# Fast-slow systems with chaotic noise

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March 28, 2014

Probability seminar, Warwick.

# Outline

Two problems :

**1** - Fast-slow systems in continuous time

**2** - Fast-slow systems in discrete time

# Fast-slow systems in continuous time

Let $\dot{Y} = g(Y)$ be some chaotic ODE with state space $\Lambda$ and invariant measure $\mu$. We consider fast-slow systems of the form

$$\begin{aligned} \frac{dX^{(\varepsilon)}}{dt} &= \varepsilon^{-1} h(X^{(\varepsilon)}, Y^{(\varepsilon)}) + f(X^{(\varepsilon)}, Y^{(\varepsilon)}) \\ \frac{dY^{(\varepsilon)}}{dt} &= \varepsilon^{-2} g(Y^{(\varepsilon)}) \ , \end{aligned}$$

where $\varepsilon \ll 1$ and $h, f : \mathbb{R}^e \times \Lambda \to \mathbb{R}^e$ and $\int h(\cdot, y)\, \mu(dy) = 0$. Also assume that $Y(0) \sim \mu$.

The aim is to characterize the **distribution** of $X^{(\varepsilon)}$ as $\varepsilon \to 0$.

# Fast-slow systems as SDEs

Consider the simplified slow equation

$$\frac{dX^{(\varepsilon)}}{dt} = \varepsilon^{-1} h(X^{(\varepsilon)}) v(Y^{(\varepsilon)}) + f(X^{(\varepsilon)})$$

where $h : \mathbb{R}^e \to \mathbb{R}^{e \times d}$ and $v : \Lambda \to \mathbb{R}^d$ with $\int v(y)\mu(dy) = 0$.

If we write $W^{(\varepsilon)}(t) = \varepsilon^{-1} \int_0^t v(Y^{(\varepsilon)}(s))ds$ then

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

where the integral is of Riemann-Lebesgue type.

# Invariance principle for $W^{(\varepsilon)}$

We can write $W^{(\varepsilon)}$ as

$$W^{(\varepsilon)}(t) = \varepsilon \int_0^{t/\varepsilon^2} v(Y(s))ds = \varepsilon \sum_{j=0}^{\lfloor t/\varepsilon^2 \rfloor - 1} \int_j^{j+1} v(Y(s))ds$$

The assumptions on $Y$ lead to **decay of correlations** for the sequence $\int_j^{j+1} v(Y(s))ds$.

One can show that $W^{(\varepsilon)} \Rightarrow W$ in the sup-norm topology, where $W$ is a multiple of Brownian motion.

# What about the SDE?

Since

$$X^{(\varepsilon)}(t) = X^{(\varepsilon)}(0) + \int_0^t h(X^{(\varepsilon)}(s))dW^{(\varepsilon)}(s) + \int_0^t f(X^{(\varepsilon)}(s))ds$$

This suggest a limiting SDE

$$X(t) = X(0) + \int_0^t h(X(s)) \star dW(s) + \int_0^t f(X(s))ds$$

But how should we interpret $\star dW$ ?

# Continuity with respect to noise (Sussmann ‘78)

Suppose that

$$X(t) = X(0) + \int_0^t h(X(s))dU(s) + \int_0^t f(X(s))ds\ ,$$

where $U$ is a smooth path.

If $d = 1$ **or** $h(x) = Id$ for all $x$, then $\Phi : U \to X$ is continuous in the sup-norm topology.

# The simple case (Melbourne, Stuart '11)

If the flow is chaotic enough so that

$$W^{(\varepsilon)} \Rightarrow W ,$$

and **either** $d = 1$ **or** $h = \mathrm{Id}$

then we have that $X^{(\varepsilon)} \Rightarrow X$ in the sup-norm topology, where

$$dX = h(X) \circ dW + f(X)ds ,$$

where the stochastic integral is of Stratonovich type.

