Homogenization for chaotic dynamical systems

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Outline of talk

- Invariance principles (turning chaos into Brownian motion)
- Homogenization of chaotic slow-fast systems
- Why rough path theory is useful

Invariance principles

Donsker's Invariance Principle I

Let $\{\xi_i\}_{i\geq 0}$ be i.i.d. random variables with $\mathbf{E}\xi_i = 0$ and $\mathbf{E}\xi_i^2 < \infty$.
Let $S_n = \sum_{j=0}^{n-1} \xi_i$ and define the path

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} \ .$$

Then Donsker's invariance principle * states that $W^{(n)} \to_w W$ in cadlag space, where W is a multiple of Brownian motion.

It's called an invariance principle because the result doesn't care what random variables you use.

Donsker's Invariance Principle II (Young 98, Melbourne, Nicol 05,08)

We can even replace $\{\xi_i\}_{i \geq 0}$ with iterations of a chaotic map.

That is, let $T : \Lambda \to \Lambda$ be a "sufficiently chaotic" map, with T-invariant ergodic measure μ on probability space $(\Lambda, \mathcal{M})$, and let $v : \Lambda \to \mathbb{R}^d$ satisfy $\int_\Lambda v \, d\mu = 0$. If

$$W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v \circ T^j ,$$

then $W^{(n)} \to_w W$ in the cadlag space, where W is Brownian motion with covariance

$$\Sigma^{\alpha\beta} = \int_\Lambda v^\alpha v^\beta d\mu + \sum_{n=1}^{\infty} \int_\Lambda v^\gamma (v^\beta \circ T^n) d\mu + \sum_{n=1}^{\infty} \int_\Lambda v^\beta (v^\gamma \circ T^n) d\mu$$

Donsker's Invariance Principle III

We can do the same in continuous time, with a chaotic flow.

That is, let $\{\phi_t\}$ be a "sufficiently chaotic" flow on Λ, with invariant measure μ. Let $v : \Lambda \to \mathbb{R}^d$ satisfy $\int_\Lambda v \, d\mu = 0$. If

$$W^{(n)}(t) = \varepsilon \int_0^{\varepsilon^{-2} t} v \circ \phi_s \, ds \ ,$$

then $W^{(n)} \to_w W$ in the sup-norm topology, where W is Brownian motion with covariance

$$\Sigma^{\alpha\beta} = \int_\Lambda v^\alpha v^\beta d\mu + \int_0^\infty \int_\Lambda v^\gamma (v^\beta \circ \phi_s) d\mu ds + \int_0^\infty \int_\Lambda v^\beta (v^\gamma \circ \phi_s) d\mu ds$$

What does "sufficiently chaotic" mean?

In the discrete time case

sufficiently chaotic $\approx$ decay of correlations

More precisely, for the above $v \in L^1(\Lambda)$ and all $w \in L^\infty(\Lambda)$, we have that

$$\left| \int_\Lambda v \; w \circ T^n d\mu \right| \lesssim \|w\|_\infty n^{-\tau} \; ,$$

for τ big enough.

This holds for

- Uniformly expanding or uniformly hyperbolic
- Non-uniformly hyperbolic maps modeled by "Young towers".

Eg. Henon-like attractors, Lorenz attractors (flows)

Invariance principle: sketch of proof I

The continuous invariance principle follows from the discrete invariance principle.

Invariance principle: sketch of proof II

The idea is to use a known invariance principle for martingales. Namely, suppose $m_1, m_2, \dots$ is a stationary, ergodic, martingale difference sequence. If

$$\sum_{i=0}^{n-1} m_i \quad \text{is a martingale, then} \quad n^{-1/2} \sum_{i=0}^{\lfloor nt \rfloor - 1} m_i \to_w BM$$

So if $\sum_{i=0}^{n-1} v \circ T^i$ were a martingale then we'd be in business.

Invariance principle: Idea of proof III

Actually, it is only a semi-martingale, with respect to the "filtration"

$$T^{-1}\mathcal{M}, T^{-2}\mathcal{M}, T^{-3}\mathcal{M}, \dots$$

where $\mathcal{M}$ is the sigma algebra from the original measure space. Moreover, we can write

$$v = m + a$$

where

$$M_n := \sum_{i=0}^{n-1} m \circ T^i \quad \text{is a martingale}$$

and

$$A_n := \sum_{i=0}^{n-1} a \circ T^i \quad \text{is bounded uniformly in } n$$

This is called a martingale approximation.

