsilon(t) - u(t), e^{imx}\rangle|^2 (1+m^2)^{-s} + \sum_{|m|\geq\varepsilon^{-\beta}} \mathbb{E}|\langle u_\varepsilon(t) - u(t), e^{imx}\rangle|^2 (1+m^2)^{-s}$$

for some $\beta \in (0,1)$. For the **high modes**

$$\sum_{|m|\geq\varepsilon^{-\beta}} \mathbb{E}|\langle u_\varepsilon(t) - u(t), e^{imx}\rangle|^2 (1+m^2)^{-s} \lesssim \varepsilon^{2\beta s} \left(\mathbb{E}\|u_\varepsilon(t)\|^2 + \mathbb{E}\|u(t)\|^2\right) .$$

If $\mathbb{E}\|u_\varepsilon(t)\|^2$ doesn't blow up too much, then we need only worry about the **low modes**.

Sketch of Proof

We have a **mild solution** given by

$$u_\varepsilon(t) = \sum_k \lambda_k \int_0^t S_\varepsilon(t-s) e^{ikx} dW_k(s)$$

where S_ε is the semigroup generated by L_ε.

Sketch of Proof

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$$u_\varepsilon(t) = \sum_k \lambda_k \int_0^t S_\varepsilon(t-s) e^{ikx} dW_k(s)$$

where S_ε is the semigroup generated by L_ε.
We take $|m| < \varepsilon^{-\beta}$ and try to approximate

$$\langle u_\varepsilon, e^{imx} \rangle = \sum_k \lambda_k \int_0^t \langle S_\varepsilon(t-s) e^{ikx}, e^{imx} \rangle dW_k(s)$$

This can be approximated using (a quantitative version of) our classical result

$$S_\varepsilon(t-s) e^{ikx} = e^{ikx - k^2(t-s)} + \mathcal{O}(\varepsilon k)$$

We have that

$$\begin{aligned}\langle u_\varepsilon(t), e^{imx}\rangle &= \lambda_m \int_0^t e^{-m^2(t-s)} dW_m(s) + \sum_k \lambda_k \int_0^t \langle R_\varepsilon^k(t-s), e^{imx}\rangle dW_k \\ &= \langle u, e^{imx}\rangle + \sum_k \lambda_k \int_0^t \langle R_\varepsilon^k(t-s), e^{imx}\rangle dW_k(s)\end{aligned}$$

Estimates for R_ε^k get bad when $k \sim \varepsilon^{-1}$, so this only works if $\lambda_k \ll k^{-1/2}$.

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$$= \langle u, e^{imx}\rangle + \sum_k \lambda_k \int_0^t \langle R_\varepsilon^k(t-s), e^{imx}\rangle dW_k(s)$$

Estimates for R_ε^k get bad when $k \sim \varepsilon^{-1}$, so this only works if $\lambda_k \ll k^{-1/2}$. A better approach is to use the **adjoint semigroup**

$$\langle u_\varepsilon(t), e^{imx}\rangle = \sum_k \lambda_k \int_0^t \langle e^{ikx}, S_\varepsilon^*(t-s)e^{imx}\rangle dW_k(s)$$

With the adjoint classical result

$$S_\varepsilon^*(t-s)e^{imx} = \rho(x/\varepsilon)e^{imx-m^2(t-s)} + \mathcal{O}(\varepsilon m)$$

We have

$$\langle u_\varepsilon(t), e^{imx}\rangle = \sum_k \lambda_k \langle e^{ikx}, \rho(x/\varepsilon)e^{imx}\rangle \int_0^t e^{-m^2(t-s)} dW_k(s) + R$$

Since $\varepsilon^{-1} \in \mathbb{N}$, only terms with $k = m + p/\varepsilon$ for any $p \in \mathbb{Z}$ will remain.

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Since $\varepsilon^{-1} \in \mathbb{N}$, only terms with $k = m + p/\varepsilon$ for any $p \in \mathbb{Z}$ will remain. We have

$$\sum_p \lambda_{m+p/\varepsilon} \langle \rho, e^{ipx}\rangle \int_0^t e^{-m^2(t-s)} dW_{m+p/\varepsilon}(s)$$

and isolating the first term

$$\lambda_m \int_0^t e^{-m^2(t-s)} dW_m(s) + \sum_{p\neq 0} \lambda_{m+p/\varepsilon} \langle \rho, e^{ipx}\rangle \int_0^t e^{-m^2(t-s)} dW_{m+p/\varepsilon}(s)$$

If $\lambda_k \to 0$, this proves the result for Q compact.

If $\lambda_k \to \bar{\lambda} \neq 0$, then these **extra terms no longer vanish**. We have that

$$\sum_p \lambda_{m+p/\varepsilon} \langle \rho, e^{ipx} \rangle \int_0^t e^{-m^2(t-s)} dW_{m+p/\varepsilon}(s)$$

is *equal in distribution* to

$$\left(\sum_p |\lambda_{m+p/\varepsilon}|^2 |\langle \rho, e^{ipx} \rangle|^2 \right)^{1/2} \int_0^t e^{-m^2(t-s)} d\hat{W}_m(s)$$

$$\to \big(|\lambda_m|^2 + |\bar{\lambda}|^2 (\|\rho\|^2 - 1) \big)^{1/2} \int_0^t e^{-m^2(t-s)} d\hat{W}_m(s)$$

This proves the result for Q bounded.

Ergodic Properties

If μ_ε and μ are (respectively) the **invariant measures** of the original and limiting SPDEs then we can show that

$$\mu_\varepsilon \to \mu \quad \text{as } \varepsilon \to 0$$

under the H^{-s} Wasserstein metric.

Extensions

Similar results hold for all SPDEs of the form

$$du_\varepsilon(t) = L_\varepsilon u_\varepsilon(t)dt + \sum_k q_k(x/\varepsilon)e^{ikx} dW_k(t)$$

for C^1 periodic functions q_k. This is the structure possessed by noise that is **cell-translation invariant**. i.e. The law of the noise is invariant if you shift it by a cell.

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Thank You!