This famously falls apart when the noise is both **multidimensional** and **multiplicative**. That is, when $d > 1$ **and** $h \neq Id$.

# Continuity with respect to rough paths (Lyons '97)

As above, let

$$X(t) = \int_0^t h(X(s))dU(s) + \int_0^t F(X(s))ds \ ,$$

where $U$ is a smooth path. Let $\mathbb{U} : [0, T] \to \mathbb{R}^{d\times d}$ be defined by

$$\mathbb{U}^{\alpha\beta}(t) \stackrel{def}{=} \int_0^t U^\alpha(s)dU^\beta(s) \ .$$

Then the map

$$\Phi : (U, \mathbb{U}) \mapsto X$$

is **continuous** with respect to the "$\rho_\gamma$ topology" . We call this the **rough path topology**.

# The rough path topology

The $\rho_\gamma$ topology is an extension of the $\gamma$-Hölder topology to the space of objects of the form $(U, \mathbb{U})$ ie. the space of **rough paths**. It has a metric

$$\rho_\gamma(U, \mathbb{U}, V, \mathbb{V}) = \sup_{s,t \in [0,T]} \left( \frac{|U(s,t) - V(s,t)|}{|s-t|^\gamma} + \frac{|\mathbb{U}(s,t) - \mathbb{V}(s,t)|}{|s-t|^{2\gamma}} \right)$$

where

$$U(s,t) = U(t) - U(s) \quad \text{and} \quad \mathbb{U}^{\beta\gamma}(s,t) = \int_s^t U^\beta(s,r)\, dU^\gamma(r)$$

In particular, it is **stronger** than the sup-norm topology.

# A general theorem for continuous fast-slow systems

Let $\mathbb{W}^{(\varepsilon),\alpha\beta}(t) = \int_0^t W^{(\varepsilon),\alpha}(s) dW^{(\varepsilon),\beta}(s)$.

Suppose that $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$ in the sup-norm topology where $W$ is Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^{\alpha}(s) \circ dW^{\beta}(s) + \lambda^{\alpha\beta} t$$

where $\lambda \in \mathbb{R}^{d \times d}$ and that $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)})$ satisfy the **tightness estimates**.

Then $X^{(\varepsilon)} \Rightarrow X$ in the sup norm topology, where

$$dX = h(X) \circ dW + \left( f(X) + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h^k_j(X) \right) dt$$

# Tightness estimates

To lift a sup-norm invariance principle to a $\rho_\gamma$ invariance principle, we use the **Kolmogorov criterion**. Let

$$W^{(\varepsilon)}(s,t) = W^{(\varepsilon)}(t) - W^{(\varepsilon)}(s)$$
$$\mathbb{W}^{(\varepsilon),\alpha\beta}(s,t) = \int_s^t W^{(\varepsilon),\alpha}(s,r)\,dW^{(\varepsilon),\beta}(r)$$

The **tightness estimates** are of the form

$$(\mathbf{E}_\mu|W^{(\varepsilon)}(s,t)|^q)^{1/q} \lesssim |t-s|^\alpha \;\text{ and }\; (\mathbf{E}_\mu|\mathbb{W}^{(\varepsilon)}(s,t)|^{q/2})^{2/q} \lesssim |t-s|^{2\alpha}$$

for $q$ large enough and $\alpha > 1/3$.

We have the following result

**Theorem (K, Melbourne '14)**

*If the fast dynamics are "sufficiently chaotic", then* $(W^{(\varepsilon)}, \mathbb{W}^{(\varepsilon)}) \Rightarrow (W, \mathbb{W})$ *where* $W$ *is a Brownian motion and*

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2}\lambda^{\alpha\beta} t$$

*where*

$$\lambda^{\beta\gamma} = \int_0^\infty \mathbf{E}_\mu(v^\beta\, v^\gamma(Y(s)) - v^\beta(Y(s))\, v^\gamma)\, ds \; .$$

# Homogenized equations

## Corollary

*Under the same assumptions as above, the slow dynamics $X^{(\varepsilon)} \Rightarrow X$ where*

$$dX = h(X) \circ dW + \left( f(X) + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h_j^k(X) \right) dt \,.$$

**Rmk.** The only case where one gets Stratonovich is when the Auto-correlation is symmetric. For instance, if the flow is **reversible**.