Invariance principle: Idea of proof IIII

So if we write

$$
\begin{aligned}
W^{(n)}(t) &= M^{(n)}(t) + A^{(n)}(t) \\
&= n^{-1/2} \sum_{i=0}^{\lfloor nt \rfloor - 1} m \circ T^i + n^{-1/2} \sum_{i=0}^{\lfloor nt \rfloor - 1} a \circ T^i
\end{aligned}
$$

Then we clearly have that

$$
W^{(n)} \to_w W \,.
$$

However ... the world isn't quite so nice, since in fact

$$
T^{-1}\mathcal{M} \supset T^{-2}\mathcal{M} \supset T^{-3}\mathcal{M} \supset \dots
$$

So we need to **reverse** time.

Using invariance principles for slow-fast systems

Slow-Fast systems in continuous time

This idea can be applied to the homogenisation of slow-fast systems. For example

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}) v(Y^{(\varepsilon)}(t)) + f(X^{(\varepsilon)}, Y^{(\varepsilon)})$$
$$\frac{dY^{(\varepsilon)}}{dt} = \varepsilon^{-2} g(Y^{(\varepsilon)}) \,,$$

where the fast dynamics $Y^{(\varepsilon)}(t) = Y(\varepsilon^{-2} t)$ with $\dot{Y} = g(Y)$ describing a chaotic flow, with ergodic measure μ and again $\int v \, d\mu = 0$. We can re-write the equations as

$$dX^{(\varepsilon)} = h(X^{(\varepsilon)}) dW^{(\varepsilon)} + f(X^{(\varepsilon)}, Y^{(\varepsilon)}) dt \quad \text{where}$$
$$W^{(\varepsilon)}(t) \stackrel{def}{=} \varepsilon^{-1} \int_0^t v(Y^{(\varepsilon)}(s)) ds = \varepsilon \int_0^{\varepsilon^{-2} t} v(Y(s)) ds$$

Fast-Slow systems in discrete time

We can do the same for discrete time systems. For example, define $X : \mathbb{N} \to \mathbb{R}^d$ and $Y : \mathbb{N} \to \Lambda$ by

$$X(n+1) = X(j) + \varepsilon h(X(n))v(Y(n)) + \varepsilon^2 f(X(n), Y(n))$$
$$Y(n+1) = TY(n)\ ,$$

where $T : \Lambda \to \Lambda$ is a chaotic map. If we let $X^{(\varepsilon)}(t) = X(\lfloor \varepsilon^{-2} t \rfloor)$ and $Y^{(\varepsilon)} = Y(\lfloor \varepsilon^{-2} t \rfloor)$ then we have

$$dX^{(\varepsilon)} = h(X^{(\varepsilon)})dW^{(\varepsilon)} + f(X^{(\varepsilon)}, Y^{(\varepsilon)})dt$$

where

$$W^{(\varepsilon)}(t) \stackrel{def}{=} \varepsilon \sum_{j=0}^{\lfloor \varepsilon^{-2} t \rfloor - 1} v \circ T^j$$

and where the integral is computed as a left Riemann sum.

For simplicity, we will focus on the more natural continuous time homogenization.

What is known? (Melbourne, Stuart '11)

If the flow is chaotic enough so that

$$W^{(\varepsilon)}(t) = \varepsilon \int_0^{\varepsilon^{-2}t} v(y(s))ds \to_w W \ ,$$

and **either** $d = 1$ **or** $h = \mathrm{Id}$

then we have that $X^{(\varepsilon)} \to X$, where

$$dX = h(X) \circ dW + F(X)dt \ ,$$

where the stochastic integral is of Stratonovich type and where $F(\cdot) = \int f(\cdot, v)d\mu(v)$.

Continuity with respect to noise (Sussmann '78)

The crucial fact that allows these results to go through is continuity with respect to noise. That is, let

$$dX = h(X)dU + F(X)dt ,$$

where U is a smooth path.

If $d = 1$ **or** $h(x) = Id$ for all x, then $\Phi : U \to X$ is continuous in the sup-norm topology.

Therefore, if $W^{(\varepsilon)} \to_w W$ then $X^{(\varepsilon)} = \Phi(W^{(\varepsilon)}) \to_w \Phi(W)$.

This famously falls apart when the noise is both **multidimensional** and **multiplicative**. That is, when $d > 1$ **and** $h \neq Id$.

