Homogenisation for multidimensional fast-slow systems

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Donsker's Invariance Principle I

Let $\{\xi_i\}_{i\geq 0}$ be i.i.d. random variables with $\mathbf{E}\xi_i = 0$ and $\mathbf{E}\xi_i^2 < \infty$.
Let $S_n = \sum_{j=0}^{n-1} \xi_i$ and define the path

$$W_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} \; .$$

Then Donsker's invariance principle * states that $W_n \to_w W$ in cadlag space, where W is a multiple of Brownian motion.

It's called an invariance principle because the result doesn't care what random variables you use.

Donsker's Invariance Principle II (Young 98, Melbourne, Nicol 05,08)

We can even replace $\{\xi_i\}_{i\geq 0}$ with iterations of a chaotic map.

That is, let $T : \Lambda \to \Lambda$ be a "sufficiently chaotic" map, with T-invariant ergodic measure μ, and let $v : \Lambda \to \mathbb{R}^d$ satisfy $\int_\Lambda v \, d\mu = 0$. Then

$$W_n(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v \circ T^j \,,$$

then $W_n \to_w W$ in cadlag space, where W is a multiple of Brownian motion.

Fast-Slow Systems

This idea can be applied to the homogenisation of slow-fast systems. For example

$$\frac{dx_\varepsilon}{dt} = \varepsilon^{-1} h(x_\varepsilon) v(y_\varepsilon(t)) + g(x_\varepsilon, y_\varepsilon)$$
$$\frac{dy_\varepsilon}{dt} = \varepsilon^{-2} f(y_\varepsilon) \ ,$$

where the fast dynamics $y_\varepsilon(t) = y(\varepsilon^{-2} t)$ with $\dot{y} = g(y)$ describing a chaotic flow, with ergodic measure μ and again $\int v \, d\mu = 0$. We can re-write the equations as

$$dx_\varepsilon = h(x_\varepsilon) dw_\varepsilon + g(x_\varepsilon, y_\varepsilon) dt \quad \text{where}$$
$$w_\varepsilon \overset{def}{=} \varepsilon^{-1} \int_0^t v(y_\varepsilon(s)) ds = \varepsilon \int_0^{\varepsilon^{-2} t} v(y(s)) ds$$

What is known? (Melbourne, Stuart '11)

If the flow is chaotic enough so that

$$w_\varepsilon(t) = \varepsilon^{-1} \int_0^t v(y_\varepsilon(s))ds \to_w W \; ,$$

and **either** $d = 1$ **or** $h = \mathrm{Id}$

then we have that $x_\varepsilon \to X$, where

$$dX = h(X) \circ dW + G(X)dt \; ,$$

where the stochastic integral is of Stratonovich type and where $G(\cdot) = \int g(\cdot, v) d\mu(v)$.

Continuity with respect to noise (Sussmann '78)

The crucial fact that allows these results to go through is continuity with respect to noise. That is, let

$$dx = h(x)dU + g(x)dt \ .$$

If $d = 1$ **or** $h = Id$, then $\Phi : U \to x$ is continuous.

Therefore, if $w_\varepsilon \to_w W$ then $x_\varepsilon = \Phi(w_\varepsilon) \to_w \Phi(W)$.

This famously **falls apart** when the noise is both **multidimensional** and **multiplicative**.

Continuity with respect to rough paths

To treat the solution map Φ when the underlying SDE is both both **multidimensional** and **multiplicative**, we use rough path theory. Given an SDE driven by a noise $U : [0, T] \to \mathbb{R}^d$. Let $\mathbb{U} : [0, T] \to \mathbb{R}^{d \times d}$ be defined by

$$\mathbb{U}^{\alpha\beta}(t) \stackrel{def}{=} \int_0^t U^\alpha(s) dU^\beta(s) \; .$$

Then the map

$$\Phi : (U, \mathbb{U}) \mapsto \text{solution of SDE}$$

is **continuous**. This is known as **continuity** with respect to the **rough path** $(U, \mathbb{U})$.

Continuity with respect to rough paths

Thus, we set

$$\mathbb{W}_{\varepsilon}^{\alpha\beta}(t) = \int_0^t w_{\varepsilon}^{\alpha}(s) d w_{\varepsilon}^{\beta}(s) ,$$

(which is defined uniquely). If we can show that

$$(w_{\varepsilon}, \mathbb{W}_{\varepsilon}) \to_w (W, \mathbb{W})$$

where $\mathbb{W}$ is some identifiable type of iterated integral of W, then we have

$$x_{\varepsilon} \to X = \Phi(W, \mathbb{W}) .$$

Convergence of the rough path

We have the following result

Theorem (Kelly, Melbourne '13)

If the fast dynamics are "sufficiently chaotic", then $(w_\varepsilon, \mathbb{W}_\varepsilon) \to_w (W, \mathbb{W})$ where W is a multiple of Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2} D^{\alpha\beta} t$$

where

$$D^{\beta,\gamma} = \int_0^\infty \int_\Lambda (v^\beta\, v^\gamma \circ \phi_s - v^\gamma\, v^\beta \circ \phi_s)\, d\mu\, ds \ ,$$

and ϕ is the flow generated by the chaotic dynamics $\dot{y} = f(y)$.

Homogenised equations

Corollary

Under the same assumptions as above, the slow dynamics $x_\varepsilon \to_w X$ *where*

$$dX_\varepsilon = h(X) \circ dW + \left(G(X) + \frac{1}{2} D^{\beta\gamma} \partial^\alpha h^\beta(X) h^{\alpha\gamma}(X) \right) dt \,.$$

Rmk. The only case where one gets Stratonovich is when the Auto-correlation is symmetric.