Now let's try **discrete** time ...

# Discrete time fast-slow systems

Suppose that $T : \Lambda \to \Lambda$ is a chaotic map with invariant measure $\mu$. We consider the discrete fast-slow system

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2} h(X_j^{(n)}, T^j) + n^{-1} f(X_j^{(n)}, T^j)$$

Now define the path $X^{(n)}(t) = X_{\lfloor nt \rfloor}^{(n)}$.

The aim is to characterize the distribution of the path $X^{(n)}$ as $n \to \infty$.

# Fast-slow systems as SDEs

Lets again simplify the slow equation to

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2} h(X_j^{(n)}) v(T^j) \; .$$

If we sum these up, we get

$$X^{(n)}(t) = X^{(n)}(0) + \sum_{j=0}^{\lfloor nt \rfloor - 1} h(X_j^{(n)}) \frac{v(T^j)}{n^{1/2}}$$

If we write $W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(T^j)$ then the path $X^{(n)}(t)$ satisfies

$$X^{(n)}(t) = X(0) + \int_0^t h(X^{(n)}(s-)) dW^{(n)}(s)$$

where the integral is defined in the "left-Riemann sum" sense.

# Invariance principle

$$W^{(n)}(t) = n^{-1/2} \sum_{j=0}^{\lfloor nt \rfloor - 1} v(T^j)$$

We still have that $W^{(n)} \Rightarrow W$ in the Skorokhod topology, where $W$ is a multiple of Brownian motion.

But $W^{(n)}$ is a step function ... so RPT doesn't really work... even if it did, you'll never satisfy the tightness estimates.

# A general theorem for discrete fast-slow systems (K 14')

Let

$$\mathbb{W}^{(n),\alpha\beta}(t) = n^{-1} \sum_{0 \le i < j < \lfloor nt \rfloor} v^{\alpha}(T^i) v^{\beta}(T^j)$$

Suppose that $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the Skorokhod topology where $W$ is Brownian motion and

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^{\alpha}(s) \circ dW^{\beta}(s) + \lambda^{\alpha\beta} t$$

where $\lambda \in \mathbb{R}^{d \times d}$ and that $(W^{(n)}, \mathbb{W}^{(n)})$ satisfy the **discrete tightness estimates**.

Then $X^{(n)} \Rightarrow X$ in the Skorokhod topology, where

$$dX(t) = h(X) \circ dW + \sum_{i,j,k} \lambda^{ik} \partial^j h^i(X) h^k_j(X) dt$$

# Discrete tightness estimates

The discrete tightness estimates are a **courser** version of the Kolmogorov criterion. Let

$$W^{(n),\alpha}(s,t) = n^{-1/2} \sum_{\lfloor ns \rfloor \le i < \lfloor nt \rfloor} v^{\alpha}(T^{i})$$

$$\mathbb{W}^{(n),\alpha\beta}(s,t) = n^{-1} \sum_{\lfloor ns \rfloor \le i < j < \lfloor nt \rfloor} v^{\alpha}(T^{i}) v^{\beta}(T^{j})$$

Then the **discrete tightness estimates** are of the form

$$(\mathbf{E}_{\mu}|W^{(n)}(\frac{j}{n},\frac{k}{n})|^{q})^{1/q} \lesssim \left|\frac{j-k}{n}\right|^{\alpha} \quad \text{and}$$

$$(\mathbf{E}_{\mu}|\mathbb{W}^{(n)}(\frac{j}{n},\frac{k}{n})|^{q/2})^{2/q} \lesssim \left|\frac{j-k}{n}\right|^{2\alpha}$$

for all $j, k = 0, \ldots, n$, for $q$ large enough and $\alpha > 1/3$.