This fact is the main motivation behind rough path theory

Continuity with respect to rough paths (Lyons '97)

As above, let

$$dX = h(X)dU + F(X)dt \; ,$$

where U is a smooth path. Let $\mathbb{U} : [0, T] \to \mathbb{R}^{d \times d}$ be defined by

$$\mathbb{U}^{\alpha\beta}(t) \stackrel{def}{=} \int_0^t U^\alpha(s) dU^\beta(s) \; .$$

Then the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is **continuous** with respect to the "d_γ topology".

This is known as **continuity** with respect to the **rough path** $(U, \mathbb{U})$.

The d_γ topology

The space of γ-rough paths is a metric space (but not a vector space).

Objects in the space are pairs of the form $(U, \mathbb{U})$ where U is a γ-Hölder path and where $\mathbb{U}$ is a natural "candidate" for the iterated integral $\int UdU$.

On the space we define the metric

$$d_\gamma(U, \mathbb{U}, V, \mathbb{V}) = \sup_{s,t \in [0,T]} \left(\frac{|U(s,t) - V(s,t)|}{|s-t|^\gamma} + \frac{|\mathbb{U}(s,t) - \mathbb{V}(s,t)|}{|s-t|^\gamma} \right)$$

where

$$U(s,t) = U(t) - U(s) \quad \text{and} \quad \mathbb{U}^{\beta\gamma}(s,t) = \int_s^t U^\beta(s,r) dU^\gamma(r)$$

Continuity with respect to rough paths

Thus, we set

$$\mathbb{W}^{(\varepsilon),\alpha\beta}(t) = \int_0^t W^{(\varepsilon),\alpha}(s) dW^{(\varepsilon),\beta}(s) ,$$

(which is defined uniquely). If we can show that

$$(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \to_w (W, \mathbb{W})$$

where $\mathbb{W}$ is some **identifiable** type of iterated integral of W, then we have

$$X^{(\varepsilon)} \to X = \Phi(W, \mathbb{W}) .$$

Convergence of the rough path

We have the following result

Theorem (Kelly, Melbourne '13)

If the fast dynamics are "sufficiently chaotic", then $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \to_w (W, \mathbb{W})$ *where* W *is a Brownian motion and*

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2} D^{\alpha\beta} t$$

where

$$D^{\beta,\gamma} = \int_0^\infty \int_\Lambda (v^\beta\, v^\gamma \circ \phi_s - v^\gamma\, v^\beta \circ \phi_s)\, d\mu\, ds\ ,$$

and ϕ *is the flow generated by the chaotic dynamics* $\dot{y} = f(y)$.

Homogenised equations

Corollary

Under the same assumptions as above, the slow dynamics $X^{(\varepsilon)} \to_w X$ *where*

$$dX = h(X) \circ dW + \left(G(X) + \frac{1}{2} D^{\beta\gamma} \partial^\alpha h^\beta(X) h^{\alpha\gamma}(X) \right) dt \; .$$

Rmk. The only case where one gets Stratonovich is when the Auto-correlation is symmetric. For instance, if the flow is **reversible**.

Idea of proof I

We will focus on how to prove the iterated invariance principle.

Theorem (Kurtz & Protter 92)

Suppose that U_n, V_n are semi-martingales and that $(U_n, V_n) \to_w (U, V)$ in cadlag space, with the limits also semi-martingales. Suppose that V_n has decomposition $V_n = M_n + C_n$ and that

1) $\sup_n \mathbf{E}[M_n]_t < \infty$, *for each* $t \in [0, T]$.

2) $\sup_n \mathbf{E}|C_n|_{TV} < \infty$

Then

$$(U_n, V_n, \int U_n dV_n) \to_w (U, V, \int U dV) \ ,$$

in cadlag space, where all the above integrals of of **Ito** *type. We say that $\{V_n\}$ is good sequence of semi-martingales.*

Idea of proof II

The sequence $W^{(n)}$ **is not** good, but the sequence $M^{(n)}$ **is** good.

Hence, to calculate $\int W^{(n)} dW^{(n)}$, we need to expand

$$\int W^{(n)} dW^{(n)} = \int M^{(n)} dM^{(n)} + \int M^{(n)} dA^{(n)} + \int A^{(n)} dM^{(n)} + \int A^{(n)} dA^{(n)}$$

The extra terms can be calculated using the **ergodic theorem**.

Extensions

- What if the slow equation is *non-product* form?

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}(t), Y^{(\varepsilon)}(t))$$

- What if the slow equation is coupled into the fast equation?

Thanks!