We have the following result

### Theorem (K, Melbourne '14)

*If the fast dynamics are "sufficiently chaotic", then* $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ *in the Skorokhod topology, where* $W$ *is a Brownian motion and*

$$\mathbb{W}^{\alpha\beta}(t) = \int_0^t W^\alpha(s) \circ dW^\beta(s) + \frac{1}{2}\kappa^{\alpha\beta} t$$

*where*

$$\kappa^{\alpha\beta} = \sum_{j=1}^{\infty} \mathbf{E}_\mu v^\alpha v^\beta (T^j)$$

# Homogenized equations

### Corollary

*Under the same assumptions as above, the slow dynamics* $X^{(n)} \Rightarrow X$ *where*

$$dX = h(X) \circ dW + \sum_{i,j,k} \frac{1}{2} \kappa^{jk} \partial^i h^j(X) h^{ik}(X) dt .$$

# Idea of proof

Recall that

$$X_{j+1}^{(n)} = X_j^{(n)} + n^{-1/2} h(X_j^{(n)}) v(T^j) \ .$$

The idea is to approximate $X^{(n)}(t) = X_{\lfloor nt \rfloor}^{(n)}$ by $\tilde{X}^{(n)}(t)$, which solves an equation driven by smooth paths.

# Idea of proof

This can be achieved by finding a (piecewise smooth) rough path $\tilde{\mathbf{W}}^{(n)} = (\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)})$ such that

$$\left(\tilde{W}^{(n)}(\frac{j}{n}), \tilde{\mathbb{W}}^{(n)}(\frac{j}{n})\right) = \left(W^{(n)}(\frac{j}{n}), \mathbb{W}^{(n)}(\frac{j}{n})\right)$$

for all $j = 0, \ldots, n$ and which is Lipschitz in between mesh points.

Then define

$$\tilde{X}^{(n)}(t) = X(0) + \int_0^t h(\tilde{X}^{(n)}(s)) d\tilde{\mathbf{W}}^{(n)}(s)$$

# Idea of proof

Alternatively we can write

$$\tilde{X}^{(n)}(t) = X(0) + \int_0^t h(\tilde{X}^{(n)}(s))d\tilde{W}^{(n)}(s) + \sum_{i,j,k} \int_0^t \frac{1}{2}\partial^i h^j(X)h^{ik}(X)dZ^{(n),jk}(s)$$

where $Z^{(n)}$ is a piecewise smooth path.

# Idea of proof

By construction, $\tilde{X}^{(n)}$ is a good approximation of $X^{(n)}$.

**Proposition**

*We have that*

$$\sup_{j=0\dots n} |X^{(n)}(j/n) - \tilde{X}^{(n)}(j/n)| \lesssim K_{n,\gamma} n^{1-3\gamma} \ ,$$

*for any $\gamma \in (1/3, 1/2]$, where the constant $K_{n,\gamma}$ depends on $n$ through the "discrete Hölder norms" of $(W^{(n)}, \mathbb{W}^{(n)})$.*

As a consequence, if $\tilde{X}^{(n)} \Rightarrow X$ then $X^{(n)} \Rightarrow X$.

# Idea of proof

But since $\tilde{X}^{(n)}$ is driven by smooth paths, we can apply the ideas from the first half of the talk.

But again by construction ...

- If $(W^{(n)}, \mathbb{W}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the Skorokhod topology then $(\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)}) \Rightarrow (W, \mathbb{W})$ in the sup-norm topology.

- If $(W^{(n)}, \mathbb{W}^{(n)})$ satisfy the discrete tightness estimates, then $(\tilde{W}^{(n)}, \tilde{\mathbb{W}}^{(n)})$ satisfy the continuous tightness estimates.

Thus $\tilde{X}^{(n)} \Rightarrow X$.

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